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DYNAMICAL STRUCTURES EMERGING IN HIGHER ORDER NONLINEAR SYSTEMS

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ABSTRACT: The polar decomposition is used to analyze a class of canonical models of differential equations over the quaternions. A spectrum of oscillatory behaviors can be observed and techniques to detect directional changes of oscillations are derived. A classification of transient oscillatory behaviors encountered is also presented.

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1. INTRODUCTION

The noncommutative ring of the quaternions provides a framework for applications in many areas of pure and applied sciences, such as physics, engineering, and mathematics, cf. Adler [2] and Kuipers [19]. This has lead to an increasing interest in quaternionic differential equations and their qualitative behaviors. See, for example, Campos and Mawhin [8], Leo and Scolarici [11], De Leo and Ducati [9], De Leo and Ducati [10], and Kravchenko and Williams [18]. The authors in the later paper explore a quaternionic generalization of the Ricatti system. In Campos and Mawhin [8], Campos and Mawhin generalize a previous work by Lloyd [25] from the complex case to the quaternion case. This motivated several questions addressed in this paper. Following a similar approach to the one in Campos and Mawhin [8], we investigate generalizations of the canonical form associated with Andronov-Hopf bifurcation in the plane: $\dot{z} = (\alpha + i)z - z|z|^2$. We propose the straightforward generalization of this system to the quaternions: $\dot{q}(t) = (\alpha + i)q(t) - q(t)\,|q(t)|^2$. More generally, we consider:

$$\dot{q(t)} = F(|q|, t)q(t),$$

and the higher order extension:

$$\dot{M(t)} = (\alpha_0 I_n + A)M(t) - M(t)M^*(t)M(t),$$

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where $F(|q|,t)$ is a time dependent real valued function, and $M(t)$ is an $n\times n$ matrix function with complex valued entries.

We observe that $F(|q|,t) = (\alpha + i - |q(t)|^2)$ yields the canonical Andronov-Hopf model over the quaternions. The main objective of our study is to understand the qualitative behavior of our systems including stability and periodicity. We use a polar factorization to identify invariant regions where all the periodic solutions lie. Under some conditions, trajectories project into an exotic "star" or "sputnik-like" spatial structure that capture directional changes of oscillations.

In Section 2, we set basic notation and derive the decomposition that plays the main role throughout the whole analysis. In Section 3 and Section 4, we restrict our system to the unit sphere. We first consider non-autonomous parameters satisfying a commutativity assumption, under which solutions have an explicit representation, see Lukes [24]. This condition allows a large spectrum of transient oscillations, reflected in the sputnik structure. We also study the non-commutative case and describe a method to construct systems exhibiting rich oscillatory behavior. Finally, in Section 5, we present a higher dimensional extension where our techniques still apply. The algebraic representation of solutions determine a criterion for the existence of periodic solutions.

2. BACKGROUND AND NOTATION

We consider the 4-dimensional system:

$$\begin{cases} \dot{\mathbf{q}}(t) = \gamma(t)\,\mathbf{q}(t) + F(|\mathbf{q}(t)|,t)\mathbf{q}(t)\,, \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases} \tag{1}$$

The underlying space, $\mathcal{H}$, is the set of all $\mathbf{q} = q_0 + q_1\,i + q_2\,j + q_3\,k$, where each coefficient q_i is a real number and the imaginary units i, j, k are defined by the relations $i^2 = j^2 = k^2 = ijk = -1$. The real-valued function F is continuously differentiable and $\gamma(t)$ is a fixed quaternion-valued function.

We list some definitions and properties of the quaternions that will be used later, see Kuipers [19].

1. The conjugate of a quaternion $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$ is denoted by $\mathbf{q}^*$ and is equal to $q_0 - q_1 i - q_2 j - q_3 k$. The imaginary and real parts are defined as $\mathcal{I}_{\mathbf{q}} = q_1 i + q_2 j + q_3 k$ and $\mathcal{R}_{\mathbf{q}} = q_0$, respectively. A quaternion with zero real part is said to be a "pure quaternion".

2. Given two quaternions $\mathbf{p}$ and $\mathbf{q}$ we define the inner product $\langle \mathbf{p}, \mathbf{q}\rangle = \mathcal{R}(\mathbf{p}\mathbf{q}^*)$. Hence the norm of $\mathbf{q}$, $|\mathbf{q}| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}$ ($\mathbf{q}$ is unitary if $|\mathbf{q}| = 1$).

We show that the system (1) can be decomposed into a scalar and a unitary systems. A similar decomposition also holds for the corresponding solutions. We denote the parameter γ by $\gamma_0 + \gamma_1 i + \gamma_2 j + \gamma_3 k$.

