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ESTIMATES OF SINGULAR SOLUTIONS OF PROTTER'S PROBLEM FOR THE 3-D HYPERBOLIC EQUATIONS

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ABSTRACT: For 3-D wave equation M. Protter formulated (1952) some boundary value problems (BVP) which are three-dimensional analogues of the Darboux problems on the plane. Protter studied these problems in a 3-D domain Ω_0, bounded by two characteristic cones Σ_1 and $\Sigma_{2,0}$, and by a plane region Σ_0. Now, 50 years later, it is well known that, for an infinite number of smooth functions in the right-hand side, these problems do not have classical solutions. The reason of this fact had been discovered in the early 90-ties: the strong power-type singularity appears in the generalized solution on the characteristic cone $\Sigma_{2,0}$. In the present paper we consider the case of the wave equation involving lower order terms and obtain some a priori estimates for the singular solutions of the third BVP. It is a strong power type singularity at the vertex O of the characteristic cone $\Sigma_{2,0}$, which is isolated and does not propagate along the cone.

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1. INTRODUCTION

We denote points in $\mathbb{R}^3$ by $(x,t) = (x_1, x_2, t)$ and consider the wave equation with some lower order terms

$$L[u] \; = \; u_{x_1x_1} + u_{x_2x_2} - u_{tt} + b_1 u_{x_1} + b_2 u_{x_2} + b u_t + cu \; = \; f, \quad (1.1)$$

in the simply connected region

$$\Omega_0 := \{(x,t) : 0 < t < 1/2, t < \sqrt{x_1^2 + x_2^2} < 1 - t\}. \quad (1.2)$$

The region $\Omega_0 \subset \mathbb{R}^3$ is bounded by the disk $\Sigma_0 := \{(x,t) : t = 0, x_1^2 + x_2^2 < 1\}$, centered at the origin $O(0,0,0)$ and the characteristic surfaces of (1.1):

$$\begin{aligned} \Sigma_1 &:= \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = 1 - t\}, \\ \Sigma_{2,0} &:= \{(x,t) : 0 < t < 1/2, \sqrt{x_1^2 + x_2^2} = t\} \end{aligned} \quad (1.3)$$

We will treat the following problem

Problem $\mathbf{P}_\alpha$. Find solutions of the equation (1.1) in Ω_0 which satisfy the conditions

$$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0 \setminus O} = 0, \quad (1.4)$$

where $\alpha \in C^1(\bar{\Sigma}_0)$.The adjoint problem to $\mathbf{P}_\alpha$ is as follows.

Problem $\mathbf{P}_\alpha^*$. Find a solution of the adjoint equation

$$L^* u \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} - (b_1 u)_{x_1} - (b_2 u)_{x_2} - (bu)_t + cu = g \quad \text{in } \Omega_0$$

with the boundary conditions:

$$u|_{\Sigma_{2,0}} = 0, \quad [u_t + (\alpha + b)u]|_{\Sigma_0} = 0. \quad (1.5)$$

Protter's Problems, see (Protter [30]). Find a solution of the wave equation

$$\Box u \equiv \Delta_x u - u_{tt} \equiv u_{x_1x_1} + u_{x_2x_2} - u_{tt} = f \quad \text{in } \Omega_0 \quad (1.6)$$

with one of the following boundary conditions

$$\begin{aligned} &P1: \quad u|_{\Sigma_0 \cup \Sigma_1} = 0, && P1^*: \quad u|_{\Sigma_0 \cup \Sigma_{2,0}} = 0\,; \\ &P2: \quad u|_{\Sigma_1} = 0, u_t|_{\Sigma_0} = 0, && P2^*: \quad u|_{\Sigma_{2,0}} = 0, u_t|_{\Sigma_0} = 0\,. \end{aligned} \quad (1.7)$$

Protter [30] formulated and investigated both Problems $P1$ and $P1^*$ in Ω_0 as multi-dimensional analogues of the Darboux problem on the plane. It is well known that the corresponding Darboux problems on $\mathbb{R}^2$ are well posed, which is not true for the Protter's problems in $\mathbb{R}^3$. The uniqueness of a classical solution of Problem $P1$ in the $3 + 1 - D$ case was proved by Garabedian [10]. For recent results concerning the Protter's problems, see

