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SOME PROPERTIES OF DIFFUSION PROCESSES WITH SINGULAR COEFFICIENTS

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CONTENTS

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1. INTRODUCTION

We report here on various properties of diffusion processes such as the existence, the uniqueness, the time spent at 0 and some asymptotic limits in the case of singular coefficients. Most of what follows concerns one-dimensional situations of the following type

$$dx_t = \sigma(x_t)dW_t + b(x_t)dt, \quad x_0 = x \in \mathbf{R}, \tag{1}$$

where W_t is a standard one dimensional Brownian motion (in a probability space (Ω, F, F_t, P)), a and b are, say, continuous functions on $\mathbf{R}$ (or on $[0, \infty)$...) growing at most linearly at infinity (for example). A typical case of interest in financial applications is the case when

$$a(x) = \nu x^\beta \text{ for } x > 0\,, \tag{2}$$

with $\nu > 0$, $0 < \beta \leq 1$. Obviously, when $\beta < 1$, σ presents a singularity at 0 which does not allow to apply the classical theory (of stochastic differential equations and diffusion processes).

The goal of this report is to recall the state of the art on this issue. More precisely, we consider in Section 2 the well-known case when $\beta \geq 1/2$ (in which case, the standard existence and uniqueness theory may be extended to include it). Section 3 is devoted to the more delicate case when $0 < \beta < 1/2$ in which case we shall prove that:

i) uniqueness in law does not hold in general,

ii) if $b(0) = 0$, there is pathwise uniqueness of **non negative** solutions x_t (assuming of course that $x \geq 0$),

iii) if $b(0) > 0$, the uniqueness in law of non negative solutions holds.

We next study in Section 4 the issue of the time spent by the process at 0. And we conclude in Section 5 with some multidimensional examples relevant for Finance (see for instance Musiela and Rutkowski [6] for the financial applications of such models).

We conclude this introduction by recalling a few facts and definitions: pathwise existence or pathwise uniqueness simply mean that the usual existence or uniqueness holds in any given probability space. Uniqueness in law means of course that the law of the process is unique and the notions in law are well-known to be equivalent to the martingale formulation (i.e. uniqueness in law is equivalent to the uniqueness of the martingale problem). And the path-wise uniqueness implies the uniqueness in law. And we refer the reader to Stroock and Varadhan [8] (for instance) for a presentation of all these facts.

Finally, under the mere assumption of continuity of the coefficients, it is also well-known (see Stroock and Varadhan [8]) that these always exists at least one solution in law (or equivalently a solution of the martingale problem). Furthermore, that solution can be selected for all x in such a way to form a strong Markov process (see also Stroock and Varadhan [8]). Therefore, the only issues that we need to address here are: i) the pathwise existence (solutions are then often called strong solutions), ii) the uniqueness (either in the pathwise sense, or in law). In addition, the singularity being at 0, it is natural (and relevant for Finance, in terms of qualitative properties of models...) to investigate the time spent by solutions at 0. Let us also mention that we do not concern ourselves here with further properties of solutions like semi-explicit formulas for the marginals of the law... These topics will be detailed elsewhere.

Fractional powers singularities are used in several models of Finance as, in particular, the "constant elasticity of variance models" (see Beckers [1], Schroder [7], Goldenberg [2], Lo et al [5], Lipton [4]; for a general presentation and analysis of mathematical modelling in Finance, we refer the reader to Musiela and Rutkowski Musiela and Rutkowski [6]). It is then natural to investigate the issues solved in this paper in order to understand and assert the well-posedness of such mathematical models.

2. $\boldsymbol{\beta \geq \frac{1}{2}}$

More generally, we consider the situation when a is Hölder continuous on $\mathbf{R}$ with an exponent $\beta \in \left[\frac{1}{2}, 1\right]$, i.e. when we have (for instance) for some $\dfrac{1}{2} \leq \beta \leq 1$, $C > 0$

$$|\sigma(x) - \sigma(y)| \leq C|x-y|^{\beta}, \quad \forall x, y, \in \mathbf{R}\,, \tag{3}$$

as is the case when σ is given by (2) (when $\frac{1}{2} \leq \beta \leq 1$). In addition, we assume that the drift satisfies the following one-sided Lipschitz condition

$$b(x) - b(y) < C(x-y) \quad \text{if} \quad x > y\,, \tag{4}$$

for some $C \geq 0$.

In that situation, the classical pathwise theory (of stochastic differential equations) may be extended to include it with the following consequences: *pathwise existence and uniqueness hold*; and the uniqueness is in fact a consequence of the following comparison property. Let x_t, y_t satisfy respectively

$$\begin{cases} dx_t \leq a(x_t)dW_t + b(x_t)dt, \quad dy_t \geq \sigma(y_t)dW_t + b(y_t)dt\,, \\ x_0 = x \leq y = y_0\,, \end{cases} \tag{5}$$

then we have for all $t \geq 0$: $x_t < y_t$, a.s.

We present here a simple proof of this fact (which easily yields the other facts mentioned above). This proof is due to Lions and Régnier [3]. Let $\varepsilon > 0$, we consider the following function on $\mathbf{R}$

$$\varphi_\varepsilon(z) = \begin{cases} 0\,, & \text{if} \quad z < \varepsilon\,, \\ z \log \dfrac{z}{\varepsilon} - (z-\varepsilon)\,, & \text{if} \quad z > \varepsilon\,. \end{cases}$$

And we apply Itô's formula to $\varphi_\varepsilon(x_{t\wedge\tau} - y_{t\wedge\tau})$, where $\tau = \inf(t \geq 0 \,/\, x_t > y_t + 1)$. We thus obtain, recalling that $\varphi_\varepsilon(x-y) = 0$, for all $t \geq 0$

$$E\left[\varphi_\varepsilon(x_{t\wedge\tau} - y_{t\wedge\tau})\right] \leq E\left[\int_0^{t\wedge\tau} \frac{(\sigma(x_s) - \sigma(y_s))^2}{x_s - y_s} 1_{x_s \geq y_s + \varepsilon}\, ds\right] + E\left[\int_0^{t\wedge\tau} (b(x_s) - b(y_s)) \log\left(\frac{x_s - y_s}{\varepsilon}\right) 1_{x_s \geq y_s + \varepsilon}\, ds\right],$$

