Communications in Applied Analysis **10** (2006), no. 1, 1–7

UPPER AND LOWER BOUNDS ON THE BINARY INPUT AWGN CHANNEL CAPACITY

John J. Komo[1] and Laurie L. Joiner[2]

[1]Department of Electrical and Computer Engineering
Clemson University
Clemson, SC 29634, USA
john.komo@ces.clemson.edu

[2]Department of Electrical and Computer Engineering
University of Alabama in Huntsville
Huntsville, AL 35899, USA
ljoiner@ece.uah.edu

Communicated by D.D. Bainov

ABSTRACT: A series expression for evaluating the channel capacity of the binary input AWGN channel is developed which precludes the necessity of numerical integration. This series expression is compact and easy to evaluate. In addition, tight upper and lower bounds are developed for the binary input AWGN channel capacity. These bounds also provide good approximations for the capacity and are within 0.1% relative error (within 0.01% for upper bound) for $C \geq 0.457$ and a corresponding $E_b/N_0 \geq 0$ dB.

AMS (MOS) Subject Classification. 94A24

1. INTRODUCTION

In order to evaluate the limit of performance of forward error correcting codes, it is important to know the channel capacity, C, which places an upper limit on the code rate Cover and Thomas [2]. Shannon's Channel Coding Theorem states that if the data rate $R \leq C$ error free communication is possible, and for $R > C$ reliable communication is not possible. For an additive white Gaussian noise (AWGN) channel with equally likely binary antipodal signaling, the channel capacity is given as Stark [4]

$$C = 1 - \frac{1}{\sqrt{2\pi}\ln(2)} \int_{-\infty}^{\infty} e^{-(y-\beta)^2/2} \ln(1 + e^{-2\beta y}) dy, \tag{1}$$

Received February 24, 2005

where $\beta = \sqrt{2E_s/N_0}$, the energy per transmitted symbol $E_s = RE_b$, R is the code rate in bits per symbol, and E_b is the energy per bit. To achieve capacity $R = C$, so the required E_b/N_0 to achieve capacity is $E_b/N_0 = \beta^2/2R$, where $R = C$ is obtained from (1). Previously, the channel capacity in (1) was only obtainable by numerical integration.

2. CHANNEL CAPACITY

The biggest problem in evaluating (1) numerically is the integration over the negative range since the exponential term goes to zero while the logarithm term goes to infinity. Letting $z = -y$ for $-\infty < y < 0$, (1) can be obtained as an integration over the positive range only, so C can then be expressed as

$$C = 1 - \frac{1}{\sqrt{2\pi}\ln(2)} \left[\int_0^\infty e^{-(y-\beta)^2/2} \ln(1+e^{-2\beta y}) dy + \int_0^\infty e^{-(z+\beta)^2/2} \ln[e^{2\beta z}(1+e^{-2\beta z})] dz \right],$$

which can be written

$$C = 1 - \frac{1}{\sqrt{2\pi}\ln(2)} \left[\int_0^\infty e^{-(y-\beta)^2/2} \ln(1+e^{-2\beta y}) dy + \int_0^\infty e^{-(z+\beta)^2/2} \ln(1+e^{-2\beta z}) dz + \int_0^\infty 2\beta z e^{-(z+\beta)^2/2} dz \right]. \tag{2}$$

Since

$$\ln(1+e^{-2\beta y}) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} e^{-2i\beta y}, \quad y \geq 0,$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_0^\infty & e^{-(y-\beta)^2/2} \ln(1+e^{-2\beta y}) dy \\ &= \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} \int_0^\infty e^{-2i\beta y} e^{-(y-\beta)^2/2} dy \\ &= \frac{1}{\sqrt{2\pi}} \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} e^{2i(i-1)\beta^2} \int_0^\infty e^{-[y+(2i-1)\beta]^2/2} dy \\ &= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} e^{2i(i-1)\beta^2} Q[(2i-1)\beta], \end{aligned} \tag{3}$$

where

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

Similarly,

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(z+\beta)^2/2}\ln(1+e^{-2\beta z})dz \\
&\quad=\frac{1}{\sqrt{2\pi}}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}\int_0^\infty e^{-2i\beta z}e^{-(z+\beta)^2/2}dz \\
&\quad=\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}e^{2i(i+1)\beta^2}Q[(2i+1)\beta].
\end{aligned}
\tag{4}
$$