(1.1), (1.4), see Grammatikopoulos et al [11] and references therein. For further publications in this area see Aldashev [1], Aldashev [2], Edmunds and Popivanov [9], Choi and Park [8], Cher [16], Popivanov and Popov [27], Popivanov and Popov [28], Popivanov and Popov [29]. Let mention some special orthogonality conditions of f_n, found in Popivanov and Popov [27], Popivanov and Popov [28], Popivanov and Popov [29], under which in the case of the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$ only regular generalized solutions of Problem $P1$ or $P2$ exist, or under only of some of them, there exist singular solutions with lower order of singularity. Unfortunately, we do not know any such conditions in the more general case of equation (1.1). On the other hand, Bazarbekov and Bazarbekov [5] gives in $\mathbb{R}^4$ another analogue of the classical Darboux problem in the corresponding to Ω four-dimensional domain. Some different statements of Darboux type problems in $\mathbb{R}^3$ or some connected with them Protter problems for mixed type equations (also studied in [31]) can be found in Aldashev [3], Aziz and Schneider [4], Bitsadze [6], Karatoprakliev [14], Kharibegashvili [15], Popivanov and Schneider [25]). In Lupo and Payne [18], Lupo and Payne [19] and Lupo et al [20] one finds results for mixed type equations including some special nonlinearity with supercritical exponent term in various situations, namely for the Frankl' and Guderley-Morawetz problem in $\mathbf{R}^2$ and for the Protter problem in $\mathbf{R}^{N+1}$ with $N \geq 2$. The existence of bounded or unbounded solutions for the wave equation in $\mathbb{R}^3$ and $\mathbb{R}^4$, as well as for the Euler-Poisson-Darboux equation has been studied in Cher [16], Choi [7], Choi and Park [8], Grammatikopoulos et al [12], Popivanov and Popov [29].

Further, we aim to find some exact a priori estimates for the singular solutions of Problem P_α and to outline the exact structure and order of singularity. For some other Protter Problems necessary and sufficient conditions for existence of solutions with fixed order of singularity were found (see Popivanov and Popov [27], Popivanov and Popov [28] in $\mathbb{R}^3$ and Popivanov and Popov [29] in $\mathbb{R}^4$).

According to the ill-possedness of Protter's Problems $P1$ and $P2$, it is interesting to find some their regularizations. A nonstandard, nonlocal regularization of Problem P1, can be found in Edmunds and Popivanov [9]. In the present paper we are looking for some other kind of regularization and formulate the following problem.

Open Question 1. Is it possible to find some conditions on the coefficients b_1, b_2, b, c and α, under which Problem P_α has only regular solutions for all smooth functions f?

Remark. If the answer to the above question is positive, then, using an operator L_k with lower order perturbations in the wave equation (1.6), we can find possible regularization for Problem $P2$. Solving the equation $L_k u_k = f$, with $L_k \to \Box$ (i.e. $b_{1k}, b_{2k}, b_k, c_k \to 0$) and $\alpha_k \to 0$, we can find an approximated sequence u_k. Due to the fact that in this case the cones Σ_1

and $\Sigma_{2,0}$ are again characteristics for L_k, this process, with respect to our boundary value problem, looks to be natural.

For Problem (1.1), (1.4), i.e. P_α and $\alpha(x) \neq 0$, there are only few publications, while for (1.1) with P_α, we refer the reader to Grammatikopoulos et al [11]. In the case of the equation (1.1), which involves either lower order terms or some other type perturbations, Problem P_α in Ω_0 with $\alpha(x) \equiv 0$ has been studied by Aldashev [1], Aldashev [2]. Finally, we point out that in the case of (1.1), with nonzero lower order terms, Karatoprakliev [14] obtained a priori estimates, but only for solutions of Problem $P1$ enough smooth in Ω_0.

Next, we formulate the following well known result Kwang-Chang [32], Popivanov and Schneider [24], presented here in the terms of the polar coordinates (ϱ, φ, t) with $x_1 = \varrho cos\varphi, x_2 = \varrho sin\varphi$.