and, thus in view of (3) and (4),

$$\begin{aligned} E\left[\varphi_\varepsilon(x_{t\wedge\tau} - y_{t\wedge\tau})\right] &\leq CE\left[\int_0^{t\wedge\tau} \frac{(x_s - y_s)^{2\beta}}{x_s - y_s} 1_{x_s \geq y_s + \varepsilon}\, ds\right] \\ &\quad + CE\left[\int_0^{t\wedge\tau} (x_s - y_s) \log\left(\frac{x_s - y_s}{\varepsilon}\right) 1_{x_s \geq y_s + \varepsilon}\, ds\right] \\ &\leq Ct + C \int_0^t E\left[\varphi_\varepsilon(x_{s\wedge\tau} - y_{s\wedge\tau})\right] ds\,, \end{aligned}$$

since $2\beta \geq 1$ and $z \log \frac{z}{\varepsilon} = \varphi_\varepsilon(z) + (z - \varepsilon) \leq \varphi_\varepsilon(z) + 1$ if $\varepsilon \leq z \leq 1$. We finally obtain, using Gronwall's Lemma, for all $t > 0$

$$E\left[\varphi_\varepsilon(x_{t\wedge\tau} - y_{t\wedge\tau})\right] \leq C\,,$$

for some $C \geq 0$ that depends on t but is independent of ε.

We now let ε go to 0_+, observing that $\varphi_\varepsilon(z) \uparrow +\infty$ as $\varepsilon \downarrow 0_+$ for all $z > 0$. Hence, we have

$$P(x_{t\wedge\tau} - y_{t\wedge\tau} > 0) = 0\,,$$

and we conclude that $x_t \leq j_t$, a.s. since $x_\tau - y_\tau = 1$ if $\tau < \infty$.

Remark. Under the general condition (3), the assumption (4) on b (namely that b' is bounded from above) is nearly optimal as can be seen in the particular case when $\sigma = 0$. On the other hand, if σ is given by (2) with $\frac{1}{2} \leq \beta \leq 1$, it is possible to allow for more singular drifts b. Indeed, the arguments developed in the following sections allow to show the following facts:

i) If $\frac{\nu^2}{2}x^{2\beta-2} - \frac{b(x)}{x}$ is bounded from above near $x = 0$ (for $x > 0$), then any solution starting at $x > 0$ never reaches 0 in finite time and is thus clearly unique (and obviously exists...).

ii) If $\overline{\lim_{x\to 0_+}} \frac{b(x)}{x^{2\beta-1}} < (2\beta - 1)\frac{\nu^2}{2}$, or if $b(x) = \mu x^{2\beta-1}$ and $\mu \le (2\beta - 1)\frac{\nu^2}{2}$, or if $\beta = \frac{1}{2}$, $b(0) = 0$ and $\frac{b(z)}{z} \in L^2(0, 1)$, then any non negative solution starting form 0 vanishes identically and thus pathwise existence and uniqueness hold for non negative solutions and initial conditions.

iii) In the borderline case $b(x) = \mu x^{2\beta-1}$ ($\frac{1}{2} \le \beta < 1$), case i) above applies if $\mu > \frac{\nu^2}{2}$ and case ii) above applies if $\mu \le (2\beta - 1)\frac{\nu^2}{2}$. Next, if $\mu < \frac{\nu^2}{2}$, any solution starting at $x > 0$ hits 0 in finite time a.s. And, if $\mu \ge \beta\frac{\nu^2}{2}$, one can show that there exists in general (i.e. for some Brownian motion) a non trivial non negative solution starting at 0. This may be shown by an explicit construction in the case when $\mu = \beta\frac{\nu^2}{2}$ and is then deduced for larger values of μ by using a construction based upon comparisons. Indeed, if $\mu = \beta\frac{\nu^2}{2}$ and B_t is a Brownian motion, we set $y_t = c\left|\left(\frac{x}{c}\right)^{1-\beta} + B_t\right|^{1-\beta}$, where $x \ge 0$, $c = (\nu(1 - \beta))^{\frac{1}{1-\beta}}$. Then, one can easily check that y_t solves (1) with $\sigma(x) = \nu x^\beta$, $b = \mu x^{2\beta-1}$ and W_t is a new Brownian motion defined by $dW_t = \text{sign}\left(\left(\frac{x}{c}\right)^{1-\beta} + B_t\right) dB_t$. In particular, y_t is non trivial if $x = 0$. Therefore, in the borderline case namely $b(x) = \mu x^{2\beta-1}$ and if $\frac{1}{2} < \beta < 1$, the only parameter range on which the uniqueness is not settled so far is $(2\beta-1)\frac{\nu^2}{2} < \mu < \beta\frac{\nu^2}{2}$, where we only know that all positive solutions reach 0 in finite time a.s. In fact, it is possible to check that there exist, in that range, non trivial non negative solutions starting at 0. We only sketch the argument: one approximates the SDE from above by (for instance)

$$dy_t = \nu y_t^\beta \, dW_t + \left(\mu y_t^{2\beta-1} + \delta\right) dt\,, \quad y_0 = y\,,$$

where $y, \delta \in (0, 1]$. A unique solution that never reaches 0 in finite time (a.s.) exists and y_t is non decreasing with respect to y and to δ. We then compute explicitely $E_y(\tau_1)$, where τ_1 denotes $\tau_1 = \inf(t \ge 0 \,/\, y_t \ge 1)$. We then let δ and y go to 0, obtaining thus a non negative solution starting at 0. The formula for $E_0(\tau_1)$ at the limit then shows that $\tau_1 < \infty$, a.s. and thus this limit solution is not identically 0.

