Communications in Applied Analysis **10** (2006), no. 1, 97–107

SOME INEQUALITIES ON TIME SCALES

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Communicated by D.D. Bainov

ABSTRACT: In this paper, we establish the Steffensen, Hardy, Grüss and Centroidal inequalities on time scales.

AMS (MOS) Subject Classification. 26B25, 26D15

1. INTRODUCTION

The purpose of this paper is to establish some renewed inequalities (for example, the inequalities of Steffensen, Hardy and Grüss) on time scales. For releted results, we refer to Agarwal et al [1], Dunkel [4], Hardy et al [5], Mitrinovic [8]. To do this, we briefly introduce the time scales theory and refer to Aulbach and Hilger [2], Hilger [6] and the books Bohner and Peterson [3] and Kaymakcalan et al [7] for further details.

Definition 1.A. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the the set $\mathbb{R}$ of all real numbers. Let $\mathbb{T}$ have the topology that it inherits from the standard

Received September 10, 2005

topology on $\mathbb{R}$. For $t \in \mathbb{T}$, if $t < \sup \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\} \in \mathbb{T},$$

while if $t > \inf \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\} \in \mathbb{T}.$$

If $\sigma(t) > t$, we say t is right scattered, while if $\rho(t) < t$, we say t is left scattered. If $\sigma(t) = t$, we say t is right dense, while if $\rho(t) = t$, we say t is left dense.

Throughout this paper, we suppose that:

(a) $\mathbb{R} = (-\infty, +\infty)$;

(b) $\mathbb{T}$ is a time scale;

(c) an interval means the intersection of a real interval with the given time scale.

Definition 1.B. If $f : \mathbb{T} \to \mathbb{R}$, then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by

$$f^\sigma(t) = f(\sigma(t))$$

for all $t \in \mathbb{T}$.

Definition 1.C. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous if it satisfies:

(A) f is continuous at each right-dense point or maximal element of $\mathbb{T}$.

(B) the left-sided limit $\lim_{s \to t^-} f(s) = f(t^-)$ exists at each left-dense point t of $\mathbb{T}$.

Let

$$C_{rd}(\mathbb{T}, \mathbb{R}) := \{f \mid f : \mathbb{T} \to \mathbb{R} \text{ is a rd-continuous function}\}$$

and

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Definition 1.D. Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}^k$, then we define $x^\Delta(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^\Delta(t)$ the *delta derivative* of $x(t)$ at t.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Definition 1.E. A function $F : \mathbb{T} \to \mathbb{R}$ is an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, we define the integral of f by

$$\int_s^t f(\tau)\ \Delta\tau = F(t) - F(s)$$

for $s, t \in \mathbb{T}^k$.

It follows from Theorem 1.74 of Bohner and Peterson [3] that every rd-continuous function has an antiderivative.

2. STEFFENSEN'S INEQUALITY

First, we establish Steffensen's inequality Mitrinovic [8] on time scales as follows.

Theorem 2.1. (Steffensen's Inequality) *Let $f, g, h \in C_{rd}([a, b], \mathbb{R})$ with*

$$\int_a^b g(t)\Delta t > 0, \quad \int_a^b h(t)\Delta t > 0$$

and

$$G(x) = \int_a^x g(t)\Delta t \quad \text{and} \quad H(x) = \int_a^x h(t)\Delta t$$

satisfy

$$\frac{G(x)}{G(b)} \leq \frac{H(x)}{H(b)} \quad \text{on} \quad [a, b]. \tag{C_1}$$

Then, the following two results hold:

(a) if f is increasing on $[a, b]$, then

$$\frac{\int_a^b f^{\sigma}(t)g(t)\Delta t}{\int_a^b g(t)\Delta t} \geq \frac{\int_a^b f^{\sigma}(t)h(t)\Delta t}{\int_a^b h(t)\Delta t}; \tag{R_1}$$

(b) if f is decreasing on $[a, b]$, then

$$\frac{\int_a^b f^{\sigma}(t)g(t)\Delta t}{\int_a^b g(t)\Delta t} \leq \frac{\int_a^b f^{\sigma}(t)h(t)\Delta t}{\int_a^b h(t)\Delta t}. \tag{R_1^*}$$