Proposition 2.1. $\mathbf{q}(t)$ *is a solution of (1) if and only if* $x(t) = |\mathbf{q}(t)|$ *and* $\mathbf{u}(t) = \frac{\mathbf{q}(t)}{|\mathbf{q}(t)|}$ *are solutions of*

$$\begin{cases} \dot{x}(t) = \gamma_0 x(t) + F(x(t), t)x(t)\,, \\ x(0) = |\mathbf{q}_0|\,, \end{cases}$$

and

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathcal{I}_\gamma(t)\,\mathbf{u}(t)\,, \\ \mathbf{u}(0) = \mathbf{u}_0\,, \end{cases}$$

respectively.

Proof. We set $\mathbf{u}(t) = \frac{\mathbf{q}(t)}{|\mathbf{q}(t)|}$. Then $\mathbf{q}(t)$ is a solution of the system (1) if and only if $\dot{|\mathbf{q}|}\,\mathbf{u} + |\mathbf{q}|\dot{\mathbf{u}} = (\gamma_0 + \mathcal{I}_\gamma)\,|\mathbf{q}|\,\mathbf{u} + F(|\mathbf{q}|, t)|\mathbf{q}|\,\mathbf{u}$, or equivalently

$$\dot{|\mathbf{q}|} + |\mathbf{q}|\,\dot{\mathbf{u}}\,\mathbf{u}^* = (\gamma_0 + \mathcal{I}_\gamma)|\mathbf{q}| + F(|\mathbf{q}|, t)|\mathbf{q}|.$$

We may assume, without loss of generality, that $|\mathbf{q}(t)| \neq 0$, for all t. Since $\dot{\mathbf{u}} = \frac{\dot{\mathbf{q}}}{|\mathbf{q}|} - \frac{\mathbf{q}\,\dot{|\mathbf{q}|}}{|\mathbf{q}|^2}$, then $\dot{\mathbf{u}}\,\mathbf{u}^* = \frac{\dot{\mathbf{q}}\,\mathbf{q}^*}{|\mathbf{q}|^2} - \frac{\dot{|\mathbf{q}|}}{|\mathbf{q}|}$ and the real parts of $\frac{\dot{\mathbf{q}}\,\mathbf{q}^*}{|\mathbf{q}|^2}$ and $\frac{\dot{|\mathbf{q}|}}{|\mathbf{q}|}$ are both equal to

$$\frac{q_0\dot{q}_0 + q_1\dot{q}_1 + q_2\dot{q}_2 + q_3\dot{q}_3}{|\mathbf{q}|^2}.$$

This implies that $\dot{\mathbf{u}}\,\mathbf{u}^*$ is a pure quaternion. The equation

$$\dot{\mathbf{q}} = (\gamma_0\ + \mathcal{I}_\gamma)\mathbf{q}\ -\ |\mathbf{q}|^2\mathbf{q}$$

can then be uniquely decomposed into its scalar and unitary components:

$$\dot{|\mathbf{q}|} = \gamma_0|\mathbf{q}| + F(|\mathbf{q}|, t)|\mathbf{q}| \ \text{ and }\ \dot{\mathbf{u}} = \mathcal{I}_\gamma\,\mathbf{u}. \qquad \square$$

The qualitative behavior of the scalar equation, $\dot{|\mathbf{q}|} = \gamma_0|\mathbf{q}| + F(|\mathbf{q}|, t)|\mathbf{q}|$, is characterized by its equilibria and their stability. We illustrate this point with $F(|\mathbf{q}|, t) = \sum_{i=0}^{n} \alpha_i\,|\mathbf{q}|^i$, where each α_i is a real number. The polynomial vector field, $p(x) = \gamma_0 + \sum_{i=0}^{n} \alpha_i\,x^i$ can be factored into a product of irreducible polynomials:

$$p(x) \\ = \alpha_n(x - p_1)\cdots(x - p_i)(x - n_1)\cdots(x - n_j)(x^2 + c_1 x + d_1)\cdots(x^2 + c_k x + d_k),$$

with $n = i+j+2k$. We assume that p has i positive solutions, j negative solutions and $2k$ complex solutions. The quantity x represents the norm of a quaternion, therefore nonnegative. A typical phase portrait with i even, $p_1 < p_2 < \cdots < p_i$ and $\alpha_n > 0$ is sketched in the Figure 1.

Each nonnegative equilibrium point of the scalar equation defines an invariant sphere in $\mathcal{H}$.

