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EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS OF A WEAKLY COUPLED SYSTEM OF REACTION-DIFFUSION EQUATIONS

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ABSTRACT: We consider the weakly coupled system of reaction-diffusion equation $(u_i)_t = \Delta u_i + |x|^{\sigma_i} u_{i+1}^{p_i}$. We have the results of condition of the nonlinear terms and the initial data for the solutions blowing up in finite time or existing in time global.

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1. INTRODUCTION

We consider nonnegative solutions of the initial value problem for a weakly coupled system

$$\begin{cases} (u_i)_t = \Delta u_i + |x|^{\sigma_i} u_{i+1}^{p_i}, & x \in \mathbf{R}^d, t > 0, i \in N^*, \\ u_i(x,0) = u_{i,0}(x), & x \in \mathbf{R}^d, i \in N^*, \end{cases} \tag{1}$$

where $N \geq 1$, $N^* = \{1, 2, \ldots N\}$, $d \geq 1$, $p_i \geq 1$ $(i \in N^*)$, $\prod_{i=1}^N p_i > 1$ and $0 \leq \sigma_i < d(p_i - 1)$ (if $p_i = 1$, we choose $\sigma_i = 0$) $(i \in N^*)$, and $u_{i,0}$ is a nonnegative bounded continuous function satisfying

$$\limsup_{|x| \to \infty} |x|^{\delta_i} u_{i,0}(x) < \infty$$

for any $i \in N^*$, where

$$\delta_i = \frac{\sigma_i + p_i \sigma_{i+1} + \ldots + p_i p_{i+1} \ldots p_{i+N-2} \sigma_{i+N-1}}{p_1 p_2 \ldots p_N - 1}. \tag{2}$$

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And the solution and others are cyclic and satisfy $u_{N+i}=u_i$, $u_{N+i,0}=u_{i,0}$, $p_{N+i}=p_i$, $\sigma_{N+i}=\sigma_i$ $(i\in N^*)$. For simply expressing, we put $u=(u_1,u_2,...,u_N)$ and $u_0=(u_{1,0},u_{2,0},...,u_{N,0})$.

Problem (1) has a unique, nonnegative and bounded solution in a suitable weighted space (see Theorem 2.4) at least locally in time. For given an initial value u_0, let $T^*=T^*(u_0)$ be the maximal existence time of the solution. If $T^*=\infty$ the solution is global. On the other hand, if $T^*<\infty$ there exists $i\in N^*$ such that

$$\limsup_{t\to T^*}\|u_i(t)\|_\infty, = \infty. \tag{3}$$

When (3) holds we say that the solution blows up in a finite time.

The purpose of the paper is to study systematically the effect of inhomogeneous term $|x|^{\sigma_i}$ on the critical blow-up exponent to the system (1) and the asymptotic behavior of global solutions for general $N\geq 1$.

In this paper, we present a unified approach to the study of blow-up and global existence of solution to the system (1) for the general $N\geq 1$ and $\sigma_i\geq 1$. Especially, we extend the previous results by Huang and Mochizuki [6] (for the case $N=2$ and $\sigma_i\geq 0$) and Umeda [16] (for the case $N\geq 3$ and $\sigma_i=0$).

Throughout this paper we shall use the following notation. We define some constants:

$$\begin{cases} \alpha_i = \dfrac{2(1+p_i+p_ip_{i+1}+\ldots+p_ip_{i+1}\ldots p_{i+N-2})}{p_1p_2\ldots p_N-1}, & i\in N^*, \\ \\ \delta_i = \dfrac{\sigma_i+p_i\sigma_{i+1}+\ldots+p_ip_{i+1}\ldots p_{i+N-2}\sigma_{i+N-1}}{p_1p_2\ldots p_N-1}, & i\in N^*, \end{cases} \tag{4}$$

which solve

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -p_{N-1} \\ -p_N & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N-1} \\ \alpha_N \end{pmatrix} = - \begin{pmatrix} 2 \\ 2 \\ \vdots \\ 2 \\ 2 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & -p_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -p_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & -p_{N-1} \\ -p_N & 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_{N-1} \\ \delta_N \end{pmatrix} = - \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{N-1} \\ \sigma_N \end{pmatrix},$$

where δ_i $(i \in N^*)$ are the same constants given by (2). These constants play an important role in our problem. Actually, we show that the number $\max_{i\in N^*}\{\alpha_i+\delta_i\}$ is the "first cutoff" which divides the blow-up case and the global existence case. This is a natural existence of the previous result in Huang and Mochizuki [6] for the case $N=2$.

We denote by BC the space of all bounded continuous functions in $\mathbf{R}^d$ and define for $a \geq 0$,

$$I^a = \{\xi \in BC; \xi(x) \geq 0 \text{ and } \limsup_{|x|\to\infty} |x|^a \xi(x) < \infty\},$$

$$I_a = \{\xi \in BC; \xi(x) \geq 0 \text{ and } \liminf_{|x|\to\infty} |x|^a \xi(x) > 0\}.$$

Let L_a^∞ be the Banach space of L^∞-functions such that

$$\|\xi\|_{\infty,a} = \sup_{x\in\mathbf{R}^d} < x >^a |\xi(x)| < \infty,$$

where $< x >= (|x|^2+1)^{1/2}$. Obviously $I^a \subset L_a^\infty$. The letter C stands for a positive generic constant which may vary from line to line. We use the notation $S(t)\xi$ to represent the solution of the heat equation with an initial value $\xi(x)$:

$$S(t)\xi(x) = (4\pi t)^{-d/2} \int_{\mathbf{R}^d} e^{-|x-y|^2/4t}\xi(y)dy. \tag{5}$$

By using the notation above, throughout paper, we suppose that initial conditions satisfy

$$u_{i,0} \in I^{\delta_i} \qquad (i \in N^*), \tag{6}$$

where δ_i is a nonnegative constant defined by (4).

Now, the results of this paper can be summarized in the following four theorems. First, we state our blow-up result for solutions to (1).

Theorem 1. *Assume that* $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$*, and* $\max_{i\in N^*}\{\alpha_i+\delta_i\} \geq d$*. Then every nontrivial solution* u *of* (1) *blows up in a finite time.*

When $\max_{i\in N^*}\{\alpha_i+\delta_i\} < d$, we show that there exists both non-global solutions and non-trivial global solution of (1). Precisely, requiring a polynomial decay of initial values u_0:

$$u_{i,0}(x) \sim C < x >^{-a_i} \qquad (i \in N^*), \tag{7}$$

where C and a_i are positive constants, we obtain the "second cutoff" $a = (a_1, a_2, ..., a_N)$ on the decay rate of initial values, namely $a_i = \alpha_i + \delta_i$ which divides the blow-up case and the global existence case when $\max_{i\in N^*}\{\alpha_i+\delta_i\} < d$.

Theorem 2. *Assume that* $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$, *and* $\max_{i\in N^*}\{\alpha_i + \delta_i\} < d$.