Proof. It follows for Theorem 1.77 of Bohner and Peterson [3] that

$$\begin{aligned}
&\frac{\displaystyle\int_a^b f^\sigma(t)g(t)\Delta t}{\displaystyle\int_a^b g(t)\Delta t} \\
&= \frac{\displaystyle\int_a^b f^\sigma(t)G^\Delta(t)\Delta t}{G(b)} \qquad (1)\\
&= \frac{1}{G(b)}[f(t)G(t)|_a^b - \int_a^b f^\Delta(t)G(t)\Delta t] \\
&= \frac{1}{G(b)}[f(b)G(b) - \int_a^b f^\Delta(t)G(t)\Delta t].
\end{aligned}$$

Similarly,

$$\frac{\displaystyle\int_a^b f^\sigma(t)h(t)\Delta t}{\displaystyle\int_a^b h(t)\Delta t} = \frac{f(b)H(b) - \displaystyle\int_a^b f^\Delta(t)H(t)\Delta t}{H(b)}. \qquad (2)$$

Thus, by (1), (2) and (C_1),

$$\frac{1}{G(b)}\int_a^b f^\sigma(t)g(t)\Delta t - \frac{1}{H(b)}\int_a^b f^\sigma(t)h(t)\Delta t = \frac{\displaystyle\int_a^b f^\Delta(t)\{G(b)H(t) - H(b)G(t)\}\Delta t}{G(b)H(b)}.$$

Using the monotonic property of f, we obtain the desired results. □

Theorem 2.2. (Hayashi) *Let $f, g \in C_{rd}([a,b],\mathbb{R})$. If $f(x)$ is nondecreasing and $g(x) \in [0, A]$ on $[a,b]$ for some constant A. Then*

$$A\int_{b-h}^b f(t)\Delta t \le A\int_a^b f(t)g(t)\Delta t \le \int_a^{a+h} f(t)\Delta t, \qquad (R_1)$$

where

$$h = \frac{1}{A}\int_a^b g(t)\Delta t.$$

3. HARDY'S INEQUALITY

In this section, we establish two Hardy's inequalities Hardy et al [5] on time scales as follows.

Theorem 3.1. (Hardy's Inequality) *Let $f \in C_{rd}([a,\infty],\mathbb{R})$ with $a > 0$. If $p > 1$, then*

$$\int_a^b \Big(\frac{1}{x}\int_a^x |f(t)|\Delta t\Big)^p < \Big(\frac{b}{a}\Big)^p\Big(\frac{p}{p-1}\Big)^p \int_a^b |f(t)|^p \Delta t$$

for all $b \geq a > 0$.

Proof. Since $F(w) = w^p$ is convex for $w \geq 0$, it follows from Jensen's inequality (Theorem 6.17 of Bohner and Peterson [3]) that

$$\Bigg(\frac{\int_a^x |f(t)|\Delta t}{\int_a^x \Delta t}\Bigg)^p \leq \frac{\int_a^x |f(t)|^p\Delta t}{\int_a^x \Delta t}$$

for all $x \geq a$. Thus

$$\begin{aligned}\Big(\int_a^x |f(t)|\Delta t\Big)^p &\leq \Big(\int_a^x \Delta t\Big)^{p-1}\Big(\int_a^x |f(t)|^p\Delta t\Big) \\ &\leq (x-a)^{p-1}\int_a^x |f(t)|^p\Delta t\end{aligned} \tag{3}$$

for all $x \geq a$. It follows from

$$(b-a)^p(p-1)^p < b^p p^p$$

that

$$\Big(\frac{b-a}{b}\Big)^p < \Big(\frac{p}{p-1}\Big)^p. \tag{4}$$

By Theorem 1.24 of Bohner and Peterson [3], we see that

$$\begin{aligned}\Big(\frac{t^p}{p}\Big)^\Delta &= \frac{1}{p}\sum_{k=0}^{p-1}(\sigma(t))^k t^{p-1-k} \\ &\geq \frac{1}{p}\sum_{k=0}^{p-1} t^k t^{p-1-k} \\ &= t^{p-1},\end{aligned}$$

which implies

$$\frac{(b-a)^p}{p} = \frac{t^p}{p}\Big|_0^{b-a} \geq \int_a^{b-a}\Big(\frac{t^p}{p}\Big)^\Delta \Delta t \geq \int_0^{b-a} t^{p-1}\Delta t. \tag{5}$$