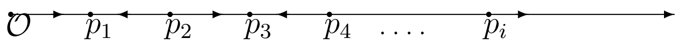

Figure 1: Qualitative behavior of a scalar system

2.1. EXAMPLE

We consider the Andronov-Hopf canonical model of the quaternions: $\dot{\mathbf{q}} = \gamma\,\mathbf{q} - |\mathbf{q}|^2\mathbf{q},\ \mathbf{q}(0) = \mathbf{q}_0$. The solution to the scalar equation $\dot{|\mathbf{q}|} = \gamma_0|\mathbf{q}| - |\mathbf{q}|^3,\ |\mathbf{q}(0)| = |\mathbf{q}_0|$ $(\gamma_0 \neq 0)$ is

$$|\mathbf{q}| = \frac{|\mathbf{q}_0|\,\sqrt{|\gamma_0|}\,\exp(\gamma_0 t)}{\sqrt{|\gamma_0 - |\mathbf{q}_0|^2 + |\mathbf{q}_0|^2\exp(2\gamma_0 t)|}},$$

provided $\gamma_0 - |\mathbf{q}_0|^2 + |\mathbf{q}_0|^2\exp(2\gamma_0 t) \neq 0$, cf. Botelho and Jamison [6]. If $\gamma_0 > 0$, the system (1) leaves invariant the 3 dimensional sphere, $S^3_{\gamma_0} = \{\mathbf{q} : q_0^2+q_1^2+q_2^2+q_3^2 = \gamma_0\}$. This sphere is an attracting set since

$$\lim_{t\to+\infty} \frac{|\mathbf{q}_0|\,\sqrt{|\gamma_0|}\,\exp(\gamma_0 t)}{\sqrt{|\gamma_0 - |\mathbf{q}_0|^2 + |\mathbf{q}_0|^2\exp(2\gamma_0 t)|}} = \sqrt{|\gamma_0|}\,.$$

The function $\exp(t\,\mathcal{I}_\gamma)$ satisfies the equation $\dot{\mathbf{u}}(t) = \mathcal{I}_\gamma\,\mathbf{u}(t)$ and therefore

$$\mathbf{q}(t) = |\mathbf{q}(t)|\exp(t\mathcal{I}_\gamma)\,\mathbf{u}_0$$

is the solution of (1). In particular, if $|\mathbf{q}_0|^2 = \gamma_0$, then $\mathbf{q}(t) = |\mathbf{q}_0|\exp(t\mathcal{I}_\gamma)\,\mathbf{u}_0$. The curve traced by $\{\exp(t\mathcal{I}_\gamma)\mathbf{u}_0\}_{t\in R}$ lies in the unit sphere S^3_1 (unitary quaternions).

As the real parameter γ_0 crosses the value $|q_0|^2$ a "circle" of radius $|q_0|$ emerges from the origin and the system undergoes the Andronov-Hopf bifurcation, cf. Perko [31].

Proposition 2.2. $\{|\mathbf{q}_0|\,\exp(t\mathcal{I}_\gamma)\,\mathbf{u}_0\}_{t\in R}$ *defines a circle of radius* $|\mathbf{q}_0|$, *centered at the origin. This circle is contained in the two dimensional subspace spanned by* $\{\mathbf{u}_0, \mathcal{I}_\gamma\mathbf{u}_0\}$.

Proof. The set $\{\mathbf{u}_0, \mathcal{I}_\gamma\mathbf{u}_0\}$ is linearly independent over the real numbers. Indeed, if $\alpha\mathbf{u}_0 + \beta\mathcal{I}_\gamma\mathbf{u}_0 = 0$, then $\alpha = 0$ and therefore $\beta = 0$, unless $\mathcal{I}_\gamma \equiv 0$.

Furthermore, $\exp(t\,\mathcal{I}_\gamma) = \cos(t|\mathcal{I}_\gamma|) + \sin(t|\mathcal{I}_\gamma|)\frac{\mathcal{I}_\gamma}{|\mathcal{I}_\gamma|}$ and $|\exp(t\,\mathcal{I}_\gamma)\,\mathbf{u}_0| = 1$. This implies that $\exp(t\,\mathcal{I}_\gamma)$ is periodic, with period $\frac{2\pi}{|\mathcal{I}_\gamma|}$. □

3. THE UNITARY SYSTEM: COMMUTATIVE CASE

Each positive equilibrium point of the scalar system gives rise to an invariant sphere of the original system. If a periodic solution exists, it must lie in one of those spheres. We now focus our study to the unitary equation:

$$\dot{\mathbf{u}}(t) = \mathcal{I}_\gamma(t)\,\mathbf{u}(t),\;\; \mathbf{u}(0) = \mathbf{u}_0. \tag{2}$$

The complete analysis of this system is quite complicated so, we first restrict ourselves to the so-called commutative case (cf. Lukes [24]):

$$\mathcal{I}_\gamma(t)\,\mathbf{I}_\gamma(t) \;=\; \mathbf{I}_\gamma(t)\,\mathcal{I}_\gamma(t),\;\; \text{where}\;\; \mathbf{I}_\gamma(t) = \int_0^t \mathcal{I}_\gamma(\xi)\,d\xi. \tag{3}$$

This can be translated in terms of the triplet $\tau = (\gamma_1,\, \gamma_2,\, \gamma_3)$ as follows:

$$g_i(t)\gamma_j(t) = \gamma_j(t)g_i(t),$$

where $g_i = \int_0^t \gamma_i(\xi)\,d\xi$. In this case, we say that τ is compatible. We examine this commutative case in great detail because solutions have an easy explicit representation and the system generates an interesting orbital structure.