(i) *Suppose that there exists some* $i \in N^*$ *such that*

$$u_{i,0} \in I_{a_i} \quad \textit{with} \quad a_i < \alpha_i + \delta_i. \tag{8}$$

Then every solution u *of* (1) *blows up in a finite time.*

(ii) *Suppose that for any* $i \in N^*$

$$u_{i,0} \in I^{a_i} \quad \textit{with} \quad a_i > \alpha_i + \delta_i \tag{9}$$

and $\|u_{i,0}\|_{\infty,a_i}$ *is small enough. Then, every solution* u *of* (1) *is global. Moreover, we have a decay estimate:*

$$u_i(x,t) \le CS(t) < x >^{-\hat{a}_i} \tag{10}$$

in $\boldsymbol{R}^d \times (0,\infty)$, *where* C *is a positive constant and* $\hat{a}_i \le a_i$ $(i \in N^*)$ *are chosen to satisfy*

$$p_i \min\{\hat{a}_{i+1}, d\} - \hat{a}_i > 2 + \sigma_i. \tag{11}$$

We also obtain the blow-up result for large initial data, even if initial data has an exponential decay.

Theorem 3. *Assume that* $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$, *and* $\max_{i\in N^*}\{\alpha_i + \delta_i\} < d$. *Suppose that there exists some* $i \in N^*$ *such that* $u_{i,0}(x) \ge Ce^{-\nu_0|x|^2}$ *for some* $\nu_0 > 0$ *and* $C > 0$ *large enough. Then every solution* u *of* (1) *blows up in a finite time.*

Remark 1.1. In particular, when $u_{i,0} \in I^{a_i}$ with $a_i > \alpha_i + \delta_i$ for any $i \in N^*$ and $\|u_{i,0}\|_{\infty,a_i}$ is large enough, every solution $u(t)$ of (1) blows up in a finite time. On the other hand, when $u_{i,0} \le Ce^{-\nu_0|x|^2}$ for any $i \in N^*$, some $\nu_0 > 0$ and C small enough, every solution of (1) is global.

Remark 1.2. We can show the results of the asymptotic behavior of the solution of (1) global in time as Huang and Mochizuki [6], Theorem 4 and Mochizuki [11], Thoerem 6.1.

We briefly recall a history of the study on blow-up and global existence of solution to the system (1). First, the blow-up and the global existence of solutions in the case $N = 1$ and $\sigma_1 = 0$,

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbf{R}^d,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d \end{cases} \tag{12}$$

was studied by Fujita [4]. Fujita proved that when $d(p-1) < 2$ the solution of (12) blows up in a finite time for any $u_0 \not\equiv 0$. On the other hand he also proved that when $d(p-1) > 2$ the solution of (12) exists globally in time if the initial value u_0 is small and has an exponential decay. The number $p = 1 + 2/d$ is called a critical blow-up exponent for (12). For the case $d(p-1) < 2$, Lee and Ni [10] studied and proved that if the initial data is large enough or decaying slowly (It contains the case that the initial data not decaying) for space, the solution blows up infinite time, and if the initial value is small enough and decaying fast, then the solution is global in time. They have the results about "second cutoff" for the case $N = 1$, too.

Fujita's results were extended by Bandle and Levine [1] for the $\sigma \geq 0$:

$$\begin{cases} u_t = \Delta u + |x|^\sigma u^p, & x \in \mathbf{R}^d, t > 0 \\ u(x,0) = u_0(x), & x \in \mathbf{R}^d, \end{cases} \tag{13}$$

and they showed that when $d(p-1) < 2+\sigma$ the solution of (13) blows up in a finite time for any $u_0 \not\equiv 0$. Hamada [5] proved the same blow-up result for the critical case $d(p-1) = 2+\sigma$ (see also Pinsky [12]).

Fujita's results were also extended by Escobedo and Herrero [2] and Mochizuki [11] to the system (1) with $N = 2$ and $\sigma_i = 0$ $(i = 1, 2)$, and by Huang and Mochizuki [6] to the system (1) with $N = 2$ and $\sigma_i \geq 0$.

Although the Fujita type critical blow-up exponent to the system (1) with $N = 2$ and $\sigma_i = 0$ was established by Escobedo and Herrero [2], their proofs were rather complicated.

Huang and Mochizuki [6] and Mochizuki [11] simplified their proof and also determined the "second cut off" on the decay rate of initial data. The asymptotic behavior of global solutions was also studied in Huang and Mochizuki [6] and Mochizuki [11] for the case $N = 2$ and $\sigma_i \geq 0$.

Our result is a natural extension of Huang and Mochizuki [6]. We emphasize that our proof gives a unified approach to show blow-up results, although the proof in Huang and Mochizuki [6] for the case $\sigma_i > 0$ is slightly different from the one for case σ_i.

For a big system (1) with $N \geq 3$ and $\sigma_i = 0$, Umeda [16] and Rencławowicz [15] (see also Rencławowicz [14]) determined independently the Fujita type critical blow-up exponent. See also Fila and Quittner [3] for large initial data. The methods in Umeda [16] and Rencławowicz [15] are different. Moreover, in Umeda [16] we also determined the "second cutoff" on the decay rate of initial data.

On results extend the results of Umeda [16], the novelty of this paper is the choice of an appropriate weighted function space in which the system (1) is locally well-possed, a unified approach to establish blow-up results and a systematic controls of solutions.

Finally, we remark on the problem to estimate the life span $T^*(u_0)$ as λ go to 0 or ∞, when the initial data has the form (7). Such problem was studied by Mochizuki [11], Pinsky [13] and Kobayashi [8], [9]. However, it is an open problem to obtain sharp estimate of the life span $T^*(u_0)$ for general $N \geq 3$ even in the case $\sigma_i = 0$.

The rest of the paper is organized as follows. In Section 2, we note preliminary results including the local existence for (1) and some useful lemmas. In Section 3, we prove the blow-up results (Theorems 1, 2 (i) and 3). In Section 4, we show the result of global existence (Theorem 2 (ii)).

2. PRELIMINARIES

First, we note useful lemmas. The lemmas are well-known and are used throughout this paper. But the proof of Lemma 2.6 is complicated for the case $N \geq 3$. We need to control the precise estimate by induction.

We set for $\gamma > 0$

$$\eta_\gamma(x,t) = S(t) < x >^{-\gamma} . \tag{14}$$

Lemma 2.1. *Let $\gamma > 0$, $0 \leq \delta \leq \min\{d, \gamma\}$. Then we have*

$$\|\eta_\gamma(x,t)\|_{\infty,\delta} \leq \begin{cases} C(1+t)^{(-\min\{d,\gamma\}+\delta)/2} & (\gamma \neq d), \\ C(1+t)^{(-d+\delta)/2}\log(2+t) & (\gamma = d). \end{cases}$$

Proof. See Huang and Mochizuki [6], Lemma 2.1 or Lee and Ni [10], Lemma 2.12. □

Lemma 2.2. (i) *The following inequality holds*

$$\eta_\gamma(x,t) \geq C \min\{< x >^{-\gamma}, (1+t)^{-\gamma/2}\}.$$

(ii) *We have in $\boldsymbol{R}^d \times (0,\infty)$*

$$|x|^\sigma \eta_a(x,t)^p \leq \begin{cases} C(1+t)^{(\sigma+b-\min\{a,d\}p)/2}\eta_b(x,t) & (a \neq d), \\ C(1+t)^{(\sigma+b-dp)/2}[\log(2+t)]^p \eta_b(x,t) & (a = d). \end{cases} \tag{15}$$

Proof. See Huang and Mochizuki [6], Lemma 4.1 and Lemma 4.2. □

Now, we establish the local solvability of the Cauchy problem (1). Basically, we follow the same argument as in Huang and Mochizuki [6].