The last term of C in (2) is evaluated as

$$
\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_0^\infty 2\beta z e^{-(z+\beta)^2/2}dz \\
&=\frac{1}{\sqrt{2\pi}}\left[2\beta\int_0^\infty (z+\beta)e^{-(z+\beta)^2/2}dz\ -2\beta^2\int_0^\infty e^{-(z+\beta)^2/2}dz\right] \\
&=\frac{2\beta e^{-\beta^2/2}}{\sqrt{2\pi}}-2\beta^2 Q(\beta).
\end{aligned}
\tag{5}
$$

Combining (3), (4), and (5) into (2) yields

$$
\begin{aligned}
C=&1-\frac{1}{\ln(2)}\left[\frac{2\beta e^{-\beta^2/2}}{\sqrt{2\pi}}-(2\beta^2-1)Q(\beta)\right. \\
&\left.+\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i(i+1)}e^{2i(i+1)\beta^2}Q[(2i+1)\beta]\right].
\end{aligned}
\tag{6}
$$

Thus, the channel capacity for the binary input AWGN channel can be obtained directly from (6) without numerical integration.

3. BOUNDS AND APPROXIMATIONS

To avoid evaluating the summation in (6), tight upper and lower bounds which can be used as approximations are developed. Let this summation be represented as $S=\sum_{i=1}^{\infty}(-1)^{i-1}c_i$, where c_i is the magnitude of the i-th term. It can be observed that

$$
\begin{aligned}
e^{2i(i+1)\beta^2}Q[(2i+1)\beta]&=e^{2i(i+1)\beta^2}\int_{(2i+1)\beta}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-t^2/2}dt \\
&=e^{4\beta^2}\int_{3\beta}^{\infty}\sqrt{\frac{u^2}{u^2+[(2i+1)^2-9]\beta^2}}\frac{1}{\sqrt{2\pi}}e^{-u^2/2}du,
\end{aligned}
\tag{7}
$$

where the change of variable $u^2 = t^2 - [(2i+1)^2 - 9]\beta^2$ has been used. Note that for $i \geq 1$ and $3\beta \leq u < \infty$, $\frac{3}{2i+1} \leq \sqrt{\frac{u^2}{u^2+[(2i+1)^2-9]\beta^2}} \leq 1$, from which it follows that

$$\frac{3}{2i+1} e^{4\beta^2} Q(3\beta) \leq e^{2i(i+1)\beta^2} Q[(2i+1)\beta] \leq e^{4\beta^2} Q(3\beta). \tag{8}$$

In a similar manner it can be shown that for $i \geq 1$

$$\begin{aligned} \frac{2i+1}{2i+3} e^{2i(i+1)\beta^2} Q[(2i+1)\beta] &\leq e^{2(i+1)(i+2)\beta^2} Q[(2i+3)\beta] \\ &\leq e^{2i(i+1)\beta^2} Q[(2i+1)\beta]. \end{aligned} \tag{9}$$

Using the upper bound of (9) it follows that

$$\begin{aligned} c_i - c_{i-1} &\geq \frac{e^{2i(i+1)\beta^2} Q[(2i+1)\beta]}{i(i+1)} - \frac{e^{2i(i+1)\beta^2} Q[(2i+1)\beta]}{(i+1)(i+2)} \\ &= \frac{2e^{2i(i+1)\beta^2} Q[(2i+1)\beta]}{i(i+1)(i+2)} \geq 0, \end{aligned}$$

the terms in S are decreasing, and S converges, since it is an alternating series. Thus S can be bounded by a finite number of terms, and the error in the bound carries the sign of the first term neglected. In general this series converges very slowly and direct upper and lower bounds are not extremely useful.

