Communications in Applied Analysis **10** (2006), no. 2, 125–131

IMPULSIVE INTEGRAL INEQUALITIES WITH DELAY

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ABSTRACT: The paper considers impulsive integral inequalities used in studying impulsive differential equations with retarded arguments.

AMS (MOS) Subject Classification: 34A37

1. INTRODUCTION

When studying impulsive differential equations with retarded arguments certain integro-summary inequalities arise. These inequalities are characterized with:

I. The function estimated is piecewise continuous and has first-kind discontinuities at the instants of impulse effect of the differential equation considered.

II. In the integrals and sums of these inequalities besides the function estimated $u(t)$ other functions of the kind $u(\tau(t))$ also participate, where $\tau(t)$ is a retarded argument: $\tau(t) \le t$, $t \in R_+$.

Such inequalities we call "*impulsive integral inequalities with delay*".

In the present paper we are going to demonstrate a way to solve some inequalities of that type.

Integral inequalities with delay and impulsive integral inequalities are considered in Bainov and Simeonov [1], Sections 14 and 16, and in the references cited therein.

Received October 10, 2005

2. MAIN RESULTS

Let $R_+ = [0, +\infty)$, $J = [t_0, +\infty)$, $J_t = [t_0, t)$ for $t \in J$ and let the sequence $\{t_k\}_1^\infty$ be such that: $t_0 < t_1 < t_2 < \dots$ and $\lim_{k\to+\infty} t_k = +\infty$. Denote by PC_0 the set of nonnegative functions $u : J \to R_+$ which are continuous for $t \in J$, $t \neq t_k$, are continuous from the left and have discontinuities of the first kind at the points $t_k \in J$. Let $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$.

Consider the functional $f : J \times PC_0 \to R_+$ satisfying the following conditions:

H1. For any $t \in J$ and $u \in PC_0$ the value $f(t, u)$ does not depend on the values $u(s)$ for $s > t$.

H2. For any fixed $u \in PC_0$ the function $\psi(t) = f(t, u)$ has finite nonnegative derivative $D\psi(t) \in PC_0$ and $\Delta\psi(t_k) \geq 0$ for $t_k \in J$.

H3. $Df(t, \lambda u) \leq \lambda Df(t, u)$ for $u \in PC_0$, $\lambda \geq 0$, $t \in J$, $t \neq t_k$.

H4. $Df(t, u) \leq Df(t, v)$ for $t \in J$, $t \neq t_k$ and $u, v \in PC_0$ such that $u(s) \leq v(s)$, $s \in J_t$.

H5. $Df(t, u + v) \leq Df(t, u) + Df(t, v)$ for $t \in J$, $t \neq t_k$ and $u, v \in PC_0$.

H6. $\Delta f(t_k, \lambda u) \leq \lambda \Delta f(t_k, u)$ for $u \in PC_0$, $\lambda \geq 0$, $t_k \in J$.

H7. $\Delta f(t_k, u) \leq \Delta f(t_k, v)$ for $t_k \in J$ and $u, v \in PC_0$ such that $u(s) \leq v(s)$, $s \in J_{t_k}$.

H8. $\Delta f(t_k, u + v) \leq \Delta f(t_k, u) + \Delta f(t_k, v)$ for $t_k \in J$ and $u, v \in PC_0$.

Theorem 1. *Let f be a functional satisfying conditions H1–H8, let $u, a, q \in PC_0$, and suppose*

$$u(t) \leq a(t) + q(t) f(t, u)\,, \qquad t \in J\,. \tag{1}$$

Then

$$u(t) \leq a(t) + q(t)\Bigg\{ f(t_0, u)E(t, t_0) + \int_{t_0}^{t} Df(s, a)E(t, s)\mathrm{d}s + \sum_{t_0 \leq t_k < t} \Delta f(t_k, a)E(t, t_k^+)\Bigg\}\,, \qquad t \in J\,, \tag{2}$$

where

$$E(t, s) = \prod_{s \leq t_k < t} (1 + \Delta f(t_k, q)) \exp\left(\int_s^t Ddf(\tau, q)\mathrm{d}\tau\right), \qquad t_0 \leq s \leq t\,. \tag{3}$$