For arbitrary $T > 0$, let

$$E_T = \{u : [0,T] \to (L^\infty)^N; \|u\|_{E_T} < \infty\}, \tag{16}$$

where

$$\|u\|_{E_T} = \sup_{t\in[0,T]} \{\sum_{i=1}^{N} \|u_i(t)\|_{\infty,\delta_i}\}.$$

We consider in E_T the related integral system

$$u_i(t) = S(t)u_{i,0} + \int_0^t S(t-s)\big(|x|^{\sigma_i} u_{i+1}^{p_i}(s)\big)ds, \tag{17}$$

where $i \in N^*$. Note that in the closed subset $P_T = \{u \in E_T; u_i \geq 0, i \in N^*\}$ of E_T, (1) is reduced to (17). Define

$$\Psi(u) = (S(t)u_{1,0} + \Phi_1(u_2), S(t)u_{2,0} + \Phi_2(u_3), ..., S(t)u_{N,0} + \Phi_N(u_1)), \tag{18}$$

where

$$\Phi_i(u_{i+1}) = \int_0^t S(t-s)\big(|x|^{\sigma_i} u_{i+1}^{p_i}(s)\big)ds \quad (i \in N^*).$$

Then a fixed point u of Ψ corresponds to a solution of (1).

Lemma 2.3 (i) *Let $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$. Then $(S(\cdot)u_{1,0}, S(\cdot)u_{2,0}, \dots, S(\cdot)u_{N,0}) \in E_T$ for any $T > 0$ and we have*

$$\|S(\cdot)u_{1,0}, S(\cdot)u_{2,0}, ..., S(\cdot)u_{N,0}\|_{E_T} \leq C \sum_{i=1}^{N} \|u_{i,0}\|_{\infty,\delta_i}.$$

(ii) *Let $u \in E_T$. Then $(\Phi_1(u_2), \Phi_2(u_3), ..., \Phi_N(u_1)) \in E_T$ and we have*

$$\|\Phi_1(u_2), \Phi_2(u_3), ..., \Phi_N(u_1))\|_{E_T} \leq CT \sum_{i=1}^{N} \|U_i\|_{E_T}^{p_i},$$

where

$$\begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{N-1} \\ U_N \end{pmatrix} = \begin{pmatrix} 0 & u_2 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & u_3 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & u_N \\ u_1 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Proof. (i) is obvious from Lemma 2.1 with $\gamma = \delta_i$ $(i \in N^*)$.

(ii) Note that

$$\int_0^t S(t-s)|\cdot|^{\sigma_i} u_{i+1}^{p_i}(s)ds$$

$$\leq \int_0^t S(t-s) <\cdot>^{\sigma_i - \delta_{i+1}p_i} ds \sup_{s\in[0,t]} \|u_{i+1}(s)\|_{\infty,\delta_{i+1}}^{p_i}.$$

By a simple calculation (see (4)) $-\sigma_i + \delta_{i+1}p_i = \delta_i < d$. Then it follows from Lemma 2.1 with $\gamma = \delta_i$ $(i \in N^*)$ that

$$\|S(t-s) < \cdot >^{\sigma_i - \delta_{i+1}p_i} \|_{\infty,\delta_i} \le C\,.$$

Thus we have

$$\left\| \int_0^t S(t-s)|\cdot|^{\sigma_i} u_{i+1}^{p_i}(s)ds \right\|_{\infty,\delta_i} \le Ct \sup_{s\in[0,t]} \|u_{i+1}(s)\|^{p_i}_{\infty,\delta_{i+1}}$$

for $i \in N^*$. These inequalities conclude the assertion (ii). □

Now we can prove the following theorem.

Theorem 2.4. *Assume that u_0 is a vector of nonnegative bounded continuous functions such that $u_{i,0} \in I^{\delta_i}$ $(i \in N^*)$. Then there exists $0 < T \le \infty$ and a unique vector $u(t) \in P_T$ which solves* (1) *in $\mathbf{R}^d \times [0,T)$.*

Proof. Let $B_R = \{u \in E_T; \|u\|_{E_T} \le R\}$. We consider two vector-valued functions

$$v_1(x,t) = (v_{1,1}(x,t), v_{1,2}(x,t), \ldots, v_{1,N}(x,t))$$

and

$$v_2(x,t) = (v_{2,1}(x,t), v_{2,2}(x,t), \ldots, v_{2,N}(x,t)).$$

For Ψ in (18), we have

$$\begin{aligned} &\|\Psi(v_1) - \Psi(v_2)\|_{E_T} \|(\Phi_1(v_{1,2}) - \Phi_1(v_{2,2}), \Phi_2(v_{1,3}) - \Phi_2(v_{2,3}) \\ &\quad , ..., \Phi_{N-1}(v_{1,N}) - \Phi_{N-1}(v_{2,N}), \Phi_N(v_{1,1}) - \Phi_N(v_{2,1}))\|_{E_T}. \end{aligned} \tag{19}$$

We consider i-th term of $\|\Psi(v_1) - \Psi(v_2)\|_{E_T}$,

$$\begin{aligned} &|\Phi_i(v_{1,i+1}) - \Phi_i(v_{2,i+1})| < x >^{\delta_i} \\ &\quad \le \int_0^t S(t-s)|x|^{\sigma_i} \Big| |v_{1,i+1}(s)|^{p_i} - |v_{2,i+1}(s)|^{p_i} \Big| ds < x >^{\delta_i}\,. \end{aligned}$$

We consider this expression in $B_R \cap P_T$ for R sufficient large. From proof of Lemma 2.3 (ii),

$$\begin{aligned} &|\Phi_i(u_{1,i+1}) - \Phi_i(u_{2,i+1})| < x >^{\delta_i} \\ &\quad \le CT \sup_{s\in[0,t]} \|v_{1,i+1}^{p_i}(s) - v_{2,i+1}^{p_i}(s)\|_{\infty,\delta_{i+1}} \\ &\quad \le CT \sup_{s\in[0,t]} \|R^{p_i-1} p_i\, (v_{1,i+1}(s) - v_{2,i+1}(s))\, \|_{\infty,\delta_{i+1}}\,. \end{aligned} \tag{20}$$

Substitute (20) into (19). Since we can put T is small enough for R, we obtain

$$\|\Psi(v_1) - \Psi(v_2)\|_{E_T} \le CTR^{\max_i\{p_i\}-1} \max_i\{p_i\}\|v_1 - v_2\|_{E_T} \le \rho\|v_1 - v_2\|_{E_T}$$

for some $\rho < 1$. Then Ψ is a strict contraction of $B_R \cap P_T$ into itself, whence there exists a unique fixed point $u \in B_R \cap P_T$ which solves (4). □

Next, we establish key estimate of solutions which will be used show blow-up results.

