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A POROSITY RESULT FOR A SADDLE POINT PROBLEM

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ABSTRACT: In this paper we consider a complete metric space $\mathfrak{M}$ of functions $f : X \times Y \to R^1$ which satisfy $\sup_{y\in Y} \inf_{x\in X} f(x, y) = \inf_{x\in X} \sup_{y\in Y} f(x, y)$, where X and Y are complete metric spaces. We establish that the set of all functions from $\mathfrak{M}$ which have a unique saddle point has a σ-porous complement.

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1. INTRODUCTION

Assume that (X, d_1) and (Y, d_2) are complete metric spaces with the metrics $d_1 : X \times X \to R^1$ and $d_2 : Y \times Y \to R^1$. For each function $f : X \times Y \to R^1$ set

$$v_f^b = \sup_{y\in Y} \inf_{x\in X} f(x, y), \quad v_f^a = \inf_{x\in X} \sup_{y\in Y} f(x, y). \tag{1.1}$$

Clearly

$$v_f^b \le v_f^a \quad \text{for each} \quad f : X \times Y \to R^1. \tag{1.2}$$

Denote by $\mathfrak{M}$ the set of all functions $f : X \times Y \to R^1$ such that for each $x \in X$ the function $y \to f(x, y)$, $y \in Y$ is continuous, for each $y \in Y$ the function $x \to f(x, y), x \in X$ is continuous and

$$-\infty < v_f^b = v_f^a < \infty. \tag{1.3}$$

For each $f \in \mathfrak{M}$ set

$$v_f = v_f^b. \tag{1.4}$$

For each $f, g \in \mathfrak{M}$ define

$$\tilde{\rho}(f, g) = \sup\{|f(x, y) - g(x, y)| : (x, y) \in X \times Y\}, \tag{1.5}$$

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$$\rho(f,g) = \tilde{\rho}(f,g)(1+\tilde{\rho}(f,g))^{-1}$$

(here we use the convention that $\infty/\infty = 1$).

Clearly $\rho : \mathfrak{M} \times \mathfrak{M} \to R^1$ is a metric and the uniformity induced by the metric ρ has the following base:

$$E(\epsilon) = \{(f,g) \in \mathfrak{M} \times \mathfrak{M} : \ |f(x,y) - g(x,y)| \leq \epsilon, \ (x,y) \in X \times Y\},$$

where $\epsilon > 0$. It is not difficult to see that the metric space $(\mathfrak{M}, \rho)$ is complete.

Denote by $\mathfrak{M}_c$ the set of all continuous function $f \in \mathfrak{M}$. Clearly $\mathfrak{M}_c$ is a closed subset of $(\mathfrak{M}, \rho)$. We consider the topological subspace $\mathfrak{M}_c \subset \mathfrak{M}$ with the relative topology.

Let $f \in \mathfrak{M}$. We say that $(x_*, y_*) \in X \times Y$ is a saddle point of f if

$$\sup_{y \in Y} f(x_*, y) = f(x_*, y_*) = \inf_{x \in X} f(x, y_*).$$

In Zaslavski [10] we showed that generic functions in the spaces $\mathfrak{M}$ and $\mathfrak{M}_c$ have unique saddle points. The study of saddle points and minimax problems is an important topic in optimization theory (see, for example, Aubin and Ekeland [1] and Ekeland and Temam [5] and the references mentioned therein). Existence results usually require special assumptions on the space and the functions. In Zaslavski [10], instead of considering the existence of a saddle point for a single function f, we investigated it for the whole space $\mathfrak{M}_c$ and showed that a saddle point exists for most of the functions f in $\mathfrak{M}_c$. Namely we established the existence of a subset $\mathcal{F}$ of $\mathfrak{M}_c$ which is a countable intersection of open everywhere dense subsets of $(\mathfrak{M}_c, \rho)$ such that for each $f \in \mathcal{F}$ there exists a unique saddle point. This approach has also been successfully applied in many areas of analysis (see, for example, De Blasi and Myjak [3], De Blasi et al [4], Reich and Zaslavski [6] and [7] and Zaslavski [9]). In the present paper we show that the complement of the set of all functions $f \in \mathfrak{M}$ for which a saddle point exists is not only of the first category, but also σ-porous in the space $(\mathfrak{M}, \rho)$. An analogous result is also established for the space $(\mathfrak{M}_c, \rho)$. Before we continue we recall the concept of porosity (see Benyamini and Lindenstrauss [2], De Blasi and Myjak [3], De Blasi et al [4], Reich and Zaslavski [6] and [7], Zajíček [8], Zaslavski [9]).