3. $\beta < 1/2$

3.1. NON UNIQUENESS

We consider here an example of the typical non uniqueness phenomena that appears when $0 < \beta < 1/2$. This example is due to Lins and Régnier [3]. We take $b \equiv$

0, $\sigma(x) = |x|^\beta$ for $x \in \mathbf{R}$ with $0 < \beta < 1/2$ (we could consider as well $\sigma(x) = |x|^\beta \mathrm{sign}(x)$). Obviously, when $x = 0$, $x_t \equiv 0$ is a solution of (1). And we are going to obtain another solution in law (i.e. another solution of the martingale problem). In order to do so, we regularize σ as follows

$$\sigma^\varepsilon(x) = \left(|x|^2 + \varepsilon^2\right)^{\beta/2} \text{ on } \mathbf{R}\,, \text{ for } \varepsilon \in (0,1]\,.$$

Let x_t^ε be the corresponding solution that is the solution of

$$dx_t^\varepsilon = \left(|x_t^\varepsilon|^2 + \varepsilon^2\right)^{\beta/2} dW_t\,, \quad x_t^\varepsilon = 0 \in \mathbf{R}\,.$$

By standard compactness arguments, the law of x_t^ε, up to the extraction of subsequences, converges to the law of a solution of

$$dx_t = |x_t|^\beta dW_t, \quad x_0 = 0 \in \mathbf{R}.$$

We claim that the law of x_t is not concentrated on 0 (in other words x_t does not vanish identically). Indeed, we observe that, if we denote by τ the first exit time of x_t^ε from $(-1,+1)$, we have

$$u_\varepsilon(0) = E^\varepsilon(\tau)\,,$$

where P^ε denotes the law of x_t^ε and u^ε solves

$$-\frac{1}{2}\left(x^2 + \varepsilon^2\right)^\beta u_\varepsilon'' = 1 \text{ on } [-1,+1]\,, \quad u_\varepsilon(\pm 1) = 0\,.$$

Obviously, u_ε is smooth, even and

$$|u_\varepsilon''| \le \frac{2}{|x|^\beta}\,.$$

Therefore, we have

$$|u_\varepsilon'| \le \frac{2}{1-\beta}|x|^{1-\beta}$$

and $u_\varepsilon(0)$ is bounded uniformly in $\varepsilon \in (0,1]$. Letting ε go to 0_+, we thus deduce that $E(\tau) < \infty$. In other words, x_t leaves $(-1,1)$ a.s. and in particular cannot be identically 0!

Remark. As we shall see below, the above non uniqueness phenomenon is closely connected to the sign of solutions. Indeed, in the above construction, we are using an approximation of x^β that does not vanish at 0, so the corresponding approximated solutions x_t^ε have no definite sign (they cross 0...). And we obtain at the limit a solution without a definite sign (and non trivial). On the other hand, as can be deduced from the analysis made in the following sections, if we use an approximation (or regularization) that vanishes at 0 such as, for example

$$\sigma_\varepsilon(x) = (x^2 + \varepsilon^2)^{\beta/2} - \varepsilon^\beta \text{ on } [0,\infty), \text{ for } \varepsilon \in (0,1].$$

Then, the solution x_t^ε of

$$d\,x_t^\varepsilon = \sigma_\varepsilon(x_t^\varepsilon)\,dWt, \quad x_t^\varepsilon = x \geq 0$$

remains non negative for all $t \geq 0$ a.s. And it does converge, as ε goes to 0_+, to the unique non negative solution of

$$dx_t = x_t^\beta\,dWt\,,\; x_0 = x \geq 0$$

(this latter fact is precisely what we shall show in the next subsection).

3.2. PATHWISE UNIQUENESS WHEN THE DRIFT VANISHES AT 0

We consider here the case when $\sigma(0) = b(0) = 0$. In that case, $x_t \equiv 0$ is obviously a solution of (1) if $x = 0$. In addition, if $x > 0$, there exists a unique positive solution until the first time it hits 0, i.e. until $\tau_0 = \lim_{\varepsilon\downarrow 0} \uparrow \tau_\varepsilon$, $\tau_\varepsilon = \inf\{t \geq 0\,/\,x_t \leq \varepsilon\}$ ($\tau_0 = +\infty$ if $x_t > 0$ for all $t \geq 0$). We shall see in Section 4 that $\tau_0 < +\infty$ a.s.

In order to simplify the presentation, we consider here the case when $\underline{\sigma(x) = \nu x^\beta}$ for $x \geq 0$ and $\underline{0 < \beta < 1/2}$ (although the argument below will clearly apply to general situations...).

We then claim that we have the following result.

Theorem 1. *i) If $x = 0$, the only non negative solution of (2) is the trivial solution $x_t \equiv 0$.*

*ii) If $x > 0$, any non negative solution x_t is, of course, the positive solution until τ_0 and $x_t 1_{(t\geq\tau_0)} = 0$ for all $t \geq 0$ a.s. (in other words, any such solution remains at 0 after τ_0). In particular, pathwise existence and uniqueness holds for **non negative** solutions (and initial conditions).*

The above two claims are easily shown once we observe that $\varphi(x_t)$ is a (local) martingale, where φ is defined by

$$\varphi = \int_0^x \exp\left(\int_0^1 \frac{2b}{\sigma^2}\right) dy.$$

In order to prove this claim, we observe that $\varphi \in C^1$ and $\varphi \in C^2([\varepsilon, \infty))$ for all $\varepsilon > 0$. Hence, we have, recalling that x_t is non negative,

$$\begin{aligned} d\varphi(x_t + \varepsilon) &= \sigma(x_t)\varphi'(x_t+\varepsilon)\,dW_t + \left(b(x_t)(\varphi'(x_t+\varepsilon) + \frac{1}{2}\sigma^2(x_t)\varphi''(x_t+\varepsilon)\right)\,dt \\ &= \sigma(x_t)\varphi'(x_t+\varepsilon)\,dW_t + \varphi'(x_t+\varepsilon)\left\{b(x_t) - b(x_t+\varepsilon)\frac{\sigma^2(x_t)}{\sigma^2(x_t+\varepsilon)}\right\}\,dt\,. \end{aligned}$$

Next, we remark that

$$b(x_t) - b(x_t + \varepsilon) \underset{\varepsilon}{\longrightarrow} 0\,,$$

$$b(x_t+\varepsilon)\left\{1 - \frac{\sigma^2(x_t)}{\sigma^2(x_t+\varepsilon)}\right\} = b(x_t+\varepsilon)\left(1 - \frac{x_t^{2\beta}}{(x_t+\varepsilon)^{2\beta}}\right) \underset{\varepsilon}{\longrightarrow} 0 \text{ a.s.}$$

since $b(0) = 0$.

And our claim on $\varphi(x_t)$ follows upon letting ε go to 0.

We may now conclude easily since we have for al $t \geq 0$

$$E\left[\varphi(x_{t\wedge\tau_0})\right] = E\left[\varphi(x_t)\right]\,,$$

where we denote by $\tau_0 = \inf(t \geq 0\,/\,x_t = 0)$ (if $x = 0$, $\tau_0 = 0$). In particular, if $x = 0$,

$$E\left[\varphi(x_t)\right] = \varphi(0) = 0\,,$$

and thus (recall that $\varphi(z) > 0$ if $z > 0$) $x_t \equiv 0$.