We define the auxiliary map

$$Z_\tau(t) = \max\{|g_1(t)|,\, |g_2(t)|,\, |g_3(t)|\},$$

whose support, $\mathcal{S}(Z_\tau)$, can be written as the disjoint union of open intervals $\{J_n\}$. For each i, there exist scalars, uniquely determined in $\mathcal{S}(Z_\tau)$, so that

$$g_i(t) = \sum_n a_{in} Z_\tau(t), \quad i = 1, 2, 3.$$

If τ is a compatible triplet then Z_τ is continuously differentiable. Moreover, when τ is restricted to a component of $\mathcal{S}(Z_\tau)$ traces a segment in the space, designated a "spike". We use the structure of the set of spikes to distinguish oscillatory behaviors of (1).

Definition 3.1. If $\tau = (f, g, h)$ is compatible, the curve $\{(f(t), g(t), h(t))\}_{t\in\mathbf{R}}$ is called a *sputnik*, and is denoted by $\mathrm{sk}(\tau)$. The degree of $\mathrm{sk}(\tau)$ is the cardinality of all the different spikes in $\mathrm{sk}(\tau)$ that are attached to the origin.

The solution to system (2) is given by

$$\mathbf{u}(t) = \exp(\mathbf{I}_\gamma(t))\mathbf{u}_0 = \cos\left(|\mathbf{I}_\gamma(t)|\right)\, \mathbf{u}_0 + \frac{\mathbf{I}_\gamma(t)}{|\mathbf{I}_\gamma(t)|}\sin\left(|\mathbf{I}_\gamma(t)|\right)\mathbf{u}_0$$

(see Lukes [24]). We observe that the curve $\{\mathbf{u}(t)\}_t$ lies in the unit 3-sphere.

We denote by $\mathbf{u}_1(t)$ the solution of the system associated with a triplet τ_1. Standard trignometric identities allows us to establish a weak form of stability, as stated in the next proposition.

Proposition 3.1. *If* $\tau = (f, g, h)$ *and* $\tau_1 = (f_1, g_1, h_1)$ *then*

$$|\mathbf{u}(t) - \mathbf{u}_1(t)| \leq |\,|\tau_1(t)| - |\tau(t)|\,| + \left| \frac{\tau_1(t)}{|\tau_1(t)|} - \frac{\tau(t)}{|\tau(t)|} \right|.$$

Proof. The product

$$\mathbf{u}_1(t)\mathbf{u}^*(t) = \left[(\cos|\tau_1(t)|)\mathbf{u}_0 + \frac{1}{|\tau_1(t)|}(\sin|\tau_1(t)|)(f_1(t)\,\mathbf{i} + g_1(t)\,\mathbf{j} + h_1(t)\,\mathbf{k})\mathbf{u}_0\right]$$
$$\times \left[(\cos|\tau(t)|)\mathbf{u}_0 + \frac{1}{|\tau(t)|}(\sin|\tau(t)|)(f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k})\mathbf{u}_0\right]^*.$$

Therefore

$$\mathcal{R}(\mathbf{u}_1(t)\mathbf{u}^*(t)) = 2\left[\cos|\tau_1(t)|\,\cos|\tau(t)| + \langle \frac{\tau_1(t)}{|\tau_1(t)|}, \frac{\tau(t)}{|\tau(t)|}\rangle \,\sin|\tau_1(t)|\,\sin|\tau(t)|\right].$$

This implies that

$$|\mathbf{u}(t) - \mathbf{u}_1(t)|^2 = 2 - \mathcal{R}(\mathbf{u}_1(t)\,\mathbf{u}^*(t)),$$

and

$$|\mathbf{u}(t) - \mathbf{u}_1(t)| \leq |\,|\tau_1(t)\,| - |\tau(t)|\,| + \left| \frac{\tau_1(t)}{|\tau_1(t)|} - \frac{\tau(t)}{|\tau(t)|} \right|. \qquad \square$$