Proof. Setting $v(t) = f(t, u)$, equation (1) takes the form

$$u(t) \leq a(t) + q(t)v(t)\,. \tag{4}$$

From H2 we conclude that $v(t)$ is nondecreasing in J. Then

$$
\begin{aligned}
Dv(t) = & Df(t,u) \\
\le & Df(t, a+qv) && \text{(by H4)} \\
\le & Df(t,a) + Df(t,qv) && \text{(by H5)} \\
\le & Df(t,a) + Df(t, qv(t)) && \text{(by H1, H2 and H4)} \\
\le & Df(t,a) + Df(t,q)v(t) && \text{(by H3)}\,,
\end{aligned}
$$

or

$$
Dv(t) \le Df(t,a) + Df(t,q)v(t)\,, \qquad t \in J\,, \quad t \ne t_k\,. \tag{5}
$$

Analogously, using H1, H2, H6–H8 we conclude that

$$
\Delta v(t_k) \le \Delta f(t_k, a) + \Delta f(t_k, q)v(t_k)\,, \qquad t_k \in J\,. \tag{6}
$$

Applying Bainov and Simeonov [2], Lemma 2.2 to the impulsive differential inequality (5), (6) and taking into account (4) we obtain (2). □

Corollary 1. *Suppose the conditions of Theorem 1 hold. Then:*

(i) *If $a(t) \equiv q(t)\varphi(t)$ and $\varphi \in PC_0$, then*

$$
\begin{aligned}
u(t) \le q(t)\Big[\varphi(t) + f(t_0,u)E(t,t_0) + \int_{t_0}^{t} Df(s,q\varphi)E(t,s)\mathrm{d}s \\
+ \sum_{t_0 \le t_k < t} \Delta f(t_k, q\varphi)E(t,t_k^+)\Big]\,, \qquad t \in J\,.
\end{aligned} \tag{7}
$$

(ii) *If $a(t) \equiv q(t)\varphi(t)$ and $\varphi \in PC_0$ is nondecreasing in J, then*

$$
u(t) \le q(t)\big[\varphi(t) + f(t_0,u)\big]E(t,t_0)\,, \qquad t \in J\,. \tag{8}
$$

(iii) *If $q(t) \equiv 1$ and $a(t)$ is nondecreasing in J, then*

$$
u(t) \le \big[a(t) + f(t_0,u)\big] \prod_{t_0 \le t_k < t} (1 + \Delta f(t_k,1)) \exp\left(\int_{t_0}^{t} Df(\tau,1)\mathrm{d}\tau\right), \qquad t \in J\,. \tag{9}
$$

Remark 1. In order to obtain (9) we use (8) and the equality

$$\int_{t_0}^{t} \lambda(s) \mathrm{e}^{\int_s^t \lambda(\tau)\mathrm{d}\tau} \prod_{s \le t_j < t} (1+\lambda_j)\mathrm{d}s + \sum_{t_0 \le t_k < t} \lambda_k \mathrm{e}^{\int_{t_k}^t \lambda(\tau)\mathrm{d}\tau} \prod_{t_k < t_j < t} (1+\lambda_j)$$
$$\equiv \mathrm{e}^{\int_{t_0}^t \lambda(\tau)\mathrm{d}\tau} \prod_{t_0 \le t_k < t} (1+\lambda_k) - 1 \,.$$

Remark 2. Since $\ln(1+x) \le x$, $x > -1$, we conclude that for $q \in PC_0$

$$E(t,s) \le \exp(f(t,q) - f(s,q)) \equiv G(t,s)\,, \qquad t_0 \le s \le t\,. \tag{10}$$

Therefore, in (2), (7)–(9) the function $E(t,s)$ can be replaced by the function $G(t,s)$ from (10).