Lemma 2.5. *Let $u_0 \not\equiv 0$ and u be the solutions of (1) with initial data u_0. Then there exist $\tau = \tau(u_0) \ge 0$ and constants $C > 0, \nu > 0$ such that*

$$u_i(x, \tau) \ge Ce^{-\nu|x|^2} \quad (i \in N^*). \tag{21}$$

Proof. (cf. Escobedo and Herrero [2], Lemma 2.4) Assume for instance that $u_{1,0} \not\equiv 0$. By shifting the origin if necessary, we may assume that there exists $R > 0$ such that $\nu = \inf\{u_{1,0}(\xi) : |\xi| \le R\} > 0$. Since $u(x,t) \ge S(t)u_{1,0}(x)$, it follows that

$$u_1(x,t) \ge \nu \exp(-\frac{|x|^2}{2t})(4\pi t)^{-d/2} \int_{|y|\le R} \exp(-|y|^2/2t)dy.$$

Define $\bar{u}_1(t) = u_1(t + \tau_1)$ for some $\tau_1 > 0$. Then, we obtain

$$\bar{u}_1(x, 0) = u_1(x, \tau_1) > c_1 \exp(-\alpha_1|x|^2) \tag{22}$$

with

$$\alpha_1 = \frac{1}{2\tau_1}, \qquad c_1 = \nu(4\pi\tau_1)^{-d/2} \int_{|y|<R} \exp\left(-\frac{|y|^2}{2\tau_1}\right) dy. \tag{23}$$

Substituting (22) in N-th equation of (17), we obtain

$$\begin{aligned} u_N(x,t) &\ge \int_0^t S(t-s)|x|^{\sigma_N} u_1^{p_N}(s)ds \\ &\ge c_1^{p_N} \int_{\tau_1}^t S(t-s)|x|^{\sigma_N} \exp(p_N\alpha_1|x|^2)ds. \end{aligned}$$

Since for $\nu > 0$ and $\sigma \ge 0$,

$$S(t)\big(|x|^\sigma e^{-\nu|x|^2}\big) \ge C_\sigma(2t)^{\sigma/2}(2\nu t + 1)^{-(d+\sigma)/2}e^{-|x|^2/2t}, \tag{24}$$

where

$$C_\sigma = (2\pi)^{-d/2} \int_{\mathbf{R}^d} |x|^\sigma e^{-|x|^2} dx \tag{25}$$

(see Huang and Mochizuki [6], Lemma 3.2), we obtain

$$
\begin{aligned}
u_N(x,t) &\geq \int_{\tau_1}^{t} \frac{c_1 C_{\sigma_N}\{2(t-s)\}^{\sigma_N/2}}{\{2\alpha_1(t-s)+1\}^{-(\sigma_N+d)/2}} \exp\left(-\frac{|x|^2}{2(t-s)}\right) ds \\
&\geq \int_{\tau_1}^{(t+\tau_1)/2} \frac{c_1 C_{\sigma_N}(t-\tau_1)^{\sigma_N/2}}{\{2\alpha_1(t-\tau_1)+1\}^{-(\sigma_N+d)/2}} \exp\left(-\frac{|x|^2}{2(t-\tau_1)}\right) ds \\
&\geq \frac{c_1 C_{\sigma_N}(t-\tau_1)^{1+\sigma_N/2}}{2\{2\alpha_1(t-\tau_1)+1\}^{-(\sigma_N+d)/2}} \exp\left(-\frac{|x|^2}{2(t-\tau_1)}\right),
\end{aligned}
$$

where C_{σ_N} is defined in (25). Define $\bar{u}_N(t) = u_N(t+\tau_N)$ for some $\tau_N > \tau_1$. Then, we obtain

$$
\bar{u}_N(0) = u_N(\tau_N) > c_N \exp(-\alpha_N |x|^2) \tag{26}
$$

with

$$
\begin{cases}
\alpha_1 = \dfrac{1}{2(\tau_N - \tau_1)}, \\
c_N = c_1 C_{\sigma_N}(\tau_N-\tau_1)^{1+\sigma_N/2}\{2\alpha_1(\tau_N-\tau_1)+1\}^{-(\sigma_N+d)/2}.
\end{cases} \tag{27}
$$

By repeating this argument, we obtain same results for $u_N, u_{N-1}, \dots, u_2$. This completes the proof. □

We suppose $\alpha_1 + \delta_1 = d$. Let $u(t) \in E_T$ be a nontrivial solution of (1). By Lemma 2.4, we may assume

$$
u_{1,0} \geq C e^{-\mu|x|^2}
$$

for some $C > 0$ and $\mu > 0$.

Lemma 2.6. *We assume $\alpha_1 + \delta_1 = d$. Then we have*

$$
u_1(x,t) \geq C t^{-d/2} e^{-|x|^2/t} \log(t/(2a)) \quad (a \leq t < T),
$$

where $a > 0$ is a small constant.

Proof. Put the sequences $\{P_l\}_{l=1}^N$ and $\{Q_l\}_{l=1}^N$ satisfies $P_N = (2+\sigma_N)/2$, $P_l = p_l P_{l+1} + (2+\sigma_l)/2$ and $Q_l = dp_l p_{l+1} \dots p_N/2$. From (17), we have

$$
u_1(x,t) \geq S(t)u_{1,0}(x) \geq C(4\mu t + 1)^{-d/2} e^{-|x|^2/(4t+1/\mu)}.
$$

Thus, we have

$$
\begin{aligned}
u_N(x,t) &\geq \int_0^t S(t-s)|x|^{\sigma_N} u_1(x,s)^{p_N} ds \\
&\geq \int_0^t (4s+1/\mu)^{-dp_N/2} S(t-s)|x|^{\sigma_N} e^{-p_N|x|^2/(4s+1/\mu)} ds.
\end{aligned}
$$