Example. We describe triplets exhibiting complicated sputnik structures. If $I = (a,\, b)$ and $c \in I$, we define $\psi_{I,c}(x) = \exp\left(-\frac{c-x}{(x-a)(x-b)}\right)^2$ for $x \in I$ and 0 otherwise. Given a positive integer n, we represent its binary expansion by $1a_1a_2\ldots a_k$, i.e. $n = 2^k + a_1\,2^{k-1} + a_2 2^{k-2} + \ldots + a_k$. Let I_n be the open interval with endpoints

$$\alpha_n = \frac{1}{2} - \sum_{i=1}^{k}(-1)^{a_i}\frac{1}{3^i} - \frac{1}{2}\frac{1}{3^{k+1}} \quad \text{and} \quad \beta_n = \alpha_n + \frac{1}{3^{k+1}},$$

for $n = 2,\, 3, \ldots$, and the middle point $c_n = \frac{\alpha_n+\beta_n}{2}$. We define the functions f and ϕ as follows:

$$f(x) = \frac{1}{3}\psi_{[\frac{1}{3},\frac{2}{3}],\frac{1}{2}}(x) + \sum_{n=2}^{\infty}\frac{1}{3^n}\psi_{I_n,c_n}(x)\,,$$

$$\phi(x) = \begin{cases} f(c_n)\,\pi & \text{if } x \in I_n\,, \\ 0 & \text{otherwise.} \end{cases}$$

The sputnik of $\tau = (f\cos(\phi), f\sin(\phi), f)$ has infinitely many spikes all connected at the origin. Different sputniks can be easily constructed by modifying the effect of the rotation $(\cos(\phi), \sin(\phi))$ via a convenient characteristic function.

Remark 3.1. The robustness of the sputnik structure associated to a triplet can be determined by the metric

$$d_*(\tau_0, \tau_1) = \sup\{\min\{\sigma_{\tau_0-\tau_1}|_K,\, 1\} : \, K \subseteq R \ \text{ and compact}\},$$

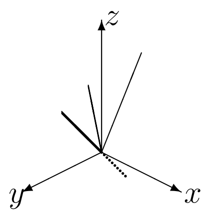

Figure 2: A Degree Three Sputnik.

where

$$\sigma_{\tau-\tau_1}|_K = \max\{|f_0(x) - f_1(x)| + |g_0(x) - g_1(x)| + |h_0(x) - h_1(x)| : \ x \in K\}.$$

We observe that given $\epsilon > 0$ and a triplet with nontrivial degree, τ, there exists τ_1 $\epsilon-$close to τ $(d_*(\tau, \tau_1) < \epsilon)$ so that $\mathrm{sk}(\tau)$ and $\mathrm{sk}(\tau_1)$ are not homeomorphic.

4. THE UNITARY SYSTEM: NON-COMMUTATIVE CASE

The general case is very hard to analyze and solutions require product integral operators, see Dollard and Friedman [12]. Here we consider a simpler illustrative example. The system $\dot{\mathbf{u}}(t) = \mathcal{I}_\gamma(t)\mathbf{u}(t)$ has a solution of the form

$$\exp(\mathbf{a}_1 t)\ \exp(\mathbf{a}_2 t)\ \ldots\ \exp(\mathbf{a}_k\, t)\mathbf{u}_0$$

($\mathbf{a}_i$ a pure quaternion and $\mathbf{a}_i\mathbf{a}_{i+1} \neq \mathbf{a}_{i+1}\mathbf{a}_i$, $i = 1, \ldots, k-1$) if and only if

$$\mathcal{I}_\gamma(t) = \sum_{i=1}^{k} \exp(\mathbf{a}_1 t) \exp(\mathbf{a}_2 t) \ldots \exp(\mathbf{a}_i t)\, \mathbf{a}_i\, \exp(-\mathbf{a}_i t) \ldots \exp(-\mathbf{a}_2 t) \exp(-\mathbf{a}_1 t).$$

In particular, if $\mathcal{I}_\gamma(t)$ reduces to the sum $\mathbf{a}_1 + \exp(\mathbf{a}_1 t)\mathbf{a}_2 \exp(-\mathbf{a}_1 t)$, then

$$|\mathcal{I}_\gamma(t)|^2 = |\mathcal{I}_\gamma(0)|^2 = -2\mathcal{R}(\mathbf{a}_1\mathbf{a}_2) + |\mathbf{a}_1|^2 + |\mathbf{a}_2|^2.$$

This implies that, for every t, the product $\mathcal{I}_\gamma(t)\mathcal{I}'_\gamma(t)$ is a pure quaternion, i.e. $\langle \mathcal{I}_\gamma(t), \mathcal{I}'_\gamma(t)\rangle = 0$. In the next proposition we collect some facts about the family $\{\mathcal{I}_\gamma(t)\}_{t\in R}$. We denote by $[\mathbf{a}, \mathbf{b}]$ the commutator of $\mathbf{a}$ and $\mathbf{b}$, i.e. $[\mathbf{a}, \mathbf{b}] = \mathbf{ab} - \mathbf{ba}$, and by $\mathcal{I}_\gamma^{(n)}(t)$ the n-th derivative of $\mathcal{I}_\gamma(t)$.