Introduce the following condition:

H9. $\tau_i(t) \in C(J,R)$, $\lim_{t\to+\infty} \tau_i(t) = +\infty$ and

$$\tau_i(t) \le t, \quad t \in J, \quad i = 0,1,\ldots,m.$$

Define the number t_{-1} and the sets E_i as follows:

$$t_{-1} = \min_{0\le i\le m} \big\{ \min_{t\in J} \tau_i(t) \big\}\,, \quad E_i = \big\{ t \in J : \tau_i(t) < t_0 \big\} \cup \big\{ t_0 \big\}\,,$$
$$i = 0,1,\ldots,m\,.$$

Now we apply Theorem 1 in order to solve concrete impulsive integral inequality with delay.

Theorem 2. *Suppose*

$$u(t) \le q(t) \Big\{ \varphi_0(t) + \sum_{i=0}^{m} \Big[\int_{t_0}^{t} b_i(s) u(\tau_i(s))\mathrm{d}s + \sum_{t_0 \le t_k < t} b_{ik} u(\tau_i(t_k)) \Big] \Big\} \tag{11}$$

for $t \in J$, and

$$u(t) = \psi(t)\,, \qquad t \in [t_{-1}, t_0]\,, \tag{12}$$

where $b_{ik} \ge 0$, $i = 0,1,\ldots,m$, $k \in N$; $u, q, \varphi_0 \in PC_0$ and $\varphi_0(t)$ is nondecreasing in J; $b_i \in PC_0$, $i = 0,1,\ldots,m$; τ_i, $i = 0,1,\ldots,m$ satisfy condition H9, and $\psi(t)$ is nonnegative piecewise continuous function in $[t_{-1}, t_0]$.

Then

$$u(t) \le q(t)\Big\{\varphi_0(t) + \sum_{i=0}^{m}\Big[\int_{J_t\cap E_i} b_i(s)\psi(\tau_i(s))\mathrm{d}s + \sum_{t_k\in J_t\cap E_i} b_{ik}\psi(\tau_i(t_k))\Big]\Big\}$$
$$\times \exp\Big\{\sum_{i=0}^{m}\Big[\int_{J_t\setminus E_i} b_i(s)q(\tau_i(s))\mathrm{d}s + \sum_{t_k\in J_t\setminus E_i} b_{ik}q(\tau_i(t_k))\Big]\Big\}, \qquad t \in J\,. \tag{13}$$

Proof. From (11) and (12) it follows that

$$u(t) \le q(t)\big[\varphi(t) + f(t,u)\big]\,, \qquad t \in J\,,$$

where

$$\varphi(t) = \varphi_0(t) + \sum_{i=0}^{m}\Big[\int_{J_t\cap E_i} b_i(s)\psi(\tau_i(s))\mathrm{d}s + \sum_{t_k\in J_t\cap E_i} b_{ik}\psi(\tau_i(t_k))\Big]$$

and

$$f(t,u) = \sum_{i=0}^{m}\Big[\int_{J_t\setminus E_i} b_i(s)u(\tau_i(s))\mathrm{d}s + \sum_{t_k\in J_t\setminus E_i} b_{ik}u(\tau_i(t_k))\Big]\,.$$

Now (13) follows from Corollary 1 (ii) and Remark 2. □

The following theorems are corollaries of Theorem 2.