In addition, if $x > 0$, we deduce from the above equality

$$E\left[\varphi(x_t)1_{(t\geq\tau_0)}\right] = 0\,,$$

and thus $x_t \equiv 0$ for $t \geq \tau_0$.

Remark. A simpler proof of the above facts may be given in the case when $\frac{b(x)}{x}$ is bounded from above on $[0,\infty)$ (or equivalently near $x = 0_+$). Indeed, let us check in that case that, for example, $x_t \equiv 0$ if $x \equiv 0$. We simply observe that we have, using the fact that $x_t \geq 0$,

$$\frac{d}{dt}E[x_t] = E[b(x_t)] \leq CE[x_t]\,,$$

hence $E[x_t] \leq 0$ for all $t \geq 0$, and thus $x_t \equiv 0$. □

3.3. UNIQUENESS IN LAW

The argument made in the preceding section does not apply anymore if $b(0) \neq 0$. Indeed, although we show in Section 4 below that solutions reach 0 in finite time with positive probability, we cannot conclude as in the previous subsection since solutions cannot "stay at 0" (we shall investigate, by the way, the nature of the time spent at 0 by solutions in Section 4). In addition, if $b(0) < 0$, solutions that reach 0 will, with positive probability, cross 0 and we have seen in the preceding subsections that, in that case, the uniqueness cannot be maintained.

This is why we only need to consider the case of non negative solutions and thus to assume that $b(0) > 0$. Not only the method introduced in Subsection 3.2 does not apply if $b(0) > 0$, as we explained above, but the pathwise uniqueness is not known in that case (even though it is highly probable). And we prove in this section the uniqueness in law by a partial differential argument. Let us also mention that this argument will also shed some light on the numerical approximation procedures that *should* (or *should not*) be used in order to compute quantities of interest for financial applications. As explained in the introduction, the existence in law (or equivalently the existence of a solution to the martingale problem) is known for arbitrary σ and b.

We thus consider solutions of (1) (in the sense of the Stroock-Varadhan martingale problems) with σ given by (2) and $0 < \beta < 1/2$. And we assume that b satisfies on $[0,1]$ for some positive constant $C \geq 0$

$$|b(x) - b(0)| < Cx^{\beta}\,, \tag{6}$$

and

$$b(0) > 0\,. \tag{7}$$

Theorem 2. *Under these conditions, there exists a unique solution x_t in law of (1) such that $x_t > 0$ a.s.*

By Girsanov transforms, the martingale problem corresponding to σ and b is, in view of (6), equivalent to the martingale problem corresponding to σ and $b = b(0)$. And we thus only have to consider the case when $b \equiv \lambda > 0$. Furthermore, as is well-known, the uniqueness in law follows from the unique determination of $u(x,t) = E[u_0(x_t)]$ for all $x \geq 0$, $t \geq 0$ and for an arbitrary smooth u_0 (say, $\in C_0^\infty([0,\infty))$).

Indeed, we claim that there exists a solution u (which is easily shown to be unique) of

$$\frac{\partial u}{\partial t} = \frac{\nu^2}{2} x^{2\beta} \frac{\partial^2 u}{\partial x^2} + \lambda \frac{\partial u}{\partial x} \text{ for } x > 0,\ t > 0;\ u|_{t=0} = u_0 \text{ for } x > 0\,, \tag{8}$$

where $u \in W^{2,1,\infty}_{x,t}([0,\infty) \times [0,\infty))$ (i.e. u, $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial t}$ are bounded in $x,t \geq 0$) and u is, by parabolic regularity, smooth for $x > 0$, $t \geq 0$. The uniqueness in law follows easily from this claim by a simple verification argument. Indeed, we have by Itô's formula for any $\varepsilon > 0$, $t > 0$, using the equation (8)

$$\frac{d}{ds} E\left[u(x_s + \varepsilon, t - s)\right] = E\left\{ \frac{\nu^2}{2} (x_s^{2\beta} - (x_s + \varepsilon)^{2\beta}) \frac{\partial^2 u}{\partial x^2}(x_s + \varepsilon, t - s) \right\}\,.$$

Hence, we deduce upon letting ε go to 0,

$$E[u_0(x_t)] = u(x,t)\,.$$

We only sketch the proof of the existence of a solution of (8) with the above regularity properties. Of course, the crucial information is the L^∞ bound on $\frac{\partial^2 u}{\partial x^2}$. And we first present a formal proof which clearly shows the origin of that bound. We differentiate (8) twice with respect to x and we find

$$\begin{cases} \dfrac{\partial}{\partial t}\left(\dfrac{\partial^2 u}{\partial x^2}\right) - \dfrac{\nu^2}{2} x^{2\beta} \dfrac{\partial^2}{\partial x^2}\left(\dfrac{\partial^2 u}{\partial x^2}\right) - \lambda \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2 u}{\partial x^2}\right) - 2\nu^2 \beta x^{2\beta-1} \dfrac{\partial}{\partial x}\left(\dfrac{\partial^2}{\partial x^2}\right) \\ +\nu^2 \beta\,(1 - 2\beta)\, x^{2\beta-2}\left(\dfrac{\partial^2 u}{\partial x^2}\right) = 0\,. \end{cases} \tag{9}$$

Formally, the bound we look for is a consequence of the maximum principle and we obtain

$$\left\| \frac{\partial^2 u}{\partial x^2} \right\|_{L^\infty_{x,t}} \le \left\| \frac{\partial^2 u_0}{\partial x^2} \right\|_{L^\infty} . \tag{10}$$

This is, in fact, not as clear as it seems because the maximum could be achieved at 0. And the reason why (10) holds is the presence in (9) of the term

$$\left[\nu^2 \beta \left(1 - 2\beta\right) x^{2\beta - 2} \frac{\partial^2}{\partial x^2} \right]$$

with a very singular term $x^{2\beta-2}$ and a positive coefficient $\nu^2\beta(1-2\beta)$ that "forces $\dfrac{\partial^2 u}{\partial x^2}$ vanish at the origin".