Proposition 4.1. *If* $\mathbf{a}$ *represents a pure quaternion and* $\{\mathcal{I}_\gamma(t)\}_{t\in R}$ *a family of pure quaternions, so that* $\exp(-\mathbf{a}t)\mathcal{I}_\gamma(t)\exp(\mathbf{a}t)$ *is constant and equal to* $\mathbf{b}$*, then:*

1. $|\mathcal{I}_\gamma(t)|^2 = |\mathbf{b}|^2$ *and* $\langle \mathcal{I}_\gamma(t),\, \mathcal{I}'_\gamma(t)\rangle = 0.$

2. $\mathcal{I}_\gamma^{(n+1)} = [\mathbf{a},\, \mathcal{I}_\gamma^{(n)}]$, $|\mathcal{I}_\gamma^{(n+1)}|$ *is constant,* $\langle \mathcal{I}_\gamma^{(n)},\, \mathcal{I}_\gamma^{(n+2k+1)}\rangle = 0,$ *and*

$$\langle \mathcal{I}_\gamma^{(n)},\, \mathcal{I}_\gamma^{(n+2k)}\rangle = (-1)^k\, |\mathcal{I}^{(n+k)}(t)|^2, \ n \text{ and } k = 0, 1, 2, \dots .$$

3. $\mathcal{I}_\gamma^{(n)} \mathcal{I}_\gamma^{(n+2k+1)}$ *is a pure quaternion, for* n *and* $k = 0, 1, 2, \dots .$

Proof. Since $\exp(-\mathbf{a}t)\mathcal{I}_\gamma(t)\exp(\mathbf{a}t) = \mathbf{b}$, then $\mathcal{I}_\gamma(t) = \exp(\mathbf{a}t)\,\mathbf{b}\,\exp(-\mathbf{a}t)$ and

$$|\mathcal{I}_\gamma(t)|^2 = \mathcal{R}\left(\exp(\mathbf{a}t)\,\mathbf{b}\,\mathbf{b}^*\,\exp(-\mathbf{a}t)\right) = |\mathbf{b}|^2.$$

This also implies that the derivative of $|\mathcal{I}_\gamma(t)|^2$ is equal to zero. Hence $\langle \mathcal{I}_\gamma(t),\, \mathcal{I}'_\gamma(t)\rangle = 0$, which proves statement (1).

Since

$$\mathcal{I}_\gamma(t) = \exp(\mathbf{a}t)\,\mathbf{b}\,\exp(-\mathbf{a}t),$$

then

$$\mathcal{I}'_\gamma(t) = \exp(\mathbf{a}t)[\mathbf{a},\, \mathbf{b}]\exp(-\mathbf{a}t)$$

and therefore

$$\mathcal{I}'_\gamma(t) = [\mathbf{a},\, \exp(\mathbf{a}t)\,\mathbf{b}\,\exp(-\mathbf{a}t)] = [\mathbf{a},\, \mathcal{I}_\gamma(t)].$$

The first equation in (2) follows by induction. The remaining statements are proved similarly. □

The next proposition describes a class of systems with solutions decomposable as a product of 2 unitary one-parameter families.

Lemma 4.1. *If* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *are two pure quaternions then,*

$$\begin{aligned}&\exp(\mathbf{a}_1 t)\mathbf{a}_2\exp(-\mathbf{a}_1 t)\\ &\qquad = \cos(2|\mathbf{a}_1|t)\mathbf{a}_2 + 2\langle \mathbf{a}_1,\, \mathbf{a}_2\rangle \sin^2(|\mathbf{a}_1|t)\frac{\mathbf{a}_1}{|\mathbf{a}_1|^2} + \frac{1}{2|\mathbf{a}_1|}\sin(2|\mathbf{a}_1|t)[\mathbf{a}_1,\, \mathbf{a}_1].\end{aligned}$$

Proof. We have that

$$\begin{aligned}&\exp(\mathbf{a}_1 t)\mathbf{a}_2\exp(-\mathbf{a}_1 t)\\ &= \left(\cos(|\mathbf{a}_1|t) + \frac{1}{|\mathbf{a}_1|}\sin(|\mathbf{a}_1|t)\mathbf{a}_1\right)\mathbf{a}_2\left(\cos(|\mathbf{a}_1|t) - \frac{1}{|\mathbf{a}_1|}\sin(|\mathbf{a}_1|t)\mathbf{a}_1\right)\\ &= \cos^2(|\mathbf{a}_1|t)\mathbf{a}_2 + \frac{1}{2|\mathbf{a}_1|}\sin(2|\mathbf{a}_1|t)[\mathbf{a}_1,\, \mathbf{a}_2] - \frac{1}{|\mathbf{a}_1|^2}\sin^2(|\mathbf{a}_1|t)\mathbf{a}_1\mathbf{a}_2\mathbf{a}_1,\end{aligned}$$

since $\mathbf{a}_1$ is a pure quaternion. The statement follows from the equality $\mathbf{a}_1\mathbf{a}_2\mathbf{a}_1 = |\mathbf{a}_1|^2\mathbf{a}_2 - 2\langle \mathbf{a}_1,\, \mathbf{a}_2\rangle\,\mathbf{a}_1$. □