Let (Y, d) be a complete metric space. For $x \in Y$, $r > 0$ set

$$B_d(x, r) = \{y \in Y : \ d(x, y) \leq r\}.$$

A set $E \subset Y$ is a porous subset of (Y, d) if there exist $\alpha \in (0, 1)$, $r_0 > 0$ such that for each $x \in Y$, each $r \in (0, r_0]$ there exists $y \in Y$ such that

$$B_d(y, \alpha r) \subset B_d(x, r) \setminus E.$$

A subset of (Y, d) is called a σ-porous subset of (Y, d) if it is a countable union of porous subsets of (Y, d).

Other notions of porosity have been used in the literature (see Benyamini and Lindenstrauss [2], Zajíček [8]). We use the rather strong notion which appears in De Blasi and Myjak [3], De Blasi et al [4], Reich and Zaslavski [6], [7] and Zaslavski [9].

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

Denote by $\mathcal{F}$ the set of all $f \in \mathfrak{M}$ for which the following conditions hold:

(i) There exist $x_f \in X$ and $y_f \in Y$ such that

$$\sup_{y \in Y} f(x_f, y) = f(x_f, y_f) = \inf_{x \in X} f(x, y_f).$$

(ii) For each $\epsilon > 0$ there exists a neighborhood U of f in $\mathfrak{M}$ and a number $\delta > 0$ such that for each $h \in U$ and each $(x_0, y_0) \in X \times Y$ satisfying

$$\sup_{y \in Y} h(x_0, y) \leq v_h^a + \delta, \quad \inf_{x \in X} h(x, y_0) \geq v_h^b - \delta$$

the inequalities

$$d_1(x_0, x_f), d_2(y_0, y_f) \leq \epsilon$$

are true.

In Zaslavski [10] it was shown that $\mathcal{F} \cap \mathfrak{M}_c$ contains a countable intersection of open everywhere dense subsets of $(\mathfrak{M}_c, \rho)$. Moreover in Zaslavski [10] it was shown that if at least one of the spaces X, Y is compact, then $\mathcal{F}$ contains a countable intersection of open everywhere dense subsets of $(\mathfrak{M}, \rho)$.

In this paper we prove the following theorem.

Theorem 1.1. *The set* $\mathfrak{M} \setminus \mathcal{F}$ *is a* σ*-porous subset of* $(\mathfrak{M}, \rho)$ *and the st* $\mathfrak{M}_c \setminus \mathcal{F}$ *is a* σ*-porous subset of* $(\mathfrak{M}_c, \rho)$.

2. AUXILIARY RESULTS

We need the following simple lemma proved in Zaslavski [10, Lemma 2.1].

Lemma 2.1. *Let $f \in \mathfrak{M}$, $\epsilon_1, \epsilon_2 > 0$, $x_0 \in X$, $y_0 \in Y$ and let*

$$\sup_{y \in Y} f(x_0, y) \le v_f^a + \epsilon_1, \quad \inf_{x \in X} f(x, y_0) \ge v_f^b - \epsilon_2.$$

Then

$$\sup_{y \in Y} f(x_0, y) - \epsilon_1 - \epsilon_2 \le v_f^a - \epsilon_2 \le f(x_0, y_0) \le v_f^b + \epsilon_1$$
$$\le \inf_{x \in X} f(x, y_0) + \epsilon_1 + \epsilon_2.$$

Denote by E the set of all $g \in \mathfrak{M}$ such that there exist $x_g \in X$ and $y_g \in Y$ which satisfy

$$\sup_{y \in Y} g(x_g, y) = g(x_g, y_g) = \inf_{x \in X} g(x, y_g).$$

Set

$$E_c = E \cap \mathfrak{M}_c.$$

The following auxiliary result was proved in Zaslavski [10, Lemma 2.3].

Lemma 2.2. *The set E is everywhere dense in $(\mathfrak{M}, \rho)$ and the set E_c is everywhere dense in $(\mathfrak{M}_c, \rho)$.*

Let $n \ge 1$ be a natural number. Denote by $\mathcal{F}_n$ the set of all $f \in \mathfrak{M}$ which have the following property:

(P1) There exist $(x_*, y_*) \in X \times Y$, $\delta > 0$ and a neighborhood V of f in $\mathfrak{M}$ such that for each $h \in V$ and each $(z, \xi) \in X \times Y$ which satisfies

$$\sup_{y \in Y} h(z, y) \le v_h^a + \delta, \ \inf_{x \in X} h(x, \xi) \ge v_h^b - \delta$$

the following inequalities hold:

$$d_1(z, x_*), \ d_2(z, y_*) \le 1/n.$$

Proposition 2.1. *$\cap_{n=1}^{\infty} \mathcal{F}_n \subset \mathcal{F}$.*

Proof. Let $f \in \cap_{n=1}^{\infty} \mathcal{F}_n$. There exist sequences $\{z_i\}_{i=1}^{\infty} \subset X$, $\{\xi_i\}_{i=1}^{\infty} \subset Y$ such that

$$\sup_{y \in Y} f(z_i, y) - v_f^a \le 1/i, \ -\inf_{x \in X} f(x, \xi_i) + v_f^b \le 1/i \tag{2.1}$$

for all integers $i \ge 1$.