It follows from (3), (4) and (5) that

$$\int_a^b (\frac{1}{x}\int_a^x |f(t)|\Delta t)^p \Delta x$$

$$\leq \int_a^b (x-a)^{p-1}\frac{1}{x^p}(\int_a^x |f(t)|^p \Delta t)\Delta x$$

$$\leq \frac{1}{a^p}(\int_a^b |f(t)|^p \Delta t)\int_a^b (x-a)^{p-1}\Delta x$$

$$= \frac{\int_a^b |f(t)|^p \Delta t}{a^p}\int_0^{b-a} u^{p-1}\Delta u$$

$$\leq \frac{1}{a^p}\frac{(b-a)^p}{p}(\int_a^b |f(t)|^p \Delta t)$$

$$= \frac{1}{p}(\frac{b}{a})^p(\frac{b-a}{b})^p(\int_a^b |f(t)|^p \Delta t)$$

$$< \frac{1}{p}(\frac{b}{a})^p(\frac{p}{p-1})^p\int_a^b |f(t)|^p \Delta t$$

$$< (\frac{b}{a})^p(\frac{p}{p-1})^p\int_a^b |f(t)|^p \Delta t.$$

Thus, we complete the proof. □

Theorem 3.2. (Hardy's Inequality for Increasing Functions) *Let $f \in C_{rd}([a,\infty),\mathbb{R})$ with $a > 0$. If $p > 1$ and $|f(x)|$ is increasing on $[a,\infty)$, then*

$$\int_a^b (\frac{1}{x}\int_a^x |f(t)|\Delta t)^p \Delta x < (\frac{p}{p-1})^p \int_a^b |f(t)|^p \Delta t$$

for $b > a > 0$.

Proof. As in the proof of Theorem 3.1, we see that inequality (3) holds for $x \geq a$. Thus, it follows from Corollary 1.68 of Bohner and Peterson [3] that, for $b > a > 0$,

$$\int_a^b (\frac{1}{x}\int_a^x |f(t)|\Delta t)^p \Delta x$$

$$\leq \int_a^b (\frac{x-a}{x})^p \frac{1}{x-a}(\int_a^x |f(t)|^p \Delta t)\Delta x$$

$$= \int_a^b (\frac{x-a}{x})^p \frac{H(x)-H(a)}{x-a}\Delta x$$

$$\leq \int_a^b (\frac{p}{p-1})^p (\frac{\sup_{t\in[a,x]} (H^{\Delta}(t))(x-a)}{x-a})\Delta x$$

$$= \int_a^b (\frac{p}{p-1})^p (\sup_{t\in[a,x]} |f(t)|^p)\Delta x$$

$$\leq (\frac{p}{p-1})^p \int_a^b |f(x)|^p \Delta x$$

by using $|f(t)|$ is increasing, where

$$H(x) := \int_a^x |f(t)|^p \Delta t \qquad \text{and}$$

$$H^{\Delta}(x) = |f(x)|^p \geq |f(t)|^p, \quad t \in [a,x].$$

This completes the proof. □

4. THE GRÜSS INEQUALITY

In this section, we establish the Grüss inequality Mitrinovic [8] on time scales as follows.

Theorem 4.1. (The Grüss Inequality) *Let* $f, g, p \in C_{rd}([a,b],\mathbb{R})$ *satisfy*

$$p(x) \geq 0,\ h \leq f(x) \leq H,\ m \leq g(x) \leq M \text{ for } x \in [a,b],$$

where h, H, m *and* M *are fixed real constants. Then*

$$|\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)f(x)g(x)\Delta x$$

$$- \frac{1}{(\int_a^b p(x)\Delta x)^2}(\int_a^b p(x)f(x)\Delta x)(\int_a^b p(x)g(x)\Delta x)|$$

$$\leq \frac{(H-h)(M-m)}{4}.$$

Proof. Let

$$D_p(f,g)=\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)f(x)g(x)\Delta x-FG,$$

where

$$F=\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)f(x)\Delta x,$$

$$G=\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)g(x)\Delta x\,.$$

It follows from

$$(H-F)(F-h)-\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)(H-f(x))(f(x)-h)\Delta x$$

$$=HF-Hh-F^2+Fh-\frac{1}{(\int_a^b p(x)\Delta x)}[H\int_a^b p(x)f(x)\Delta x-Hh\int_a^b p(x)\Delta x$$

$$-\int_a^b p(x)f^2(x)\Delta x+h\int_a^b p(x)f(x)\Delta x]$$

$$=\frac{1}{\int_a^b p(x)dx}\int_a^b p(x)f^2(x)\Delta x-(\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)f(x)\Delta x)^2$$

$$=D_p(f,f)$$

and

$$D_p(g,g)\;=\;(M\;-\;G)(G\;-\;m)\;-\;\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)(M\;-\;g(x))(g(x)\;-\;m)\Delta x\,,$$