Remark 1. In fact, if u_0 satisfies for some $C_0 \ge 0$

$$\left| \frac{\partial^2 u_0}{\partial x^2} \right| \le C_0 x^{1-2\beta} \text{ for } x \in [0,1] \,,$$

then we claim that we have for all t

$$\left| \frac{\partial^2 u}{\partial x^2}(x,t) \right| \le C x^{1-2\beta}, \text{ for some } C = C(T) \ (\forall T \in (0,\infty)) \,.$$

Indeed, one can check that $\varphi = e^{Bt} \log(1 + Ax^{1-2\beta})$ is a supersolution of (8) for some $A, B > 0$ to be determined. Indeed, we have

$$\begin{aligned}
&\frac{\partial \varphi}{\partial t} - \frac{\nu^2}{2} x^{2\beta} \frac{\partial^2 \varphi}{\partial x^2} - 2\beta \nu^2 x^{2\beta - 1} \frac{\partial \varphi}{\partial x} - \lambda \frac{\partial \varphi}{\partial x} + \nu^2 \beta (1-2\beta) x^{2\beta-2} \varphi \\
&\quad = B\varphi + \nu^2 \beta (1-2\beta) e^{Bt} x^{2\beta-2} \log\left(1 + Ax^{1-2\beta}\right) - \frac{e^{Bt}\nu^2 A}{x(1 + Ax^{1-2\beta})} \beta(1-2\beta) \\
&\qquad + e^{Bt} \frac{\nu^2}{2} \frac{A^2(1-2\beta)x^{-2\beta}}{(1+Ax^{1-2\beta})^2} - \lambda e^{Bt} \frac{A(1-2\beta)}{(1+Ax^{1-2\beta})} x^{-2\beta} \\
&\qquad \ge e^{Bt} \left\{ B \frac{Ax^{1-2\beta}}{1+Ax^{1-2\beta}} + \frac{\nu^2}{2} \frac{A^2(1-2\beta)x^{-2\beta}}{(1+Ax^{1-2\beta})^2} - \lambda \frac{A(1-2\beta)x^{-2\beta}}{(1+Ax^{1-2\beta})} \right\},
\end{aligned}$$

where we used the inequality $\log(1+z) \ge \frac{z}{1+z}$ for all $z \ge 0$. We then choose $A = \frac{4\lambda}{\nu^2}$ and observe that the difference of the last two terms is clearly non negative if $x \le x_0 = A^{-\frac{1}{2-2\beta}}$. Next, if $x \ge x_0$, we choose $B = \frac{\lambda(1-2\beta)}{x_0}$, and our claim follows. The inequality for $\frac{\partial^2 u}{\partial x^2}$ then follows choosing $D > 0$ large enough such that

$$\left| \frac{\partial^2 u_0}{\partial x^2} \right| \le D\varphi \text{ on } [0,1] \,, \quad D \log\left(1 + A\right) \ge \left\| \frac{\partial^2 u_0}{\partial x^2} \right\|_{L^\infty} . \qquad \square$$

There only remains to justify this formal argument and we do so by an approximation argument where we replace $x^{2\beta}$ by $(x+\varepsilon)^{2\beta} - \varepsilon^{2\beta}$ ($\varepsilon \in (0,1]$), i.e. we look for a solution (with the regularity stated above at least) of

$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \dfrac{\nu^2}{2}\left((x+\varepsilon)^{2\beta} - \varepsilon^{2\beta}\right)\dfrac{\partial^2 u^\varepsilon}{\partial x^2} - \lambda \dfrac{\partial u^\varepsilon}{\partial x} = 0\,, \text{ for } x \geq 0\,,\ t \geq 0\,, \\ u^\varepsilon|_{t=0} = u_0\,, \text{ for } x \geq 0\,. \end{cases}$$

In fact, we build directly $w^\varepsilon = \frac{\partial^2 u^\varepsilon}{\partial x^2}$ by solving

$$\begin{cases} \dfrac{\partial w^\varepsilon}{\partial t} - \dfrac{\nu^2}{2}\left((x+\varepsilon)^{2\beta} - \varepsilon^{2\beta}\right)\dfrac{\partial^2 w^\varepsilon}{\partial x^2} - 2\nu^2\beta(x+\varepsilon)^{2\beta-1}\dfrac{\partial w^\varepsilon}{\partial x} \\ +\nu^2\beta(1-2\beta)(x+\varepsilon)^{2\beta-2}w^\varepsilon - \lambda\dfrac{\partial w^\varepsilon}{\partial x} = 0\,, \text{ for } x \geq 0\,,\ t > 0\,. \end{cases}$$

This is still a degenerate problem at $x = 0$ but the corresponding coefficient namely $\nu((x+\varepsilon)^{2\beta} - \varepsilon^{2\beta})^{1/2}$ is Hölder continuous with an exponent equal to 1/2 (in particular, Section 2 applies to that case). And we obtain easily a solution w^ε satisfying $\|w^\varepsilon\|_{L^\infty_{x,t}} \leq \|\frac{\partial^2 u_0}{\partial x^2}\|_{L^\infty}$ decaying rapidly at infinity, smooth for $x > 0$ and $t \geq 0$... We then introduce $v^\varepsilon = -\int_x^{+\infty} w^\varepsilon(y,t)\,dy$ which solves for $x > 0$ and $t \geq 0$

$$\begin{cases} \dfrac{\partial v^\varepsilon}{\partial t} = \dfrac{\nu^2}{2}\left((x+\varepsilon)^{2\beta} - \varepsilon^{2\beta}\right)\dfrac{\partial^2 v^\varepsilon}{\partial x^2} + \nu^{2\beta}(x+\varepsilon)^{2\beta-1}\dfrac{\partial v^\varepsilon}{\partial x} + \lambda\dfrac{\partial v^\varepsilon}{\partial x} \\ v^\varepsilon|_{t=0} = \dfrac{\partial u_0}{\partial x}\,. \end{cases}$$

One then easily checks that v^ε decays rapidly at infinity and we finally set $u^\varepsilon = -\int_x^{+\infty} v^\varepsilon(y,t)\,dy$ in order to conclude.

Remark 2. For financial applications, it is sometimes necessary to compute numerically the solution u of (8) with $\lambda \geq 0$. Such a computation is delicate in view of the non uniqueness phenomena shown above. In order to avoid the singularity, it is natural to solve the equation on $[\delta, +\infty)$ (instead of $[0,\infty)$) since the problem is then no more singular nor degenerate. However, one needs to impose boundary conditions at $x = \delta$. If $\lambda = 0$, this is easy and one may simply set

$$u|_{x=\delta} = u_0(\delta)\,.$$

If $\lambda > 0$, such a boundary condition yields a spurious (i.e. false) solution: indeed, let us assume for instance that u_0 vanishes near 0. Then, for small enough, the solution u^δ is given by

$$u^\delta(x,t) = E\left[u_0(x_{t\wedge\tau_\delta})\right]\,,$$

where $\tau_\delta = \inf(t \geq 0\,,\ x_t < \delta)$. And, as δ goes to 0, u^δ converges to $E\left[u_0(x_{t\wedge\tau_0})\right]$ which is, in generaly, different from the quantity we wish to compute namely $E\left[u_0(x_t)\right]$!