Theorem 1.1. *For all $n \in N$, $n \geq 4$; a_n, b_n arbitrary constants, the functions*

$$v_n(\varrho, \varphi, t) = t\varrho^{-n} \left(\varrho^2 - t^2\right)^{n-\frac{3}{2}} (a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.8}$$

are classical solutions of the homogeneous problem $P1^*$ and the functions

$$w_n(\varrho, \varphi, t) = \varrho^{-n} \left(\varrho^2 - t^2\right)^{n-\frac{1}{2}} (a_n \cos n\varphi + b_n \sin n\varphi) \tag{1.9}$$

are classical solutions of the homogeneous problem $P2^*$.

This theorem shows that for the classical solvability (see Bitsadze [6]) of the problem $P1$ (respectively, $P2$) the function f at least must be orthogonal to all smooth functions (1.8) (respectively, (1.9)). The reason of this fact has been found by Popivanov and Schneider [24], where they announced for Problems $P1$ and $P2$ that there exist singular solutions for the wave equation (1.6) with power type isolated singularities even for very smooth functions f. Using Theorem 1.1, Popivanov and Schneider [26] proved the existence of generalized solutions of Problems $P1$ and $P2$, which have at least power type singularities at the vertex O of the cone $\Sigma_{2,0}$. Considering Problems $P1$ and $P2$, Popivanov and Schneider [24] announced the existence of singular solutions for both wave and degenerate hyperbolic equation (see Popivanov and Schneider [25]). First a priori estimates for singular solutions of Protter's Problems $P1$ and $P2$, concerning the wave equation in R^3, were obtained in Popivanov and Schneider [26]. On the other hand, for the case of the wave equation in R^{m+1}, Aldashev [1] shows that there exist solutions of Problem $P1$ (respectively, $P2$) in the domain Ω_ε, which grow up on the cone $\Sigma_{2,\varepsilon}$ like $\varepsilon^{-(n+m-2)}$ (respectively, $\varepsilon^{-(n+m-1)}$), when $\varepsilon \to 0$ and the cone $\Sigma_{2,\varepsilon} := \{\varrho = t + \varepsilon\}$ approximates $\Sigma_{2,0}$. It is obvious that for $m = 2$ this results can be compared with the estimate (1.11) of Theorem 1.3 below. For the homogeneous Problem P^*_α (except the case $\alpha \equiv 0$, i.e. except Problem $P2^*$), even for the wave equation, we do not know nontrivial solutions analogous to (1.8) and (1.9). Anyway, in Grammatikopoulos et al [11] under appropriate conditions for the coefficients of the general equation (1.1), we derive results which ensure the existence of many singular solutions of Problem P_α. Here we refer also to Cher [16], who gives some nontrivial solutions for the homogeneous Problems $P1^*$ and $P2^*$, but in the case of Euler-Poisson-Darboux equation. These results are closely connected to the such ones of Theorem 1.1.

To formulate known results for Problem P_α we first give the definition of generalized solutions

Definition 1.2. A function $u = u(x_{1,}x_2, t)$ is called a generalized solution of Problem P_α in Ω_0, if

(1) $u \in C^1(\bar{\Omega}_0\backslash O)$, $[u_t + \alpha(x)u]\big|_{\Sigma_0\backslash O} = 0 \; u\big|_{\Sigma_1} = 0$,

(2) the equality

$$\int_{\Omega_0} [u_t v_t - u_{x_1}v_{x_1} - u_{x_2}v_{x_2} + (b_1u_{x_1} + b_2u_{x_2} + bu_t + cu - f)v]dx_1dx_2dt$$
$$= \int_{\Sigma_0} \alpha(x)(uv)(x, 0)dx_1dx_2$$

holds for all v from

$$V_0 := \{v \in C^1(\bar{\Omega}_0) : [v_t + (\alpha + b)v]\big|_{\Sigma_0} = 0,$$
$$v = 0 \text{ in a neighborhood of } \Sigma_{2,0}\}.$$

The Definition 1.2 assures that generalized solutions of Problem P_α may have singularities on the cone $\Sigma_{2,0}$.

M.K. Grammatikopoulos, T. Hristov, N. Popivanov proved in Grammatikopoulos et al [11] the following existence theorem for solutions of Problem P_α which have singularities on $\Sigma_{2,0}$.