Remark 4.1. If $\mathbf{a}$ and $\mathbf{b}$ are orthogonal pure quaternions, then $\{\mathbf{a}, \mathbf{b}, [\mathbf{a}, \mathbf{b}]\}$ is orthogonal and spans the set of all pure quaternions. We observe that $|[\mathbf{a}, \mathbf{b}]|^2 = 4\,|\mathbf{a}|^2\,|\mathbf{b}|^2 - 4\langle \mathbf{a}, \mathbf{b}\rangle$. We also have that

$$\mathcal{I}_\gamma(t) = \alpha_0(t)\,\mathbf{a} + \alpha_1(t)\,\mathbf{b} + \alpha(t)\,[\mathbf{a}, \mathbf{b}],$$

where

$$\alpha_0(t) = \frac{1}{|\mathbf{a}|^2}\langle \mathcal{I}_\gamma(t), \mathbf{a}\rangle, \quad \alpha_1(t) = \frac{1}{|\mathbf{b}|^2}\langle \mathcal{I}_\gamma(t), \mathbf{b}\rangle$$

and

$$\alpha_2(t) = \frac{1}{|[\mathbf{a}, \mathbf{b}]|^2}\langle \mathcal{I}_\gamma(t), [\mathbf{a}, \mathbf{b}]\rangle.$$

If the unitary system has solutions of the form $\mathbf{u}(t) = \exp(\mathbf{a}_1 t)\exp(\mathbf{a}_2 t)\mathbf{u}_0$ then $\mathcal{I}_\gamma(t)$ has the decomposition $\mathbf{a}_1 + \exp(\mathbf{a}_1 t)\mathbf{a}_2\exp(\mathbf{a}_1 t)$. Furthermore, $\mathbf{a}_1$ and $\mathbf{a}_2$ can be determined from $\mathcal{I}_\gamma(0)$ and $\mathcal{I}'_\gamma(0)$, as shown in the next proposition.

Proposition 4.2. *If there exist* $\mathbf{a}_1$ *and* $\mathbf{a}_2$ *pure quaternions so that*

$$\mathbf{u}(t) = \exp(\mathbf{a}_1 t)\exp(\mathbf{a}_2 t)\mathbf{u}_0$$

is the solution to $\dot{\mathbf{u}}(t) = \mathcal{I}_\gamma(t)\mathbf{u}(t)$ *(*$\mathbf{u}(0) = \mathbf{u}_0$*), then*

$$\mathbf{a}_1 = \frac{1}{4|\mathcal{I}_\gamma(0)|^2}[\mathcal{I}_\gamma(0), \mathcal{I}'_\gamma(0)] + \lambda\mathcal{I}_\gamma(0)$$

and $\mathbf{a}_2 = \mathcal{I}_\gamma(0) - \mathbf{a}_1$, *with* λ *some real number.*

Proof. We have $\mathcal{I}_\gamma(t) = \mathbf{a}_1 + \exp(\mathbf{a}_1 t)\mathbf{a}_2\exp(-\mathbf{a}_1 t)$. In particular, at $t = 0$, we have

$$\begin{cases} \mathbf{a}_1 + \mathbf{a}_2 = \mathcal{I}_\gamma(0)\,, \\ [\mathbf{a}_1, \mathbf{a}_2] = \mathcal{I}'_\gamma(0). \end{cases} \tag{4}$$

Therefore,

$$\mathcal{I}'_\gamma(0) = [\mathbf{a}_1, \mathcal{I}_\gamma(0)] \equiv 2\mathbf{a}_1 \times \mathcal{I}_\gamma(0),$$

where $\times$ denotes the standard cross product in R^3. This implies that $\mathcal{I}_\gamma(0)$ and $\mathcal{I}'_\gamma(0)$ are orthogonal and

$$\mathbf{a}_1 = \frac{\mathcal{I}_\gamma(0) \times \mathcal{I}'_\gamma(0)}{2|\mathcal{I}_\gamma(0)|^2} + \lambda\,\mathcal{I}_\gamma(0),$$

for some real number λ. Therefore, $\mathbf{a}_2 = \mathcal{I}_\gamma(0) - \mathbf{a}_1$. □

Remark 4.2. Proposition 4.2 states a necessary condition for the solutions to the unitary system to be decomposable into the product of two one-parameter groups. In this case, if the ratio between $|\mathbf{a}_1|$ and $|\mathbf{a}_2|$ is rational, the trajectories exhibit oscillatory behavior, or more precisely, $u(t) = \exp(\mathbf{a}_1 t)\exp(\mathbf{a}_2 t)$ is periodic.

The following theorem is an immediate consequence of the preceding discussion.