An approximation, a_i, to c_i can be obtained using the lower bound of (8) as

$$a_i = \frac{3e^{4\beta^2} Q(3\beta)}{i(i+1)(2i+1)}. \tag{10}$$

The i-th error term is then given as $e_i = c_i - a_i$, with $c_1 = a_1$ and $e_1 = 0$. Now,

$$\begin{aligned} e_i - e_{i+1} =& \frac{(i+2)e^{2i(i+1)\beta^2} Q[(2i+1)\beta] - ie^{2(i+1)(i+2)\beta^2} Q[(2i+3)\beta]}{i(i+1)(i+2)} \\ &- \frac{18e^{4\beta^2} Q(3\beta)}{i(i+2)(2i+1)(2i+3)}. \end{aligned} \tag{11}$$

Consider

$$\begin{aligned} &(i+2)e^{2i(i+1)\beta^2} Q[(2i+1)\beta] - ie^{2(i+1)(i+2)\beta^2} Q[(2i+3)\beta] \\ &= (i+2)e^{2i(i+1)\beta^2} \int_{(2i+1)\beta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt - ie^{2(i+1)(i+2)\beta^2} \\ &\quad \times \int_{(2i+3)\beta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \\ &= e^{4\beta^2} \int_{3\beta}^{\infty} f(u) \frac{1}{\sqrt{2\pi}} e^{-u^2/2} du\,, \end{aligned} \tag{12}$$

where

$$f(u) = (i+2)\sqrt{\frac{u^2}{u^2+[(2i+1)^2-9]\beta^2}} - i\sqrt{\frac{u^2}{u^2+[(2i+3)^2-9]\beta^2}}.$$

Since for $i \geq 2$, $f(0) = 0$,

$$f(3\beta) = \frac{18(i+1)}{(2i+1)(2i+3)} < 2, \quad f(\infty) = 2,$$

and only one maximum at $u > 3\beta$ (no minimum), $f(3\beta)$ yields the minimum value of $f(u)$ for $3\beta \leq u < \infty$. Thus $f(u) \geq \frac{18(i+1)}{(2i+1)(2i+3)}$, and (12) can be lower bounded as

$$\begin{aligned} &(i+2)e^{2i(i+1)\beta^2}Q[(2i+1)\beta] - ie^{2(i+1)(i+2)\beta^2}Q[(2i+3)\beta] \\ &\geq \frac{18(i+1)e^{4\beta^2}Q(3\beta)}{(2i+1)(2i+3)}. \end{aligned} \tag{13}$$

Putting (13) in (11) yields $e_i - e_{i+1} \geq 0$ for $i \geq 2$, which shows that the series obtained from e_i and a_i alternate and converge. Thus S can be upper bounded by replacing c_i by a_i for $i \geq 2$ ($i = 2$ term is negative), which yields

$$\begin{aligned} S &= \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i(i+1)} e^{2i(i+1)\beta^2} Q[(2i+1)\beta] \\ &\leq 3e^{4\beta^2}Q(3\beta)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i(i+1)(2i+1)} = 3(\pi-3)e^{4\beta^2}Q(3\beta)\,, \end{aligned} \tag{14}$$

where Jolley [3]

$$\begin{aligned} \sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i(i+1)(2i+1)} &= \sum_{i=1}^{\infty}(-1)^{i-1}\left[\frac{1}{i}+\frac{1}{i+1}-\frac{4}{2i+1}\right] \\ &= \ln(2) + [1-\ln(2)] - 4\left(1-\frac{\pi}{4}\right) = \pi - 3 \end{aligned}$$

has been used. Correspondingly, inserting (14) into (6), a lower bound for C is obtained as

$$C \geq 1 - \frac{1}{\ln(2)}\left[\frac{2\beta e^{-\beta^2/2}}{\sqrt{2\pi}} - (2\beta^2-1)Q(\beta) + 3(\pi-3)e^{4\beta^2}Q(3\beta)\right]. \tag{15}$$

This bound is also a good approximation, evaluates the channel capacity using just three terms, and is quite accurate over a large range of values. For $E_b/N_0 = 0$ dB ($\beta = 0.956$ and $C = 0.457$) the relative error using (15) is 0.101% and for $E_b/N_0 =$ 3.00 dB ($\beta = 1.882$ and $C = 0.887$) the relative error is 0.00231%. Note that $Q(x)$ can be accurately approximated Borjesson and Sundberg [1], thus precluding the need for numerical integration.