Since

$$S(t)|x|^{\sigma_N} e^{-p_N|x|^2/(4s+1/\mu)} \geq Ct^{\sigma_N} \left\{ \frac{2p_N t}{4s+1/\mu} + 1 \right\}^{-(d+\sigma_N)/2} e^{-|x|^2/2t},$$

we obtain

$$u_N(x,t) \geq C \int_{t/4}^{t/2} (4s+1/\mu)^{-dp_N/2} (t-s)^{\sigma_N/2} e^{-|x|^2/2(t-s)} ds$$

$$\geq Ct^{P_N}(t+1)^{-Q_N} e^{-|x|^2/t}.$$

Substitute this into $u_{N-1}(x,t) \geq \int_0^t S(t-s)|x|^{\sigma_{N-1}} u_N(x,s)^{p_{N-1}}$. Then we have

$$u_{N-1}(x,t) \geq \int_0^t s^{p_{N-1}P_N+\sigma_{N-1}/2}(s+1)^{-p_{N-1}Q_N}$$

$$\left\{ \frac{2p_{N-1}(t-s)}{s} + 1 \right\}^{-(d+\sigma_{N-1})/2} e^{-|x|^2/(t-s)} ds$$

$$\geq Ce^{-|x|^2/t} \int_{t/4}^{t/2} (s+1)^{-Q_{N-1}} s^{p_{N-1}P_N+\sigma_{N-1}/2} ds$$

$$\geq C(t+1)^{-Q_{N-1}} t^{P_{N-1}} e^{-|x|^2/t}$$

by (24) again. By repeating this argument, we have

$$u_2 \geq C(t+1)^{-Q_2} t^{P_2} e^{-|x|^2/t}$$

by using (24) again. Thus we obtain

$$u_1(x,t) \geq C \int_0^t (s+1)^{-p_1 Q_2} s^{p_1 P_2+\sigma_2/2}$$

$$\times \left\{ \frac{2p_1(t-s)}{s} + 1 \right\}^{-(d+\sigma_1)/2} e^{-|x|^2/(t-s)} ds$$

$$\geq C(t+1)^{-d/2} e^{-|x|^2/t} \int_a^{t/2} s^{-Q_1+P_1-1} ds$$

for small $a > 0$. Since $\alpha_1 + \delta_1 = d$ and $Q_1 = P_1$, we have

$$u_1(x,t) \geq C(t+1)^{-d/2} e^{-|x|^2/t} \int_a^{t/2} s^{-1} ds \geq Ct^{-d/2} e^{-|x|^2/t} \log(t/2a)$$

for $a < t < T$. □

3. PROOF OF BLOW-UP RESULTS

In this section we summarize several blow-up conditions which follow from Theorem 3.2. Here, we take the same strategy as in Huang and Mochizuki [6] and Mochizuki [11]. Actually, we can deduce our blow-up problem to the one for the systems of ordinary differential equations with a parameter $\epsilon > 0$. We found a nice scaling to reduce the problem furthermore to the one for a simpler (ϵ-independent) system of ordinary differential equations. This gives us a uniform treatment of our blow up results.

Let $\rho_\epsilon(x) = (\epsilon/\pi)^{d/2} e^{-\epsilon|x|^2}$, $\epsilon > 0$. For a solution $u(t) \in E_T$ of (1) we put

$$F_{i,\epsilon}(t) = \int_{\mathbf{R}^d} u_i(x,t)\rho_\epsilon(x)dx \quad (i \in N^*). \tag{28}$$

Since $-\Delta\rho_\epsilon(x) \le 2d\epsilon\rho_\epsilon(x)$, the pair $\{2d\epsilon, \rho_\epsilon(x)\}$ is regarded as an approximate principal eigensolution of $-\Delta$ in $\mathbf{R}^d$. With this fact and Jensen's inequality we easily have

$$F'_{i,\epsilon}(t) \ge -2d\epsilon F_{i,\epsilon}(t) + C_{p_i}\epsilon^{-\sigma_i/2} F_{i+1,\epsilon}(t)^{p_i} \quad (i \in N^*), \tag{29}$$

where

$$C_{p_i} = \left(\pi^{-d/2}\int_{\mathbf{R}^d} |x|^{-\sigma_i/(p_i-1)} e^{-|x|^2} dx\right)^{-p_i+1}$$

for $p_i > 1$ and $C_{p_i} = 1$ for $p_i = 1$.

Let us consider the system of ordinary differential equations

$$\begin{cases} f'_{i,\epsilon}(t) = -2d\epsilon f_{i,\epsilon}(t) + C_{p_i}\epsilon^{-\sigma_i/2} f_{i+1,\epsilon}(t)^{p_i} \quad (i \in N^*), \\ f_{i,\epsilon}(0) = F_{i,\epsilon}(0), \quad (i \in N^*). \end{cases} \tag{30}$$

By the scaling with (4)

$$f_i(t) = \frac{(C_{p_i} C_{p_{i+1}}^{p_i} C_{p_{i+2}}^{p_i p_{i+1}} \dots C_{p_{i+N-1}}^{p_i p_{i+1} \dots p_{i+N-2}})^{1/(p_1 p_2 \dots p_N - 1)}}{2d^{\alpha_i/2}\epsilon^{(\alpha_i+\delta_i)/2}} f_{i\epsilon}\left(\frac{t}{2d\epsilon}\right)$$

for $i \in N^*$, we obtain the simpler system of equations

$$f'_i(t) = -f_i(t) + f_{i+1}(t)^{p_i} \quad (i \in N^*). \tag{31}$$

Lemma 3.1. *Let $f(t) = (f_1(t), f_2(t), ..., f_N(t))$ be the solution to* (31) *with the initial data*

$$f_1(0) = f_0 > 1, \quad f_j(0) = 0 \ \ (j \in N^* \backslash \{1\}).$$

If f_0 is sufficiently large, then $f(t)$ blows up in a finite time. Moreover, the life span T_0 of $f(t)$ is estimated from above by

$$T_0 \leq t_0 + \int_{\prod_{i=1}^N f_i(t_0)}^{\infty} \{C_1(p)\xi^{C_2(p)+1} - N\xi\}^{-1} d\xi, \tag{32}$$

where

$$C_1(p) = \prod_{i=1}^{N} \frac{1}{\beta_i^{\beta_i}} \qquad \left(\beta_i = \frac{\alpha_{i+1}}{\sum_{j=1}^N \alpha_j} \quad (i \in N^*)\right),$$

$$C_2(p) = \frac{2}{\sum_{i=1}^N \alpha_i},$$

and $0 < t_0 < T_0$ *is chosen to satisfy* $\{\prod_{i=1}^N f_i(t_0)\}^{C_2(p)} > N$.

Proof. We take the same strategy as in Mochizuki [11], Lemma 2.2. Multiplying e^t on the both sides of (31), we have

$$f_l(t) = e^{-t} \int_0^t e^s f_{l+1}^{p_l}(s) ds\,, \tag{33}$$

for $l \in N^*$, and iteration these equation, we have

$$\begin{aligned} f_1(t) =& e^t f_0 + e^{-t} \int_0^t e^{(1-p_1)s_1} \Bigg[\int_0^{s_N} e^{(1-p_2)s_2} \times \ldots \times \Bigg(\int_0^{s_3} e^{(1-p_{N-1})s_2} \\ & \times \Bigg\{ \int_0^{s_2} e^{s_1} f_1(s_1)^{p_N} ds_1 \Bigg\}^{p_{N-1}} ds_2 \Bigg)^{p_{N-2}} \ldots ds_{N-1} \Bigg]^{p_2} ds_N. \end{aligned} \tag{34}$$