Let $\epsilon > 0$. Choose a natural number n such that

$$1/n < \epsilon/8. \tag{2.2}$$

Since $f \in \mathcal{F}_n$ it follows from property (P1) that there exist $(x_n, y_n) \in X \times Y$, $\delta_n > 0$ and a neighborhood V_n of f in $\mathfrak{M}$ such that the following property holds:

(P2) For each $h \in V_n$ and each $(z, \xi) \in X \times Y$ which satisfies

$$\sup_{y \in Y} h(z, y) \leq v_h^a + \delta_n, \ \inf_{x \in X} h(x, \xi) \geq v_h^b - \delta_n$$

the following inequality holds:

$$d_1(z, x_n), \ d_2(\xi, y_n) \leq 1/n < \epsilon/2.$$

It follows from property (P2) and (2.1) that

$$d_1(z_i, x_n), \ d_2(\xi_i, y_n) \leq 1/n < \epsilon/2 \tag{2.3}$$

for all sufficiently large natural numbers i.

Since ϵ is an arbitrary positive number we conclude that $\{z_i\}_{i=1}^{\infty}$ and $\{\xi_i\}_{i=1}^{\infty}$ are Cauchy sequences.

Let

$$z_f = \lim_{i \to \infty} z_i, \ \xi_f = \lim_{i \to \infty} \xi_i. \tag{2.4}$$

In view of (2.3) and (2.4)

$$d_1(z_f, x_n), d_2(\xi_f, y_n) \leq 1/n < \epsilon/2. \tag{2.5}$$

It follows from (2.1) and Lemma 2.1 that for each integer $i \geq 1$

$$\sup_{y \in Y} f(z_i, y) - 2/i \leq v_f^a - 1/i \leq f(z_i, \xi_i) \leq v_f^b + 1/i$$

$$\leq \inf_{x \in X} f(x, \xi_i) + 2/i. \tag{2.6}$$

By (2.4), (2.6) and (1.3)

$$\sup_{y \in Y} f(z_f, y) \leq \liminf_{i \to \infty} (\sup_{y \in Y} f(z_i, y)) \leq v_f^a = v_f^b \tag{2.7}$$

$$\leq \limsup_{i \to \infty} (\inf_{x \in X} f(x, \xi_i)) \leq \inf_{x \in X} f(x, \xi_f).$$

On the other hand

$$\sup_{y \in Y} f(z_f, y) \geq f(z_f, \xi_f) \geq \inf_{x \in X} f(x, \xi_f).$$

Combined with (2.7) this inequality implies that

$$\sup_{y \in Y} f(z_f, y) = f(z_f, \xi_f) = \inf_{x \in X} f(x, \xi_f). \tag{2.8}$$

Relations (2.8), (1.2) and (1.1) imply that

$$f(z_f, \xi_f) = v_f^a = v_f^b. \tag{2.9}$$

Now assume that

$$h \in V_n,\ (z,\xi) \in X \times Y, \tag{2.10}$$

$$\sup_{y \in Y} h(z,y) \le v_h^a + \delta_n,\ \inf_{x \in X} h(x,\xi) \ge v_h^b - \delta_n.$$

By (P2) and (2.10)

$$d_1(z,x_n),\ d_2(\xi,y_n) \le 1/n.$$

Combined with (2.5) and (2.2) these inequalities imply that

$$d_1(z,z_f),\ d_2(\xi,\xi_f) \le 2/n < \epsilon.$$

Thus $f \in \mathcal{F}$ and Proposition 2.1 is proved. □

3. PROOF OF THEOREM 1.1

By Proposition 2.1 in order to prove the theorem it is sufficient to show that for each natural number n the set $\mathfrak{M} \setminus \mathcal{F}_n$ is a porous subset of $(\mathfrak{M}, \rho)$ and the set $\mathfrak{M}_c \setminus \mathcal{F}_n$ is a porous subset of $(\mathfrak{M}_c, \rho)$.