Denoting $a_1 := b_1 cos\varphi + b_2 sin\varphi, a_2 := \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi)$, we assume for the coefficients of the equation (1.1) in polar coordinates:
Cd1: a_1, a_2, b, c are functions of $(|x|, t)$; $\alpha = \alpha(|x|)$.

Theorem 1.3. (Theorem 6.1 in Grammatikopoulos et al [11]) *Let* $\alpha \geq 0$; a_1, b, $c \in C^1(\bar{\Omega}_0\backslash O)$, $a_2 \equiv 0$ *and*

$$a_1(|x|, t) \geq |b|(|x|, t), \quad a_1(|x|, t) \geq 2|x|c(|x|, t), \quad (x, t) \in \Omega_0.$$

Then for each function

$$f_n(x, t) = |x|^{-n}(|x|^2 - t^2)^{n-1/2} \cos n(\arctan \frac{x_2}{x_1})$$
$$\in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0), \quad (1.10)$$

$n \in N, n \geq 4$, the corresponding generalized solution u_n of Problem P_α belongs to $C^2(\bar{\Omega}_0 \backslash O)$ and satisfies the estimate

$$|u_n(x,t)|_{t=|x|} \;\geq\; c_0|x|^{-n}|\cos n(\arctan\frac{x_2}{x_1})|, \quad 0 \;<\; |x| \;<\; 1/2\,, \qquad (1.11)$$

where $c_0 = const. > 0$.

Remark 1.4. The reason to use the functions $f_n(x,t)$ from (1.10) for a right-hand side f of the equation (1.1) one finds in Theorem 1.1 (see (1.9)).

In case of the wave equation for Problem P_α the authors prove in Grammatikopoulos et al [11] lower and upper estimates for generalized solutions with singularities on $\Sigma_{2,0}$.

Theorem 1.5. (Theorem 7.1 in Grammatikopoulos et al [11]) *For the wave equation*

$$\Box u = u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0 \quad in\ \Omega_0,$$

$$u|_{\Sigma_1} = 0, \quad [u_t + \alpha u]|_{\Sigma_0\backslash O} = 0$$

let $\alpha \in C^\infty(0,1] \cap C[0,1]$ be an arbitrary function. Then:

(i) *for each $n \in N$, $n \geq 4$ define*

$$g_n(x,t) \;:=\; f_n(x,t)\exp\{\Lambda(|x|+t)\} \;\in\; C^{n-2}(\bar{\Omega}_0) \;\cap\; C^\infty(\Omega_0), \qquad (1.12)$$

with $f_n(x,t)$ from (1.10) and $\Lambda = const.$ Then there exists an appropriate constant $\Lambda_0 > 0$, such that for $\Lambda \geq \Lambda_0$ the corresponding generalized solution u_n of Problem P_α for $f = g_n$ belongs to $C^n(\bar{\Omega}_0\backslash O)$ and satisfies

$$|u_n(x,t)|_{t=|x|} \quad\geq\quad \frac{1}{2}|u_n(2x,0)| \;+\; c_0|x|^{-n}|\cos n(\arctan\frac{x_2}{x_1})|, \qquad (1.13)$$

where $c_0 = const. > 0$.

(ii) *in case $\alpha \leq 0$ an upper estimate holds*

$$|u_n(x_1,x_2,t)| \leq C_\mu|x|^{-1/2}\Big(\frac{|x|}{x_1^2+x_2^2-t^2}\Big)^{n-\frac{1}{2}}|\cos n(\arctan\frac{x_2}{x_1})|, \qquad (1.14)$$

for

$$(|x|,t) \in D_1^\mu := \{(\varrho,t) : 0 < \varrho - t \leq \varrho + t \leq \mu(\varrho - t)\}\,,\ \mu < 2^{\frac{2n+1}{2n-1}} - 1,$$

where $\mu < 2^{\frac{2n+1}{2n-1}} - 1$, and C_μ is a constant.