Let $f_0 > 1$ be chosen large enough to satisfy

$$\inf_{t_0 > 0} \left\{ e^{t_0} f_0 + 2^{p_1 p_2 \ldots p_N} e^{-t_0} g(t_0, 0) \right\} \geq 2^{p_1 p_2 \ldots p_N} - \delta, \tag{35}$$

where $\delta > 0$ is a small constant satisfying $\delta < 2^{p_1 p_2 \cdots p_N} - 2$, and

$$\begin{aligned} g(t_a, t_b) = & \int_{t_b}^{t_a} e^{(1-p_1)s_1} \Bigg[\int_{t_b}^{s_1} e^{(1-p_2)s_2} \times \ldots \times \Bigg(\int_{t_b}^{s_{N-3}} e^{(1-p_{N-2})s_{N-2}} \\ & \Bigg\{ \int_{t_b}^{s_{N-2}} e^{s_{N-1}} (1 - e^{-s_{N-1}})^{p_{N-1}} ds_{N-1} \Bigg\}^{p_{N-2}} ds_{N-2} \Bigg)^{p_{N-3}} \ldots ds_2 \Bigg]^{p_2} ds_1. \end{aligned}$$

We shall first show that under this condition $f_1(t) > 2$ for any $0 < t < T_0$. Assume the contrary that there exist $0 < t_1 < T_0$ such that $f_1(t) > 2$ in $0 \leq t < t_1$ and $f_1(t_1) = 2$. Then it follows from (34) and (35) that

$$2 = f_1(t_1) \geq e^{t_1} f_0 + 2^{p_1 p_2 \ldots p_N} e^{-t_1} g(t_1, 0)$$

and a contradiction occurs. Next, we shall show that $\lim_{t \to T_0} f_1(t) = \infty$ $(T_0 \leq \infty)$. Assume to the contrary that there exist a sequence $\{t_j\}$ such that

$$\lim_{t_j\to\infty} f_1(t_j) = M \text{ for some } 2 \le M < \infty.$$

We choose $\epsilon > 0$ and $t_* > 0$ to satisfy $M < (M-\epsilon)^{p_1p_2\dots p_N}$ and $f_1(t) > M-\epsilon$ in $t_* < t < T$. It then follows from (34) that

$$f_1(t_j) \ge e^{t_j} f_0 + 2^{p_1p_2\dots p_N} e^{-t_j} g(t_*, 0) + (M-\epsilon)^{p_1p_2\dots p_N} e^{-t_j} g(t_j, t_*)$$

$$\to (M-\epsilon)^{p_1p_2\dots p_N} > M \quad (t_j \to \infty)$$

and we have contradiction and $\lim_{t\to T_0} f_1(t) = \infty$. Noting (33), we now conclude

$$\lim_{t\to T_0} f_2(t) = \lim_{t\to T_0} f_3(t) = \dots = \lim_{t\to T_0} f_N(t) = \infty \quad (T_0 \le \infty). \tag{36}$$

To complete the assertion we put $h(t) = f_1(t)f_2(t)\dots f_N(t)$. Then by (31) and Young's inequality,

$$h'(t) \ge -Nh(t) + C_1(p)h(t)^{C_2(p)+1}. \tag{37}$$

Integrating this, we obtain

$$t - t_0 \le \int_{h(t_0)}^{h(t)} \left\{C_1(p)\xi^{C_2(p)+1} - N\xi\right\}^{-1} d\xi.$$

Since $p_1p_2\dots p_N > 1$ and $C_2(p)+1 > 1$, this and (36) show that $h(t)$ blows up in a finite time and the life span T_0 is estimated by (32). □

Let us consider the solution $f_\epsilon(t) = (f_{1\epsilon}(t), f_{2\epsilon}(t), ..., f_{N\epsilon}(t))$ of (30). As is shown in Lemma 3.1, there exists $A_i > 0$ for some $i \in N^*$ such that if

$$F_{i,\epsilon}(0) > A_i(2d\epsilon)^{(\alpha_i+\delta_i)/2}, \tag{38}$$

then $F_\epsilon = (F_{1,\epsilon}(t), F_{2,\epsilon}(t), ..., F_{N,\epsilon}(t))$ blows up in a finite time. Moreover, its life span is estimated from above by $(2d\epsilon)^{-1}T_0$.

Theorem 3.2. *Let $F_\epsilon(t)$ satisfy differential inequalities* (29). *If* (38) *is satisfied for some $\epsilon > 0$, then $F_\epsilon(t)$ blows up in finite time. Moreover, its life span is estimated from above by $(2d\epsilon)^{-1}T_0$. Then, we obtain*

$$T^*(u_0) \le (2d\epsilon)^{-1}T_0. \tag{39}$$

Proof of Theorem 1. First, we consider the noncritical case as $\max_{i\in N^*}\{\alpha_i+\delta_i\} > d$. Without loss of generality, we can let $\alpha_2+\delta_2 > d$. By means of a comparison principle and Lemma 2.5, we can assume $u_{2,0} \in L^1(\mathbf{R}^d)$ and

$$\int_{\mathbf{R}^d} u_{2,0}(x)dx > 0.$$

The Lebesgue's Dominated Convergence Theorem then shows the existence of ϵ_0 such that

$$F_{2,\epsilon}(0) = \left(\frac{\epsilon}{\pi}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} u_{2,0}(x) e^{-\epsilon|x|^2} dx \geq \frac{1}{2}\left(\frac{\epsilon}{\pi}\right)^{\frac{d}{2}} \int_{\mathbf{R}^d} u_{2,0}(x) dx$$

for any $0 < \epsilon \leq \epsilon_0$. Since $\alpha_2 + \delta_2 > d$ by the assumption, this implies that the condition (38) of Theorem 3.2 is satisfied if ϵ is sufficiently small. Thus, $F_\epsilon(t)$ blows up in a finite time.

Next, we consider the critical case as $\max_{i\in N^*}\{\alpha_i + \delta_i\} = d$. For each nontrivial solution $u(t) \in E_T$ of (1), it follows from Lemma 2.6 that

$$S(t)u_1(0,t) \geq Ct^{-d/2}\log(t/2a) \int_{\mathbf{R}^d} e^{-5|x|^2/4t} dx \geq Ct^{-d/2}\log(t/2a) \tag{40}$$

in $a < t < T^*$. Contrary to the conclusion, assume that u is global. Then by Theorem 3.2 it holds that

$$F_{1,\epsilon}(t) = (\epsilon/\pi)^{d/2} \int_{\mathbf{R}^d} u_1(x,t) e^{-\epsilon|x|^2} dx \leq A_1 \epsilon^{(\alpha_1+\delta_1)/2}$$

for any $t \geq 0$ and $\epsilon > 0$. Thus, choosing $\epsilon = (4t)^{-1}$, we obtain

$$F_{1,1/4t}(t) = S(t)u_1(0,t) \leq A_1(4t)^{-(\alpha_1+\delta_1)/2} = A_1(4t)^{-d/2}.$$

This and (40) contradict to each other if $T^* = \infty$.