Let n be a natural number. Let

$$0 < \alpha < (128n)^{-1}. \tag{3.1}$$

Assume that

$$f \in \mathfrak{M},\ r \in (0,1]. \tag{3.2}$$

Put

$$\gamma = 16n\alpha r \in (0, 8^{-1}r). \tag{3.3}$$

By Lemma 2.2 there exists $f_0 \in \mathfrak{M}$ such that

$$f_0 \in E, \quad \rho(f,f_0) \le r/8, \tag{3.4}$$

$$\text{if } \ f \in \mathfrak{M}_c, \quad \text{then} \quad f_0 \in \mathfrak{M}_c.$$

Since $f_0 \in E$ there exist $\bar{x} \in X$, $\bar{y} \in Y$ such that

$$\sup_{y \in Y} f_0(\bar{x},y) = f_0(\bar{x},\bar{y}) = \inf_{x \in X} f_0(x,\bar{y}). \tag{3.5}$$

Define a function $f_1 : X \times Y \to R^1$ by

$$f_1(x,y) = f_0(x,y) + \gamma \min\{1, d_1(x,\bar{x})\} - \gamma \min\{1, d_2(y,\bar{y})\},\ (x,y) \in X \times Y. \tag{3.6}$$

It is not difficult to see that

$$f_1(\bar{x},\bar{y}) = f_0(\bar{x},\bar{y}), \tag{3.7}$$

$$\sup_{y \in Y} f_1(\bar{x}, y) = f_1(\bar{x}, \bar{y}) = \inf_{x \in X} f_1(x, \bar{y}), \tag{3.8}$$

$$v^b_{f_1} = f_1(\bar{x}, \bar{y}) = v^a_{f_1},$$

$$f_1 \in E \text{ and } f_1 \in E_c \text{ if } f \in E_c. \tag{3.10}$$

Clearly

$$\rho(f_0, f_1) \le 2\gamma. \tag{3.11}$$

Now assume that $z \in X$, $\xi \in Y$,

$$\sup_{y \in Y} f_1(z, y) \le v^a_{f_1} + (2n)^{-1}\gamma, \tag{3.12}$$

$$\inf_{x \in X} f_1(x, \xi) \ge v^b_{f_1} - (2n)^{-1}\gamma.$$

By (3.9), (3.12), (3.6), (3.5) and (3.7)

$$f_1(\bar{x}, \bar{y}) + (2n)^{-1}\gamma = v^a_{f_1} + (2n)^{-1}\gamma \ge \sup_{y \in Y} f_1(z, y)$$

$$= \sup_{y \in Y}\{f_0(z, y) + \gamma \min\{1, d_1(z, \bar{x})\} - \gamma \min\{1, d_2(y, \bar{y})\}\}$$

$$= \gamma \min\{1, d_1(z, \bar{x})\} + \sup_{y \in Y}\{f_0(z, y) - \gamma \min\{1, d_2(y, \bar{y})\}\}$$

$$\ge \gamma \min\{1, d_1(z, \bar{x})\} + f_0(z, \bar{y})$$

$$\ge \gamma \min\{1, d_1(z, \bar{x})\} + f_0(\bar{x}, \bar{y}) = f_1(\bar{x}, \bar{y}) + \gamma \min\{1, d_1(z, \bar{x})\}.$$

This implies that

$$d_1(z, \bar{x}) \le (2n)^{-1}. \tag{3.13}$$

It follows from (3.9), (3.12), (3.6), (3.5) and (3.7) that

$$f_1(\bar{x}, \bar{y}) - (2n)^{-1}\gamma = v^b_{f_1} - (2n)^{-1}\gamma \le \inf_{x \in X} f_1(x, \xi)$$

$$= \inf_{x \in X}\{f_0(x, \xi) + \gamma \min\{1, d_1(x, \bar{x})\} - \gamma \min\{1, d_2(\xi, \bar{y})\}\}$$

$$= -\gamma \min\{1, d_2(\xi, \bar{y})\} + \inf_{x \in X}\{f_0(x, \xi) + \gamma \min\{1, d_1(x, \bar{x})\}\}$$

$$\le -\gamma \min\{1, d_2(\xi, \bar{y})\} + f_0(\bar{x}, \xi) \le -\gamma \min\{1, d_2(\xi, \bar{y})\} + f_0(\bar{x}, \bar{y})$$

$$= f_1(\bar{x}, \bar{y}) - \gamma \min\{1, d_2(\xi, \bar{y})\}.$$

This inequality implies that $d_2(\xi, \bar{y}) \le (2n)^{-1}$. Therefore we have shown that the following property holds:

(P3) If $z \in X$, $\xi \in Y$ satisfy (3.12), then

$$d_1(z, \bar{x}) \le (2n)^{-1},\ d_2(\xi, \bar{y}) \le (2n)^{-1}.$$

Assume that $g \in \mathfrak{M}$ and

$$\rho(f_1, g) \le \alpha r. \tag{3.14}$$

It follows from (3.14), (3.11), (3.4), (3.3) and (3.1) that

$$\rho(g, f) \le \rho(g, f_1) + \rho(f_1, f_0) + \rho(f_0, f) \le \alpha r + 2\gamma + r/8 < r. \tag{3.15}$$

In view of (3.15), (3.2), (1.5), (3.14) and (3.1)

$$\tilde{\rho}(f_1, g) = \rho(f_1, g)(1 - \rho(f_1, g))^{-1} \le \alpha r(1 - \alpha r)^{-1} \le 2\alpha r,$$

$$|f_1(x, y) - g(x, y)| \le 2\alpha r \quad \text{for all} \quad (x, y) \in X \times Y. \tag{3.16}$$

Assume that $z \in X$, $\xi \in Y$ satisfy

$$\sup_{y \in Y} g(z, y) \le v_g^a + \gamma(4n)^{-1}, \ \inf_{x \in X} g(x, \xi) \ge v_g^b - (4n)^{-1}\gamma. \tag{3.17}$$

It follows from (3.16), (3.17), (3.1), (3.2) and (3.3) that

$$\sup_{y \in Y} f_1(z, y) \le \sup_{y \in Y} g(z, y) + 2\alpha r \le v_g^a + (4n)^{-1}\gamma + 2\alpha r$$

$$\le v_{f_1}^a + (4n)^{-1}\gamma + 4\alpha r \le v_{f_1}^a + (2n)^{-1}\gamma, \tag{3.18}$$

$$\inf_{x \in X} f_1(x, \xi) \ge \inf_{x \in X} g(x, \xi) - 2\alpha r \ge v_g^b - (4n)^{-1}\gamma - 2\alpha r$$

$$\ge v_{f_1}^b - 4\alpha r - (4n)^{-1}\gamma \ge v_{f_1}^b - (2n)^{-1}\gamma. \tag{3.19}$$

By (3.18), (3.19) and (P3)

$$d_1(z, \bar{x}), \ d_2(\xi, \bar{y}) \le (2n)^{-1}.$$

We have shown that

$$\{g \in \mathfrak{M} : \ \rho(g, f_1) \le \alpha r/2\} \subset \mathcal{F}_n.$$

Combined with (3.15) this inclusion implies that

$$\{g \in \mathfrak{M} : \ \rho(g, f_1) \le \alpha r/2\} \subset \mathcal{F}_n \cap \{h \in \mathfrak{M} : \ \rho(h, f) \le r\}.$$

Therefore $\mathfrak{M} \setminus \mathcal{F}_n$ is a porous set in $(\mathfrak{M}, \rho)$ and $\mathfrak{M}_c \setminus \mathcal{F}_n$ is a porous set in $(\mathfrak{M}_c, \rho)$. Theorem 1.1 is proved.

REFERENCES

[1] J.P. Aubin and I. Ekeland, *Applied Nonlinear Analysis,* Wiley-Interscience, New York, 1984.

[2] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis,* Amer. Math. Soc., Providence, RI, 2000.

[3] F.S. De Blasi and J. Myjak, Sur la porosité des contractions sans point fixe, *C.R. Acad. Sc. Paris,* **308** (1989), 51-54.

[4] F.S. De Blasi, J. Myjak, and P.L. Papini, Porous sets in best approximation theory, *London Math. Soc.,* **44** (1991), 135-142.

[5] I. Ekeland and R.Temam, *Analyse Convexe et Problemes Variationnels,* Dunod, Paris, 1974.

[6] S. Reich and A.J. Zaslavski, The set of divergent descent methods in a Banach space is σ-porous, *SIAM J. Optim.*, **11** (2001), 1003-1018.

[7] S. Reich and A.J. Zaslavski, The set of noncontractive mappings is σ-porous in the space of all nonexpansive mappings, *C.R. Acad. Sci. Paris,* **333** (2001), 539-544.

[8] L. Zajíček, Porosity and σ-porosity, *Real Analysis Exchange,* **13** (1987), 314-350.

[9] A.J. Zaslavski, Well-posedness and porosity in optimal control without convexity assumptions, *Calculus of Variations and Partial Differential Equations,* **13** (2001), 265-293.

[10] A.J. Zaslavski, Generic existence of a saddle point, *Communications in Applied Analysis*, **8** (2004), 143-151.