It turns out that the above analysis yields the correct boundary condition. Indeed, if we write (8) at $x = 0$, or more generally using the bound on $\frac{\partial^2 u}{\partial x^2}$, we have

$$\left|\frac{\partial u}{\partial t} - \lambda \frac{\partial u}{\partial x}\right| \leq C x^{2\beta},$$

or even $\left|\frac{\partial u}{\partial t} - \lambda \frac{\partial u}{\partial x}\right| \leq Cx$ (see Remark 1 above). We may thus require the following boundary condition at δ

$$\frac{\partial u}{\partial t} - \lambda \frac{\partial u}{\partial x}\bigg|_{x=\delta} = 0$$

(this is a meaningful maximum principle preserving boundary condition, that could also be used at $x = 0$ with different approximations where we replace x^β by a smooth, positive function approximating it such as $(x + \delta)^\beta$ for example...). One can then easily check that the solution u_δ we build with this boundary condition satisfies: $\left|u^\delta - u\right|_{L^\infty} \leq C\delta$ for some $C > 0$.

We finally point out that, in the case when $\lambda > 0$, these does not seem to be any simple evaluation of u or $\frac{\partial u}{\partial x}$, at $x = 0$ for $t > 0$ (the limit $\beta \longrightarrow \frac{1}{2}$ – shows that the values at 0 are in fact globally determined from the intial data $u_0(x)$ for all $x \geq 0$).

4. LOCAL TIME AT 0

We have seen in the previous sections that the behavior at 0 is crucial. This is why we investigate here the nature of the time spent by solutions of

$$dx_t = \nu x_t^\beta \, dW_t + \lambda \, dt \,, \quad x_0 = x \geq 0 \,, \tag{11}$$

where $0 < \nu$, $0 < \beta < 1$, $\lambda \geq 0$.

First of all, if $x > 0$ and $\lambda = 0$, we claim that the (non negative if $\beta < 1/2$...) solution of (11) reaches 0 in finite time a.s. and stays there. Next, if $x > 0$ and $\lambda > 0$, we claim that such a solution:

i) does not reach 0 in finite time (a.s.) if $\beta > \frac{1}{2}$ or $\beta = \frac{1}{2}$, $\lambda \geq \frac{\nu^2}{2}$.

ii) reaches 0 in finite time (a.s.) if $\beta = \frac{1}{2}$, $\lambda < \frac{\nu^2}{2}$,

iii) reaches 0 in finite time with positive probability if $\beta < 1/2$.

All these facts are easily shown by a classical argument: let $x > 0$, we consider for $\varepsilon > 0$ small enough and for $R > 0$ large enough the following probability

$$P_x\left(\tau_\varepsilon < \tau^R\right) = w(x)\,,$$

where $\tau_\varepsilon = \inf(t \geq 0, x_t < \varepsilon)$, $\tau^R = \inf(t \geq 0, x_t > R)$.

Then, w solves the following ordinary differential equation

$$\frac{\nu^2}{2} x^{2\beta} w'' + \lambda w' = 0\,, \text{ on } [\varepsilon, R]\,, \quad w(\varepsilon) = 1\,,\ w(R) = 0\,.$$

And we find easily that w is given by (if $\lambda > 0$)

$$w(x) = \frac{\int_x^R \psi(y)\, dy}{\int_\varepsilon^R \psi(y)\, dy}\,,$$

where

$$\psi(x) = \exp\left(-\frac{2\lambda}{\nu^2}\frac{x^{1-2\beta}}{1-2\beta}\right)\,, \quad \text{if } \ \beta < \frac{1}{2}\,,$$

$$\psi(x) = \exp\frac{1}{x^{\frac{2\lambda}{\nu^2}}}\,, \quad \text{if } \ \beta = \frac{1}{2}\,,$$

$$\psi(x) = \exp\left(\frac{2\lambda}{\nu^2}\frac{x^{1-2\beta}}{2\beta-1}\right)\,, \quad \text{if } \ \beta > \frac{1}{2}\,,$$

We then let ε go to 0_+ and we see that w goes to 0 as ε goes to 0 if $\beta > \frac{1}{2}$, or if $\lambda \geq \frac{\nu^2}{2}$ since ψ is not integrable at 0. Hence, $P_x(\tau_0 < \tau^R) = 0$ for all $R > 0$ and letting R go to $+\infty$, we deduce in these cases that $P_x(\tau_0 < \infty) = 0$. On the other hand, if $\beta < \frac{1}{2}$, or if $\beta = \frac{1}{2}$ and $\lambda < \frac{\nu^2}{2}$, we obtain

$$P_x(\tau_0 < \tau^R) = \frac{\int_x^R \psi(y)\, dy}{\int_0^R \psi(y)\, dy}\,.$$

Next, if $\beta = \frac{1}{2}$ and $\lambda < \frac{\nu^2}{2}$, $\int_0^\infty \psi\, dy = +\infty$ and thus

$$P_x(\tau_0 < \infty) = 1\,.$$

Finally, if $\beta < 1/2$, we deduce

$$P_x(\tau_0 < \infty) = \frac{\int_x^\infty \exp\left(-\frac{2\lambda}{\nu^2}\frac{y^{1-2\beta}}{1-2\beta}\right) dy}{\int_0^\infty \exp\left(-\frac{2\lambda}{\nu^2}\frac{y^{1-2\beta}}{1-2\beta}\right) dy}\,.$$

We now consider the case when $x = 0$ and investigate the time spent at 0 by the solution in the case when $\lambda > 0$ (otherwise, $x_t \equiv 0$!). We claim that we have

$$\begin{cases} E\displaystyle\int_0^\infty 1_{(x_t=0)}\, dt = 0\,, \text{ if } \beta \geq \dfrac{1}{2}\,, \\ E\displaystyle\int_0^T 1_{(x_t=0)}\, dt = E\displaystyle\int_0^{x_T} \psi(y)\, dy\,, \text{ if } \beta < \dfrac{1}{2}\,. \end{cases}$$