Remark 1.6. We see from Theorem 1.5 that in the case of Problem P_α for the wave equation with $\alpha(\varrho) \leq 0$ the corresponding generalized solution with a special function f has lower bounds (1.13) and upper bounds (1.14) in D_1^μ. Note also, that the generalized solutions in Theorems 1.3 and 1.5 have singularities at the vertex O of the cone $\Sigma_{2,0}$ and that these singularities do not propagate in the direction of the bicharacteristics on the characteristic cone $\Sigma_{2,0}$. For results, concerning the propagation of singularities for solutions of second order operators see Hörmander [13], Chapter 24.5.

In this paper we treat Problem P_α in polar coordinates, i.e.

$$L[u] \;=\; \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + a_2 u_\varphi + b u_t + cu \;=\; f, \quad (1.15)$$

$\big(a_1 = b_1\cos\varphi + b_2\sin\varphi, a_2 = \varrho^{-1}(b_2\cos\varphi - b_1\sin\varphi)\big)$ in case function $f = f(\varrho,\varphi,t)$ is of the form

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi \quad (1.16)$$

and ask for generalized solution of form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi. \quad (1.17)$$

Following Grammatikopoulos et al [11], in Section 2, Problem P_α is reduced to a two-dimensional problem (2.3) for functions $\{u_n^{(1)}(\varrho,t), u_n^{(2)}(\varrho,t)\}$, called Problem $P_{\alpha,1}$. Using characteristic coordinates $\xi = 1-\varrho-t,\ \eta = 1-\varrho+t$ and new functions

$$u_n^{(i)}(\xi,\eta) := z_n^{(i)}(\varrho,t) := \varrho^{\frac{1}{2}}u_n^{(i)}(\varrho,t),\ i=1,2, \quad (1.18)$$

we obtain system (2.6) for $\{u_n^{(1)}, u_n^{(2)}\}(\xi,\eta)$ in D_ε^1, $\varepsilon > 0$, called Problem $P_{\alpha,2}$. Note that for the Problems $P_{\alpha,1}$ and $P_{\alpha,2}$ in Section 2 the index n is omitted. See also Remark 3.1 at the and of Section 3.

In Section 3 we construct the equivalent integral equation system (3.13) - (3.15) of Problem $P_{\alpha,2}$. This system is then solved in Section 4 by a successive approximation method (4.1). Following some ideas of Popivanov and Popov [28] for Problem P1, we obtain in Lemma 4.1 a basic estimate for a singular solution of Problem P_α. Using this result, Theorem 4.4 gives the existence of classical solution $\{U_n^{(1)}, U_n^{(2)}\}(\xi,\eta) \in C^1(\bar{D}_\varepsilon^1)$, $U_{n,\xi\eta}^{(i)} \in C(\bar{D}_\varepsilon^1)$ of Problem $P_{\alpha,2}$ and some upper bounds (4.15) for $|U_n^{(i)}(\xi,\eta)|$ and the derivatives $|U_{n,\xi}^{(i)}|$, $|U_{n,\eta}^{(i)}|$. The exact condition of the order of possible singularity appear naturally there. Going back to the $(\varrho,t)-$ coordinates we obtain generalized solution $u_n^{(i)}(\varrho,t)$ of Problem $P_{\alpha,1}$ and thus a generalized solution of the original Problem P_α in Ω_0. One finds some a priori estimates of all solutions of Problem P_α, including singular ones, in Theorem 4.6. In Theorem 4.7 these a priori estimates are extended in the general case when the function $f(x,t)$ is not only a trigonometric polynomial. Especially, in Theorem 4.7 one finds some sufficient conditions on the Fourier coefficients of the function $f(x,t)$, under which the existence of the generalized solution is given and some a priori estimates are proved. In the a priori estimate the possible singularity $(|x|^2+t^2)^{-\alpha_n/2}$ is of order $\alpha_n = n + C$, where the constant C depends only of the bounds of the coefficients of the first derivatives in the equation (1.1). Let mention also here the big difference between the conditions on the Fourier coefficients in the present paper and in Popivanov and Schneider [26], Aldashev [1], Grammatikopoulos et al [11] (see Remark 4.10).