Theorem 3. *Suppose*

$$u(t) \le a(t) + \int_{t_0}^{t}\big[b(s)u(s) + c(s)u(\tau(s))\big]\mathrm{d}s + \sum_{t_0\le t_k<t}\big[b_k u(t_k) + c_k u(\tau(t_k))\big] \tag{14}$$

for $t \in J$*, and*

$$u(t) = \psi(t)\,, \qquad t \in [t_{-1}, t_0]\,, \tag{15}$$

where $b_k \ge 0$*,* $c_k \ge 0$*,* $k \in N$*;* $u, a, b, c \in PC_0$ *and* $a(t)$ *is nondecreasing in* J*;* τ *satisfies condition H9, and* $\psi(t)$ *is nonnegative piecewise continuous function in* $[t_{-1}, t_0]$*,* $t_{-1} = \min\{\tau(t) : t \in J\}$*.*

Then

$$u(t) \le \Big[a(t) + \int_{J_t\cap E} c(s)\psi(\tau(s))\mathrm{d}s + \sum_{t_k\in J_t\cap E} c_k\psi(\tau(t_k))\Big]$$
$$\times \exp\Big[\int_{t_0}^{t} b(s)\mathrm{d}s + \sum_{t_0\le t_k<t} b_k + \int_{J_t\setminus E} c(s)\mathrm{d}s + \sum_{t_k\in J_t\setminus E} c_k\Big] \tag{16}$$

for $t \in J$, *where* $E = \{t \in J : \tau(t) < t_0\} \cup \{t_0\}$.

Proof. Indeed, (16) is a particular case of (13) with $m = 1$, $q(t) \equiv 1$, $\tau_0(t) = t$, $\tau_1(t) = \tau(t)$, $E_0 = \{t_0\}$, $E_1 = E$, $b_0(t) = b(t)$, $b_1(t) = c(t)$, $b_{0k} = b_k$, $b_{1k} = c_k$, $\varphi_0(t) = a(t)$. □

Corollary 2. *Suppose the conditions of Theorem 3 hold. Then*

$$u(t) \le \left[a(t) + \int_E c(s)\psi(\tau(s))\mathrm{d}s + \sum_{t_k \in E} c_k\psi(\tau(t_k))\right] \times \exp\left[\int_{t_0}^{t}(b(s) + c(s))\mathrm{d}s + \sum_{t_0 \le t_k < t}(b_k + c_k)\right], \qquad t \in J\,. \quad (17)$$

Remark 3. Estimate (17) is worse but simpler than estimate (16).

Theorem 4. *Suppose*

$$u(t) \le a(t) + \int_{t_0}^{t}\big[b(s)u(s) + c(s)u(s-r)\big]\mathrm{d}s + \sum_{t_0 \le t_k < t}\big[b_k u(t_k) + c_k u(t_k - r)\big] \quad (18)$$

for $t \in J$, *and*

$$u(t) = \psi(t)\,, \qquad t \in [t_0 - r, t_0]\,, \quad (19)$$

where $b_k \ge 0$, $c_k \ge 0$, $k \in N$; $u, a, b, c \in PC_0$ *and* $a(t)$ *is nondecreasing in* J; $r > 0$ *and* $\psi(t)$ *is a nonnegative piecewise continuous function in* $[t_0 - r, t_0]$.

Then

$$u(t) \le \left[a(t) + \int_{t_0}^{t} c(s)\psi(s-r)\mathrm{d}s + \sum_{t_0 \le t_k < t} c_k\psi(t_k - r)\right] \times \exp\left[\int_{t_0}^{t} b(s)\mathrm{d}s + \sum_{t_0 \le t_k < t} b_k\right], \qquad t_0 \le t \le t_0 + r \quad (20)$$

and

$$u(t) \le \left[a(t) + \int_{t_0}^{t_0+r} c(s)\psi(s-r)\mathrm{d}s + \sum_{t_0 \le t_k < t_0+r} c_k\psi(t_k - r)\right] \times \exp\left[\int_{t_0}^{t} b(s)\mathrm{d}s + \sum_{t_0 \le t_k < t} b_k + \int_{t_0+r}^{t} c(s)\mathrm{d}s + \sum_{t_0+r \le t_k < t} c_k\right], \qquad t > t_0 + r\,. \quad (21)$$

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