Indeed, if $\beta < 1/2$, we consider $u(x) = \int_0^x \psi(y)\,dy$ and we apply Itô's formula with $u(x+\varepsilon)$, namely

$$E[u(x_T+\varepsilon)] = u(\varepsilon) + E\left[\int_0^T \frac{\nu^2 x_s^{2\beta}}{2} u''(x_s+\varepsilon) + \lambda u'(x_s+\varepsilon)\,ds\right]$$

$$= u(\varepsilon) + \lambda E\left[\int_0^T u'(x_s+\varepsilon)\left(1 - \frac{x_s^{2\beta}}{(\varepsilon+x_s)^{2\beta}}\right)\right] ds\,.$$

Then, if we let ε go to 0, we deduce

$$E[u(x_T)] = \lambda E\left[\int_0^T \psi(0) 1_{(x_s=0)}\,ds\right] = \lambda E\left[\int_0^T 1_{(x_s=0)}\,ds\right]\,.$$

The case $\beta > 1/2$ or $\beta = \frac{1}{2}$ and $\lambda \geq \frac{\nu^2}{2}$ is easy to handle since solutions starting from $x > 0$ never reach 0 (a.s.). We thus only detail the case when $\beta = \frac{1}{2}$ and $\lambda < \frac{\nu^2}{2}$. We then set $u(x) = \left(1 - \frac{2}{\lambda}\right)^{-1} x^{1-\frac{2\lambda}{\nu^2}}$ for $x \geq 0$ and argue as above. We find

$$E\left[u(x_T+\varepsilon)\right] = u(\varepsilon) + \lambda E\left[\int_0^T \frac{1}{(x_s+\varepsilon)^a}\left(1 - \frac{x_s^{2\beta}}{(x_s+\varepsilon)^{2\beta}}\right)\,ds\right]\,,$$

where $a = \frac{2\lambda}{\nu^2}$. Hence,

$$\frac{\lambda}{\varepsilon^a} E\left[\int_0^T 1_{(x_s=0)}\,ds\right] \leq \frac{1}{1-a} E\left(x_T+\varepsilon\right)^{1-a}\,,$$

and we conclude upon letting ε go to 0_+.

5. THE LIMIT $\beta \to 0_+$

We investigate in this section the limit, as β goes to 0_+, of solutions of (1) with 0 given by (2) and b satisfying either (4) and $b(0) = 0$, or (7) and b Lipschitz (for instance). We shall not recall these conditions below and we simply refer to the case when $b(0) = 0$ and when $b(0) > 0$.

We begin with the case $b(0) = 0$. In that case, we know that the non negative solution starting from 0 stays at 0, while if it starts from $x > 0$ is the unique "classical" solution until it hits 0 and stays there. It is then straightforward to check that, as β goes to 0_+, X_t converges as to $X^0_{t\wedge\tau_0}$ (where τ_0 is, as before, the first time X^0_t reaches 0), where x^0_t solves

$$d\,X^0_t = \nu\,dW_t + b(x^0_t)\,dt\,, \quad X^0_0 = x \geq 0\,. \tag{12}$$

The case when $b(0) > 0$ is more delicate. Heuristically, x_t converges (in law) to x_t^0 which solves uniquely (in law)

$$d\,x_t^0 = \nu 1_{(x_t>0)}\, dWt + b(x_t^0)\, dt\,, \quad x_t^0 > 0 \text{ a.s. } x_0^0 = x \geq 0\,.$$

As we shall see below, this guess is indeed correct and not only does (13) has a unique solution in law but the law of x_t does converge (weakly in the sense of probability measures) to the law of x_t^0. And these two claims follow easily from some straightforward partial differential equations considerations.

Indeed, we only have to understand the behavior as β goes to 0_+ of the solution u_β (shown to exist in the previous section) of

$$\frac{\partial u_\beta}{\partial t} - \frac{\nu^2}{2} x^{2\beta} \frac{\partial^2 u_\beta}{\partial x^2} - b(x)\frac{\partial u_\beta}{\partial x} = 0 \text{ for } \lambda, t > 0\,, \tag{14}$$

$$u_\beta|_{t=0} = \varphi(t)\,, \tag{15}$$

where φ belongs say, to $W^{3,\infty}(0,\infty)$. Let us recall that for financial applications u_β corresponds to the price of an european option (with maturity t) whose pay-off is given by φ and that $\frac{\partial u_\beta}{\partial x}$ corresponds to the hedge ("delta"). In fact, $\frac{\partial u_\beta}{\partial x} = v_\beta$ solves

$$\frac{\partial v_\beta}{\partial t} - \frac{\nu^2}{2} x^{2\beta} \frac{\partial^2 v_\beta}{\partial x^2} - \beta\nu^2 x^{2\beta-1}\frac{\partial v_\beta}{\partial x} - b(x)\frac{\partial v_\beta}{\partial x} - b'(x) v_\beta = 0 \text{ for } x, t > 0\,, \tag{16}$$

$$v_\beta|_{t=0} = \varphi'(t)\,, \tag{17}$$

In order to simplify the presentation, we only detail the argument in the case when b is constant ($b > 0$) and we observe that, in view of Girsanov transforms, it is enough to prove our claims in this case. Then, if b is constant, we have shown in the previous section that if $|\varphi(x)| < C_0 x$ for $x \in [0,1]$, then we have for some positive constant $C = C(T)$ which is independent on β

$$\left|\frac{\partial^2 u_\beta}{\partial x^2}\right| = \left|\frac{\partial v_\beta}{\partial x}\right| \leq C x^{1-2\beta}\,, \text{ for } \theta \leq t \leq T\,. \tag{18}$$

It is then straightforward to check that v_β converges uniformly (locally on $[0,\infty)^2$) to the solution of the Neumann problem

$$\frac{\partial v}{\partial t} - \frac{\nu^2}{2}\frac{\partial^2 v}{\partial x^2} - \lambda\frac{\partial v}{\partial x} = 0\,, \text{ for } x, t > 0\,; \quad \frac{\partial v}{\partial x}|_{x=0} = 0\,, \text{ for } t > 0\,, \tag{19}$$

$$v|_{t=0} = \varphi'(x)\,. \tag{20}$$

By a simple density argument, this conclusion remains valid if $\varphi \in C_b^1([0,\infty))$.