In Example 5.1, Section 5, Problem P_α for the wave equation (1.6) is considered. If in Problem $P2$ the function $f \in C^1(\bar{\Omega}_0)$ has an Fourier expansion

$$f(\varrho,\varphi,t) = \sum_{n=0}^{\infty} \{ f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi \}, \tag{1.19}$$

Theorem 5.1 gives some appropriate conditions on the Fourier coefficients $f_n^{(1)}$ and $f_n^{(2)}$of f, such that the corresponding generalized solution $u(x,t)$ exists and some upper bounds hold. Some a priori estimates of $u(x,t)$ one finds also there.

In Example 5.2 the Problem P_α is considered for the wave equation with lower order terms (see Remark 5.2), in the case $c \neq 0$ in the equation (1.1). Theorem 4.4 gives some existence results and a priori estimates with order α_n of possible singularity approximately n.

In Example 5.3 one finds a result for existence of singular solutions of Problem P_α and some below bounds for them. In the special case of singular at the point O coefficients b_1, b_2 or c in the equation (1.1), are found some right-hand side functions $f_n(x,t)$, the corresponding generalized solution for which has singularity of order $\alpha_n > n+1$.

Open Questions 2. Can one find some regular coefficients b_1, b_2, b, c, for which the same phenomena of the order of singularity $\alpha_n > n+1$ appear?

3. Can one find some function $f \in C^1(\bar{\Omega}_0)$ such that the corresponding generalized solution has an exponential growth as $(x,t) \to O$?

4. Can one find some orthogonality conditions in the case of the equation (1.1), under which we have an lower order of singularity α_n, instead of the estimate (1.11)?

2. PRELIMINARIES

We consider (1.1) in polar coordinates (see (1.15)) in case

$$f(\varrho,\varphi,t) = f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi,\ n \in \mathbb{N}, \tag{2.1}$$

and ask for the generalized solution to be of the form

$$u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi. \tag{2.2}$$

Following Grammatikopoulos et al [11], from (1.1) we obtain the system

$$\begin{cases} \frac{1}{\varrho}(\varrho u^{(1)}_{n,\varrho})_\varrho - u^{(1)}_{n,tt} + a_1 u^{(1)}_{n,\varrho} + b_3 u^{(1)}_{n,t} + (c - \frac{n^2}{\varrho^2})u^{(1)}_n + n a_2 u^{(2)}_n \\ \qquad = f^{(1)}_n, \\ \frac{1}{\varrho}(\varrho u^{(2)}_{n,\varrho})_\varrho - u^{(2)}_{n,tt} + a_1 u^{(2)}_{n,\varrho} + b_3 u^{(2)}_{n,t} + (c - \frac{n^2}{\varrho^2})u^{(2)}_n - n a_2 u^{(1)}_n \\ \qquad = f^{(2)}_n. \end{cases} \tag{2.3}$$

To deal with singularities on $t = \varrho$, especially at $(0,0)$, we consider (2.3) in the domain

$$G_\varepsilon = \{(\varrho, t) : t > 0, \varepsilon + t < \varrho < 1 - t\},\ \varepsilon > 0,$$

which is bounded by the disc $S_0 = \{(\varrho, t) : t = 0, 0 < \varrho < 1\}$, and

$$S_1 = \{(\varrho, t) : \varrho = 1 - t\}, \qquad S_{2,\varepsilon} = \{(\varrho, t) : \varrho = t + \varepsilon\}$$

and treat the following problem (omitted the index n):
Problem $P_{\alpha,1}$. Find solutions $u = (u^{(1)}, u^{(2)})$ of system (2.3) which satisfy

$$u^{(i)}\big|_{S_1\cap\partial G_\varepsilon} = 0, \quad [u^{(i)}_t + \alpha(\varrho)u^{(i)}]\big|_{S_0\cap\partial G_\varepsilon} = 0, \quad i = 1, 2.$$

Definition 2.1. A function $u = (u^{(1)}, u^{(2)})(\varrho, t)$ is called a generalized solution of Problem $P_{\alpha,1}$ in G_ε, $\varepsilon > 0$, if:

(1) $u \in C^1(\bar{G}_\varepsilon)$, $[u^{(i)}_t + \alpha(\varrho)u^{(i)}]\big|_{S_0\cap\partial G_\varepsilon} = 0$, $u^{(i)}\big|_{S_1\cap\partial G_\varepsilon} = 0$, $i = 1, 2$;
(2) The equalities

$$\int_{G_\varepsilon} \big[u^{(1)}_t v_{1,t} - u^{(1)}_\varrho v_{1,\varrho} + \big(a_1 u^{(1)}_\varrho + b u^{(1)}_t + (c - \frac{n^2}{\varrho^2})u^{(1)} + n a_2 u^{(2)} - f^{(1)}\big)v_1\big]\,\varrho d\varrho\, dt = \int_{S_0\cap\partial G_\varepsilon} \alpha(\varrho)u^{(1)} v_1 \varrho\, d\varrho,$$

$$\int_{G_\varepsilon} [u^{(2)}_t v_{2,t} - u^{(2)}_\varrho v_{2,\varrho} + \big(a_1 u^{(2)}_\rho + b u^{(2)}_t + (c - \frac{n^2}{\varrho^2})u^{(2)} - n a_2 u^{(1)} - f^{(2)}\big)v_2\big]\,\varrho d\varrho\, dt = \int_{S_0\cap\partial G_\varepsilon} \alpha(\varrho)u^{(2)} v_2 \varrho\, d\varrho$$

hold for all

$$v_1, v_2 \in V_\varepsilon^{(1)} = \{v \in C^1(\bar{G}_\varepsilon) \\ : [v_t + (\alpha + b)v]\big|_{S_0 \cap \partial G_\varepsilon} = 0, v\big|_{S_{2,\varepsilon} \cap \partial G_\varepsilon} = 0\}.$$

Introducing a new function

$$z^{(i)}(\varrho, t) = \varrho^{\frac{1}{2}} u^{(i)}(\varrho, t) = z^{(i)}(\varrho(\xi, \eta), t(\xi, \eta)) =: U^{(i)}(\xi, \eta), \; i = 1, 2, \quad (2.4)$$

in characteristic coordinates

$$\xi = 1 - \varrho - t, \quad \eta = 1 - \varrho + t \quad (2.5)$$

we obtain the system

$$\begin{cases} U^{(1)}_{\xi\eta} - A_1 U^{(1)}_\xi - B_1 U^{(1)}_\eta - C_1 U^{(1)} - D_1 U^{(2)} = F^1(\xi, \eta) & \text{in } D_\varepsilon, \\ U^{(2)}_{\xi\eta} - A_2 U^{(2)}_\xi - B_2 U^{(2)}_\eta - C_2 U^{(2)} - D_2 U^{(1)} = F^2(\xi, \eta) & \text{in } D_\varepsilon, \end{cases} \quad (2.6)$$

where $D_\varepsilon = \{(\xi, \eta) : 0 < \xi < \eta < 1 - \varepsilon\}$ and

$$F^{(i)}(\xi, \eta) = \frac{1}{4\sqrt{2}}(2 - \eta - \xi)^{\frac{1}{2}} f^{(i)}(\varrho(\xi, \eta), t(\xi, \eta)), \; i = 1, 2, \quad (2.7)$$

$$A_1 = A_2 = \frac{1}{4}(a_1 + b), \quad B_1 = B_2 = \frac{1}{4}(a_1 - b), \\ D_2 = -D_1 = \frac{1}{4} n a_2, \; C_1 = C_2 = \frac{1}{4}\left\{\frac{4n^2 - 1}{(2 - \xi - \eta)^2} + \frac{a_1}{2 - \xi - \eta} - c\right\}. \quad (2.8)$$

Note, that Problem $P_{\alpha,1}$ is reduced to the Darboux-Goursat problem for the system (2.6) in D_ε.

To investigate the smoothness or the singularities of solutions at the original Problem P_α on $\Sigma_{2,0}$, we are looking for classical solutions for the system (2.6) not only in the domain D_ε, but also in the domain

$$D_\varepsilon^{(1)} := \{(\xi, \eta) : 0 < \xi < \eta < 1, 0 < \xi < 1 - \varepsilon\}, \quad \varepsilon > 0,$$

where $D_\varepsilon \subset D_\varepsilon^{(1)}$. Thus we come to the following

Problem $P_{\alpha,2}$. Find solutions $(U^{(1)}, U^{(2)})(\xi, \eta)$ of system (2.6) in $D_\varepsilon^{(1)}$, which satisfy the boundary conditions

$$U^{(i)}(0, \eta) = 0, (U^{(i)}_\eta - U^{(i)}_\xi)(\xi, \xi) + \alpha(1 - \xi) U^{(i)}(\xi, \xi) = 0, \quad (2.9)$$

$i = 1, 2, \; \xi \in (0, 1 - \varepsilon), \eta \in (0, 1)$.