The proof of Theorem 1 is thus complete. □

Proof of Theorem 2. (i) If $u_{1,0} \in I_{a_1}$ with $a_1 < \alpha_1 + \delta_1 < d$, we have

$$F_{1,\epsilon}(0) = (\epsilon/\pi)^{d/2} \int_{\mathbf{R}^d} u_{1,0}(x) e^{-\epsilon|x|^2} dx = \pi^{-d/2} \int_{\mathbf{R}^d} u_{1,0}(\epsilon^{-1/2}x) e^{-|x|^2} dx.$$

Then it follows that

$$\epsilon^{-(\alpha_1+\delta_1)/2} F_{1,\epsilon}(0) \geq C\epsilon^{-(\alpha_1+\delta_i-a)/2} \pi^{-d/2} \int_{\mathbf{R}^d} |x|^{-a_1} e^{-|x|^2} dx > A_1$$

for sufficiently small $\epsilon > 0$. If $i \in N^*\backslash\{1\}$, we can obtain a similar estimate for $F_{i,\epsilon}$. Thus $F_\epsilon(t)$ blows up in a finite time by Theorem 3.2. □

Proof of Theorem 3. We then have for any $i \in N^*$,

$$F_{i,\epsilon} \geq C(\epsilon/\pi)^{d/2} \int_{\mathbf{R}^d} e^{-(\epsilon+\nu_0)|x|^2} dx = C\left(\frac{\epsilon}{\epsilon+\nu_0}\right)^{d/2}.$$

So, if we choose $\epsilon = 1$ and $C > (2\pi)^{d/2}\max_{i\in N^*}\{A_i\}(1+\nu_0)^{d/2}$, the condition of Theorem 3.2 is also satisfied in this case. □

4. PROOF OF GLOBAL EXISTENCE

In this section we require $\max_{i\in n^*}\{\alpha_i+\delta_i\}<d$, and treat the existence of global solutions of (1), and we show Theorem 2 (ii).

First note that condition (9) can be replaced by $u_{i,0}\in I^{\hat{a}_i}$ $(i\in N^*)$ since we have $I^{a_i}\subset I^{\hat{a}_i}$ $(i\in N^*)$. Then, to establish Theorem 2 (ii), we have only to consider the special case $\hat{a}_i=a_i$ $(i\in N^*)$. As is easily seen, in this case condition (11) is equivalent to

$$p_i a_{i+1,d}-a_i>2+\sigma_i \qquad (i\in N^*), \tag{41}$$

where $a_{j,d}=\min\{a_j,d\}$.

Using η defined in (14), we define the Banach spaces E_η and X as

$$E_\eta=\left\{u;\|u\|_{E_\eta}\equiv\sum_{i=1}^{N}(|||u_i/\eta_{a_i}|||_\infty)<\infty\right\},$$

and

$$X=\left\{v;|||v/\eta_{a_N}|||_\infty<\infty\right\},$$

where

$$|||w|||_\infty=\sup_{(x,t)\in\mathbf{R}^d\times(0,\infty)}|w(x,t)|.$$

(17) is reduced to

$$u_N(t)=V(t)(u_0,u_N), \tag{42}$$

where $V(t)$ is made by iteration and

$$V(t)(u_0,v)=S(t)u_{N,0}+\int_0^t S(t-s_1)|x|^{\sigma_N}\Bigg(S(s_1)u_{1,0}$$
$$+\int_0^{s_1}S(s_1-s_2)|x|^{\sigma_1}\Bigg\{\dots|x|^{\sigma_{N-2}}\Bigg[S(s_{N-1})u_{N-1,0}$$
$$+\int_0^{s_{N-1}}S(s_{N-1}-s_N)v^{p_{N-1}}(s_N)ds_N\Bigg]^{p_{N-2}}\dots\Bigg\}^{p_1}ds_2\Bigg)^{p_N}ds_1.$$

Here, if V is a strict contraction, its fixed point yields a solution of (1). Moreover, using that $(a+b)^p\le 2^{p-1}(a^p+b^p)$ for $a>0,b>0,p\ge 1$,

$$V(t)(u_0,v)\le T(t)(u_0)+\Gamma(t)(v),$$

where

$$T(t)(u_0) = S(t)u_{N,0} + 2^{p_N-1}\int_0^t S(t-s_1)|x|^{\sigma_N}(S(s_1)u_{1,0})^{p_N}ds$$

$$+ 2^{p_N p_N p_1 - 2}\int_0^t S(t-s_1)|x|^{\sigma_N}$$

$$\times \left(\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\{S(s_2)u_{2,0}\}^{p_1}\,dr\right)^{p_N} ds + \ldots$$

$$+ 2^{p_N + p_N p_1 + \ldots + p_N p_1 \ldots p_{N-2} - N + 1}\int_0^t S(t-s_1)|x|^{\sigma_N}$$

$$\left[\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\left\{\ldots |x|^{\sigma_{N-3}}\left(\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})|x|^{\sigma_{N-2}}\right.\right.\right.$$

$$\left.\left.\left.\times\,[S(s_{N-1})u_{N-1,0}]^{p_{N-2}}ds_{N-1}\right)^{\sigma_{N-3}}\ldots\right\}^{p_1} ds_2\right]^{p_N} ds_1,$$

and

$$\Gamma(t)(v) = 2^{p_N + p_N p_1 + \ldots + p_N p_1 \ldots p_{N-2} - N + 1}$$

$$\times \int_0^t S(t-s_1)|x|^{\sigma_N}\left(\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_1}\left\{\ldots |x|^{\sigma_{N-2}}\right.\right.$$

$$\left.\left.\left\{\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\right\}^{p_{N-2}}\ldots\right\}^{p_1} ds_2\right)^{p_N} ds_1.$$

Lemma 4.1. (i) *Let u_0 satisfy* (9). *Then $T(\cdot)(u_0) \in X$ and*

$$|||T(\cdot)(u_0)/\eta_{a_N}(\cdot)|||_\infty$$
$$\le C\left(\|u_{N,0}\|_{\infty,a_N} + \|u_{1,0}\|_{\infty,a_1}^{p_N} + \|u_{2,0}\|_{\infty,a_2}^{p_N p_1} + \ldots + \|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_N p_1 \ldots p_{N-2}}\right).$$

(ii) Γ *maps X into itself and*

$$|||\Gamma(v)/\eta_{a_N}|||_\infty \le C|||v/\eta_{a_N}|||_\infty^{p_1 p_2 p_3 \ldots p_N}.$$

Proof. (i) (cf. Huang and Mochizuki [6], Lemma 4.3) By (14) and (15) in Lemma 2.2, we obtain $T(t)(u_0) = I_1 + I_2 + \ldots + I_N$, where

$$I_1 \le \|u_{N0}\|_{\infty,a_N}\eta_{a_N}(t),$$

$$I_2 \le 2^{p_N-1}\int_0^t S(t-s)|x|^{\sigma_N}(\eta_{a_1}\|u_{1,0}\|_{\infty,a_1})^{p_N}ds \le C\|u_{1,0}\|_{\infty,a_1}^{p_N}\eta_{a_N}(t),$$

and by same argument, we have

$$I_3 \le C\|u_{2,0}\|_{\infty,a_1}^{p_1p_N}\eta_{a_N}(t),$$

$$\vdots$$

$$I_N \le C\|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_Np_1\dots p_{N-2}}\eta_{a_N}(t).$$