A good upper bound on C, with four terms, can be obtained starting with the bound, $i \geq 2$ (similar to (8))

$$\frac{5}{2i+1} e^{12\beta^2} Q(5\beta) \leq e^{2i(i+1)\beta^2} Q[(2i+1)\beta] \leq e^{12\beta^2} Q(5\beta). \tag{16}$$

The approximation that leads to an upper bound (similar to (10)) is given as

$$a_i = \frac{5e^{12\beta^2} Q(5\beta)}{i(i+1)(2i+1)}, \quad i \geq 2, \tag{17}$$

which results in

$$S \geq 5e^{12\beta^2} Q(5\beta) \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{i(i+1)(2i+1)} = -5\left(\frac{19}{6} - \pi\right) e^{12\beta^2} Q(5\beta)\,, \tag{18}$$

where Jolley [3]

$$\begin{aligned} \sum_{i=2}^{\infty} \frac{(-1)^{i-1}}{i(i+1)(2i+1)} &= \sum_{i=2}^{\infty} (-1)^{i-1} \left[\frac{1}{i} + \frac{1}{i+1} - \frac{4}{2i+1}\right] \\ &= [\ln(2) - 1] + \left[1 - \ln(2) - \frac{1}{2}\right] - 4\left(1 - \frac{\pi}{4} - \frac{1}{3}\right) \\ &= -\frac{19}{6} + \pi \end{aligned}$$

has been used. Inserting (18) into (6), an upper bound for C is given as

$$\begin{aligned} C \leq 1 - \frac{1}{\ln(2)} \Bigg[& \frac{2\beta e^{-\beta^2/2}}{\sqrt{2\pi}} - (2\beta^2 - 1) Q(\beta) + \frac{1}{2} e^{4\beta^2} Q(3\beta) \\ & -5\left(\frac{19}{6} - \pi\right) e^{12\beta^2} Q(5\beta) \Bigg]. \end{aligned} \tag{19}$$

This upper bound is a better approximation to C than the lower bound of (15). For $E_b/N_0 = 0$ dB ($\beta = 0.956$ and $C = 0.457$) the relative error using (19) is 0.0101% and for $E_b/N_0 = 3.00$ dB ($\beta = 1.882$ and $C = 0.887$) the relative error is 0.000205%. Even though the lower bound approximation of C has a relative error of 9.54% at $E_b/N_0 = -1.49$ dB ($\beta = 0.3$ and $C = 0.0635$), the upper bound approximation has only a relative error of 1.55%.

4. CONCLUSIONS

A useful series expression for evaluating the channel capacity of the binary input AWGN channel has been obtained in (6). This series expression eliminates the necessity of evaluating the channel capacity of (1) by numerical integration. Tight upper and lower bounds on C in (6) for the binary input AWGN channel capacity are developed in (15) and (19). These bounds also yield good approximations for most ranges of E_b/N_0 and corresponding C.

ACKNOWLEDGEMENTS

This work was supported by the DoD Multidisciplinary University Research Initiative (MURI) program administered by the Office of Naval Research under Grant N00014-00-1-0565.

REFERENCES

[1] P.O. Borjesson and C.E.W. Sundberg, Simple approximations of the error function $Q(x)$ for communication applications, *IEEE Transactions on Communications*, **27** (1979), 639-643.

[2] T.M. Cover and J.A. Thomas, *Elements of Information Theory*, Wiley Interscience, 1991.

[3] L.B.W. Jolley, *Summation of Series*, Second Edition, Dover, 1961.

[4] W.E. Stark, Unpublished course notes, University of Michigan, 2000.