that

$$D_p(f,f)\leq(H-F)(F-h)$$

and

$$D_p(g,g)\leq(M-G)(G-m).$$

But

$$\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)(f(x)-F)(g(x)-G)\Delta x$$

$$=\frac{1}{\int_a^b p(x)\Delta x}[\int_a^b p(x)f(x)g(x)\Delta x - G\int_a^b p(x)f(x)\Delta x$$

$$-F\int_a^b p(x)g(x)\Delta x + FG\int_a^b p(x)\Delta x]$$

$$=\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)f(x)g(x)\Delta x - FG$$

$$=D_p(f,g).$$

By the Cauchy inequality (Theorem 6.15 of Bohner and Peterson [3]),

$$D_p^2(f,g)\leq \frac{1}{(\int_a^b p(x)\Delta x)^2}[\int_a^b p(x)(f(x)-F)^2\Delta x][\int_a^b p(x)(g(x)-G)^2\Delta x].$$

However,

$$\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)(f(x)-F)^2\Delta x$$

$$=\frac{1}{\int_a^b p(x)\Delta x}[\int_a^b (p(x)f^2(x)-2Ff(x)p(x)+p(x)F^2)\Delta x]$$

$$=\frac{\int_a^b p(x)f(x)\Delta x}{\int_a^b p(x)\Delta x}-2F\frac{\int_a^b p(x)f^2(x)\Delta x}{\int_a^b p(x)\Delta x}+F^2$$

$$=\frac{\int_a^b p(x)f^2(x)\Delta x}{\int_a^b p(x)\Delta x}-F^2$$

$$=D_p(f,f).$$

Similarly,

$$\frac{1}{\int_a^b p(x)\Delta x}\int_a^b p(x)(g(x)-G)^2\Delta x = D_p(g,g).$$

Hence

$$D_p^2(f,g) \le D_p(f,f)D_p(g,g)$$
$$\le (H-F)(F-h)(M-G)(G-m).$$

But

$$(H-h)^2 = (H-F+F-h)^2$$
$$= (H-F)^2 + (F-h)^2 + 2(H-F)(F-h)$$
$$\ge 4(H-F)(F-h).$$

Similarly,

$$(M-m)^2 \ge 4(M-G)(G-m).$$

Thus

$$D_p(f,g)^2 \le \frac{1}{16}(H-h)^2(M-m)^2,$$

which implies

$$D_p(f,g) \le \frac{1}{4}(H-h)(M-m).$$

□

5. THE CENTROIDAL INEQUALITY

In this section, we establish the Centroidal inequality Mitrinovic [8] on time scales as follows.

Theorem 5.1. (The Centroidal Inequality, see Mitrinovic [8]) *Let*

$$f, g \in C_{rd}([a,b],[0,\infty))$$

with $\int_a^b f(x)\Delta x > 0$ *and* $\int_a^b f(x)g(x)\Delta x > 0$. *If* g *is decreasing (increasing) on* $[a,b]$, *then*

$$\frac{\int_a^b xf(x)g(x)\Delta x}{\int_a^b f(x)g(x)\Delta x} \le (\ge) \frac{\int_a^b xf(x)\Delta x}{\int_a^b f(x)\Delta x} \qquad (R_3)$$

Proof. Let the right-hand side of (R_3) be denoted by A, then

$$\int_a^b (x-A)f(x)\Delta x = 0.$$

It follows from $a \le x \le b$ and the decreasing property of g that

$$\int_a^A (x-A)f(x)g(x)\Delta x \le g(A)\int_a^A (x-A)f(x)\Delta x,$$

$$\int_A^b (x-A)f(x)g(x)\Delta x \le g(A)\int_A^b (x-A)f(x)\Delta x.$$

The above two inequalities imply

$$\int_a^b (x-A)f(x)g(x)\Delta x \le g(A)\int_a^b (x-A)f(x)\Delta x = 0,$$

and hence

$$\int_a^b xf(x)g(x)\Delta x \le A\int_a^b f(x)g(x)\Delta x.$$

This completes the proof of (R_3). □

Remark. Under the assumptions of Theorem 5.1, if $g = f$ on $[a, b]$, then (R_3) is reduced to

$$\frac{\displaystyle\int_a^b xf^2(x)\Delta x}{\displaystyle\int_a^b f^2(x)\Delta x} \le (\ge) \frac{\displaystyle\int_a^b xf(x)\Delta x}{\displaystyle\int_a^b f(x)\Delta x}.$$

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