(ii) By (14) and (15)

$$\begin{aligned}\Gamma(v) \le& C\||v/\eta_{a_N}|\|_\infty^{p_1p_2\dots p_N}\int_0^t S(t-s_1)|x|^{\sigma_2}\left(\int_0^{s_1} S(s_1-s_2)|x|^{\sigma_3}\Big\{\dots|x|^{\sigma_N}\right.\\ &\left.\Big\{\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_1}\eta_{a_N}(s_N)^{p_1}ds_N\Big\}^{p_N}\dots\Big\}^{p_3}ds_2\right)^{p_2}ds_1\\ \le& C\||v/\eta_{a_N}|\|_\infty^{p_1p_2\dots p_N}\int_0^t \eta_{a_1}(s)^{p_N}ds \le C\||v/\eta_{a_N}|\|_\infty^{p_1p_2\dots p_N}\eta_{a_N}.\end{aligned}$$

□

Proof of Theorem 2. (ii) Let

$$C\left(\|u_{N,0}\|_{\infty,a_N}+\|u_{1,0}\|_{\infty,a_1}^{p_N}+\|u_{2,0}\|_{\infty,a_2}^{p_Np_1}+\dots+\|u_{N-1,0}\|_{\infty,a_{N-1}}^{p_Np_1\dots p_{N-2}}\right)\le m,$$

$\|u_i\|_{\infty,a_i} \le m$ $(i \in N^*)$, $B_m = \{v \in X : \||v/\eta_{a_3}|\|_\infty \le 2m\}$ and $P = \{u \in X; u \ge 0\}$. Here the constant C is the one appeared in Lemma 4.1. Then we shall show that $V(u_0, v)$ is a strict contraction of $B_m \cap P$ into itself provided m is small enough.

It is trivial that V maps P into P. We shall show that V maps $B_m \to B_m$. If m is small enough, then

$$V(t)(u_0,v)/\eta_{a_N} \le m + C(2m)^{p_1p_2\dots p_N} \le 2m.$$

This proves that V maps $B_m \to B_m$.

Now, we show that $V(u_0, v)$ is a strict contraction on $B_m \cap P$. Using $|a^p - b^p| \le p(a+b)^{p-1}|a-b|$ for $a > 0$, $b > 0$ and $p \ge 1$, with $v = \max\{v_1, v_2\}$, we can estimate as follows.

$$\begin{aligned}&|V(t)(u_0,v_1)-V(t)(u_0,v_2)|\\ &\qquad\le C\int_0^t S(t-s_1)\times J_1\times\int_0^{s_1} S(s_1-s_2)\times J_2\times\\ &\qquad\dots\times\int_0^{s_{N-2}} S(s_{N-2}-s_{N-1})\times J_{N-1}\times J_N ds_{N-1}\dots ds_2ds_1\,,\end{aligned}$$

where

$$J_l(x,r) = 2|x|^{\sigma_{l-1}}\Bigg(S(r)u_{l,0}(x) + \int_0^r S(r-s_{l+1})|x|^{\sigma_l}\Bigg\{S(s_{l+2})u_{l+1,0}$$

$$+\int_0^{s_{l+1}}(x)S(s_{l+1}-s_{l+2})\ldots|x|^{\sigma_{N-2}}\Bigg[S(s_{N-1})u_{N,0}(x)$$

$$+\int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}v^{p_{N-1}}(s_N)ds_N\Bigg]^{p_{N-2}}\ldots ds_{l+2}\Bigg\}^{p_l} ds_{l+1}\Bigg)^{p_{l-1}-1},$$

with $l = 1, 2, \ldots, N-1$, and

$$J_N = \int_0^{s_{N-1}} S(s_{N-1}-s_N)|x|^{\sigma_{N-1}}|v_1^{p_{N-1}}(s_N) - v_2^{p_{N-1}}(s_N)|^{p_{N-1}}ds_N.$$

Noting $(a+b)^p \le 2^{\max\{p-1,0\}}(a^p+b^p)$ for $a>0$, $b>0$ and $p\ge 0$, we find

$$\left[S(s_{k-1})u_{k,0}(x) + \int_0^{k-1} S(s_{k-1}-s_k)|x|^{\sigma_{k-1}}v^{p_{k-1}}ds_k\right]^q$$

$$\le C\left\{[S(s_{k-1})u_{k,0}(x)]^q + \left[\int_0^{k-1} S(s_{k-1}-s_k)|x|^{\sigma_{k-1}}v^{p_{k-1}}ds_k\right]^q\right\}$$

$$\le C\|u_{k_0}\|^q_{\infty,a_{k-1}}\eta^q_{a_{k-1}}(s_{k-1})$$

$$\times\left(\int_0^{s_{k-1}} S(s_{k-1}-s_k)|x|^{\sigma_{k-1}}\|v\|^{p_{k-1}}_{\infty,a_k}\eta^{p_{k-1}}_{a_k}(s_k)ds_k\right)^q.$$

For some $q>0$ and $v\in B_m$, by Lemma 2.2 (ii) and (41),

$$\left[S(s_{k-1})u_{k,0}(x) + \int_0^{k-1} S(s_{k-1}-s_k)|x|^{\sigma_{k-1}}v^{p_{k-1}}ds_k\right]^q$$

$$\le C\left(m^q + C(2m)^{p_{k-1}q}\right)\eta^q_{a_{k-1}}(s_{k-1}).$$

Then, by this fact and using Lemma 2.2 (ii) and (41) some times, we have

$$J_l \le Cm^{p_{l-1}-1}|x|^{\sigma_{l-1}}\eta_{a_l}^{p_{l-1}-1}(s_l)$$

for $l = 1, 2, \ldots, N-1$, and

$$J_N \le Cm^{p_{N-1}-1}(|v_1-v_2|/\eta_{a_N})\eta_{a_{N-1}}.$$

Thus, we obtain for some $C>0$.

$$\|V(t)(u_0,v_1) - V(t)(u_0,v_2)\|_{\eta_{a_N}} \le Cm^{p_1+p_2+\ldots+p_N-N}\|v_1-v_2\|_{\eta_{a_N}}.$$

Since $p_i\ge 1$ ($i\in N^*$) and $p_1p_2\ldots p_N>1$, $V(t)$ is a strict contraction of $B_m\cap P$ into itself provided m is small enough. Hence, there exists a unique fixed point

$v = (u_N) \in X$ which solves (42). We substitute $v = u_N$ into (17). Then the vector u solves (17). Moreover, since $u_N \in B_m$, we find

$$u_N \leq CS(t) < x >^{-a_N} .$$

By the same reason as in the proof of Lemma 4.3, we have

$$|u_{N-1}(x,t)| \leq \eta_{a_{N-1}}(x,t) \left\{ \|u_{N-1,0}\|_{\infty,a_{N-1}} + C\| |u_N/\eta_{a_N}| \|_\infty \right\} ,$$

and

$$|u_l(x,t)| \leq \eta_{a_l}(x,t) \left\{ \|u_{l,0}\|_{\infty,a_l} + C\| |u_{l+1}/\eta_{a_{l+1}}| \|_\infty \right\} ,$$

for $l = N-2, N-1, \dots, 2, 1$. Then $u_i \in B_m (i \in N^*)$ and the proof of Theorem 3 is completed. □

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