Theorem 4.1. *Solutions of the system* $\dot{\mathbf{u}}(t) = \mathcal{I}_\gamma(t)\mathbf{u}(t)$ *are written as the product of two unitary operators if and only if* $\mathcal{I}_\gamma(t)$ *is real analytic and there exists a pure quaternion* $\mathbf{a}_1$ *such that* $\mathcal{I}_\gamma^{(n)}(0) = [\mathbf{a}_1, \mathcal{I}_\gamma^{(n-1)}(0)]$*, for* $n = 1, 2, \dots$.

5. HIGHER DIMENSIONAL ANALOG

The canonical form associated with Andronov-Hopf bifurcation

$$\dot{z} = (\alpha + i)z - z|z|^2$$

can be extended to higher dimensions the following way:

$$\dot{M(t)} = (\alpha_0 I + A)M(t) - M(t)M^*(t)M(t),$$

with $M(t)$ is an $n \times n$ matrix function with complex valued entries. If we assume an initial condition of the form $M(0) = U_0 M_0$, where U_0 and M_0 are the factors of its polar decomposition. There are circumstances when a solution has a relatively simple representation and conditions for periodic solutions can be expressed in terms of the spectrum of A.

The splitting techniques used in Section 2 can be carried out in this setting. In fact, if $M(t)$ is written as follows:

$$M(t) = U(t)[M(t)]\,,$$

where $U(t)$ is the unitary part and $[M(t)]$ denotes the unique non-negative square root of $M(t)M^*(t)$. The differential equation is now written:

$$U^*(t)\{\dot{U(t)} - AU(t)\}\,[M(t)] + \left([\dot{M(t)}] - \alpha\,[M(t)] + [M(t)]^3\right) = 0.$$

If we set $U(t) = e^{At}U_0$, then the system reduces to

$$[\dot{M(t)}] - \alpha_0\,[M(t)] + [M(t)]^3 = 0.$$

The matrix M_0 is non-negative and Hermitian, therefore there is an orthonormal basis of $\mathbf{C}^n$, $\{\phi_k\}$, such that

$$M_0 = \sum_{k=1}^{m} \lambda_k \phi_k \otimes \phi_k.$$

Now we search for a solution of the form

$$[M(t)] = \sum_{k=1}^{m} \lambda_k(t) \phi_k \otimes \phi_k.$$

The orthogonality of the $\{\phi_j\}$ reduces the problem to the scalar equations:

$$\dot{\lambda_k}(t) = \alpha_0\lambda_k(t) - \lambda_k^3(t), \quad \lambda_k(0) = \lambda_k, \quad k = 1, \ldots, m.$$

Consequently,

$$M(t) = e^{At}U_0 \left(\sum_{k=1}^{m} \frac{\lambda_k(0)\sqrt{|\alpha_0|}e^{\alpha_0 t}}{\sqrt{|\alpha_0 - \lambda_k^2(0) + \lambda_k(0)^2 e^{2\alpha_0 t}|}} \phi_k \otimes \phi_k \right).$$

We notice that if $\lambda_k^2(0) = \alpha_0$, then $M(t) = Ce^{At}U_0$, with C is a constant. The existence of a periodic solution of period T implies that

$$e^{A(t+T)} = e^{At},$$

for all t. This is equivalent to

$$e^{AT} = I,$$

which clearly constrains the spectrum of A. If A is assumed to be skew Hermitian, e^{At} is a commuting family of unitary matrices and has a representation of the form

$$e^{At} = \sum_{j=1}^{n} \mu_j(t)\psi_j \otimes \psi_j \,,$$

where $\{\psi_j\}$ is a basis for $\mathbf{C}^n$, and $\mu_j(t)$ is an eigenvalue of e^{At}. For each j, $\mu_j(t)$ is a complex number with modulus one, hence $\mu_j(t) = e^{\xi_j t i}$, where ξ_j is an eigenvalue of A. Therefore $e^{AT} = I$ if and only if $\xi_j T = \pm(2k_j)\pi$.

6. CONCLUSIONS

The main focus of this paper is the study of transient oscillatory behaviors that occur in a nonlinear system of differential equations over the quaternions. Some of the techniques used extend to more general settings, such as Clifford algebras. We used a polar decomposition of matrices to decompose our system into a scalar and a unitary components. This is consistent with the approach followed in the complex case. The scalar equation plays a crucial role in defining invariant regions and determining their stability. The unitary system leaves invariant the unit sphere and, under some commutativity assumption, the orbit structure can be classified via a 3-D structure, designated sputnik. Sputniks detect oscillatory behavior of solutions and also the spatial direction where oscillations occur. We present a strategy to generate examples where the solutions to the unitary system are decomposed as a product of two noncommuting unitary groups of operators. Oscillatory behavior now depends on the type of synchronization of two periods. Solutions decomposable in a product of more than two unitary groups, consecutively noncommutative, and their associated systems were not considered but similar technique seems to apply. The existence of solutions that cannot be written as a product of unitary systems was left for future investigation.

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