We may now go back to up and deduce that u_β converges uniformly (locally on $[0,\infty)^2$) to some u which is smooth for $t > 0$, $x \geq 0$ and solves (15) and

$$\frac{\partial u}{\partial t} - \frac{\nu^2}{2}\frac{\partial^2 u}{\partial x^2} - \lambda\frac{\partial u}{\partial x} = 0\,, \text{ for } x \geq 0,\ t > 0\,; \quad \frac{\partial^2 u}{\partial x^2}|_{x=0} = 0\,, \text{ for } t > 0\,, \tag{21}$$

And this conclusion is, once more by density, valid for any $\varphi \in C_b([0,\infty))$. Let us observe that the boundary condition $\frac{\partial^2 u}{\partial x^2}|_{x=0} = 0$ is in fact equivalent to

$$\frac{\partial u}{\partial t} - b(0)\frac{\partial u}{\partial x}\bigg|_{x=0} = 0\,, \ \text{for } t > 0\,, \tag{22}$$

which is indeed a meaningful maximum principle preserving boundary condition.

We conclude by checking that, if x_t^0 solves (13), then we have

$$u(x,t) = E\left[\varphi(x_t^0)\right]\,. \tag{23}$$

In order to do so, we apply Itô's formula to $u(x_s^0, t-s)$ for $0 \le s < t$ and we obtain using (21)

$$\begin{aligned}
\frac{d}{ds}E\left[u(x_s^0, t-s)\right] &= E\left[\left(-\frac{du}{dt} + \frac{\nu^2}{2}1_{(x_s^0>0)}\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial u}{\partial x}\right)(x_s^0, t-s)\right] \\
&= E\left[\left(-\frac{\partial u}{\partial t} + \lambda\frac{\partial u}{\partial x}\right)(0, t-s)1_{(x_s^0=0)}\right] \\
&= 0\,,
\end{aligned}$$

in view of (22). And our claim follows upon integrating cases $[0,t]$.

6. MULTIDIMENSIONAL EXTENSIONS

Except for very particular multidimensional situations, the possibility of handling Hölder continuous diffusion parameters (that are not Lipschitz) is purely one dimensional.

For example, let us consider the following natural multidimensional extension

$$dx_t = |x_t|^\beta\, dW_t\,, \quad x_0 = x \in \mathbf{R}^N\,, \tag{24}$$

where $0 < \beta < 1$, W_t is a N-dimensional Brownian motion and $N > 2$. We claim that if $x = 0$, there exists a solution of (12) that does not vanish identically. Indeed, we approximate (12) by

$$d\,x_t^\varepsilon = \left(|x_t^\varepsilon|^2 + \varepsilon^2\right)^{\beta/2} dW_t\,, \quad x_t^\varepsilon = x \in \mathbf{R}^N\,.$$

And we observe that $u^\varepsilon(x) = E_x(\tau_\varepsilon^1)$, where $\tau^1 = \inf(t \ge 0\,,\ |x_t^\varepsilon| > 1)$ solves

$$\begin{cases} -\dfrac{1}{2}\left(|x|^2 + \varepsilon^2\right)^\beta \Delta u^\varepsilon = 1 \ \text{ in } B\,, \\ u^\varepsilon|_{\partial B} = 0\,, \end{cases}$$

where $B = \{x \in R^N\,,\ |x| < 1\}$.

Obviously, $|\Delta u^\varepsilon| < \frac{2}{|x|^\beta} \in L^p(B)$ for $p > \frac{N}{2}$. Hence, u^ε is, uniformly in ε, Hölder continuous and we obtain, at the limit as ε goes to 0_+, a solution (in law) x_t for which

$$E_x \tau^1 < \infty \,,$$

and thus x_t does not vanish identically.

On the other hand, some positive results exist for particular systems of the following type (for example)

$$\begin{cases} dx_t = x_t^\beta y_t^\alpha \, dW_t \,, \quad x_0 = x \geq 0 \,, \\ dy_t = \sigma(y_t)\, dB_t + b(y_t)\, dt \,, \end{cases}$$

when $0 < \beta < 1$, $\alpha > 0$ and W_t, B_t are two correlated Brownian motions with $[dW_t, dB_t] = \rho \, dt$. We are not concerned here with the possible blow up of solutions (issue that we addressed in another work ...). We assume that the equation for y_t yields a unique solution either pathwise or in law (possibly with sign restrictions as we saw above). Then, we claim that the above system admits a unique solution either pathwise or in law. More precisely, let as first assume that σ and b are such that y_t is uniquely defined pathwise (possibly with the condition $y_t \geq 0$ for all $t \geq 0$, a.s.). Then, the proofs made in the previous sections carry over to the equation for x_t and, if $\beta \geq 1/2$, we obtain a unique (pathwise) solution x_t that remains non negative, while if $0 < \beta < 1/2$ the pathwise uniqueness of a non negative x_t holds.

In the second case, namely, when σ and b are such that y_t is uniquely defined in law (possibly with the condition $y_t > 0$ for all $t \geq 0$, a.s.), the uniqueness in law of (x_t, y_t) (with $x_t \geq 0$ for all $t \geq 0$, a.s. if $0 < \beta < 1/2$) holds. Indeed, we may write $dW_t = \rho \, dB_t + \sqrt{1-\rho^2}\, dZ_t$, where (W_t, Z_t) is a standard two dimensional Brownian motion. And the law of (y_t, W_t) is uniquely determined. It is then easy to check that x_t is also unique in law or more precisely that the law of (x_t, y_t) is uniquely determined. Indeed, it suffices to show this claim replacing x_t by $x_{t\wedge\tau_0}$ and thus, by approximation, by $x_{t\wedge\tau_\varepsilon}$, where $\tau_\varepsilon = \inf(t \geq 0\,,\ x_t < \varepsilon)$ $(\varepsilon \in (0,1])$. By a further approximation, we may even replace x_t^β by $(x_t+\delta)^\beta - \delta^\beta = \alpha_\delta(x_t)$ for $\delta > 0$. Then, as is well-known, the corresponding solution may be obtained iteratively solving for example

$$dx_t^{n+1} = \alpha_\delta\left(x_t^n\right) y_t^\alpha \, dW_t \,,\ x_0^{n+1} = x \,, \quad \text{for} \quad n \geq 0 \,,$$

and $x_t^0 \equiv x$. Then, we check by induction that the law of $\left(\left(x_t^k\right)_{1\leq k\leq n}, y_t, W_t\right)$ is uniquely determined (approximating, for instance, stochastic integrals by non anticipating Riemann sums ...). And we conclude easily.

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