3. A SYSTEM OF INTEGRAL EQUATIONS FOR PROBLEM $P_{\alpha,2}$

We consider a point $(\xi_0, \eta_0) \in D_\varepsilon^{(1)}$ and rectangle R, triangle T defined by

$$R := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi_0 < \eta < \eta_0\},$$

$$T := \{(\xi, \eta) : 0 < \xi < \xi_0, \xi < \eta < \xi_0\}.$$

By use of Green's theorem in

$$\begin{aligned} I_R^{(i)} &:= \iint_R U_{\xi\eta}^{(i)}(\xi, \eta)\, d\xi d\eta = \int_0^{\xi_0} \left(\int_{\xi_0}^{\eta_0} U_{\xi\eta}^{(i)}(\xi, \eta)\, d\eta \right) d\xi\,, \\ I_T^{(i)} &:= \iint_T U_{\xi\eta}^{(i)}(\xi, \eta)\, d\xi d\eta = \int_0^{\xi_0} \left(\int_{\xi}^{\xi_0} U_{\xi\eta}^{(i)}(\xi, \eta)\, d\eta \right) d\xi, \end{aligned} \tag{3.10}$$

$i = 1, 2$, and the boundary conditions (2.9) we obtain

$$I_R^{(i)} + 2I_T^{(i)} = U^{(i)}(\xi_0, \eta_0) - \int_0^{\xi_0} \alpha(1 - \xi) U^{(i)}(\xi, \xi)\, d\xi. \tag{3.11}$$

We set $p^{(i)} := U_\xi^{(i)}$, $q^{(i)} := U_\eta^{(i)}$ and define (see (2.6))

$$\begin{aligned} E^{(1)}(\xi, \eta) &:= [F^1 + A_1 p^{(1)} + B_1 q^{(1)} + C_1 U^{(1)} + D_1 U^{(2)}](\xi, \eta), \\ E^{(2)}(\xi, \eta) &:= [F^2 + A_2 p^{(2)} + B_2 q^{(2)} + C_2 U^{(2)} + D_2 U^{(1)}](\xi, \eta). \end{aligned} \tag{3.12}$$

Using (3.10) - (3.12) and (2.6) we obtain six integral equations (i=1,2)

$$\begin{aligned} U^{(i)}(\xi_0, \eta_0) = \int_0^{\xi_0} \left(\int_{\xi_0}^{\eta_0} E^{(i)}(\xi, \eta)\, d\eta \right) d\xi + 2 \int_0^{\xi_0} \left(\int_0^{\eta} E^{(i)}(\xi, \eta)\, d\xi \right) d\eta \\ + \int_0^{\xi_0} \alpha(1 - \xi) U^{(i)}(\xi, \xi)\, d\xi, \end{aligned} \tag{3.13}$$

$$\begin{aligned} p^{(i)}(\xi_0, \eta_0) = \int_0^{\xi_0} E^{(i)}(\xi, \xi_0)\, d\xi + \int_{\xi_0}^{\eta_0} E^{(i)}(\xi_0, \eta)\, d\eta \\ + \alpha(1 - \xi_0) U^{(1)}(\xi_0, \xi_0), \end{aligned} \tag{3.14}$$

$$q^{(i)}(\xi_0, \eta_0) = \int_0^{\xi_0} E^{(i)}(\xi, \eta_0)\, d\xi\,. \tag{3.15}$$

The system (3.13) - (3.15) is equivalent to the system (2.6) with the boundary conditions (2.9).

Remark 3.1. We remind that in Section 2 the index n in system (2.3) was omitted. We see that in (2.6) the coefficients C_i, D_i $(i = 1, 2)$ depend on n, where on the right-hand side we have

$$F^{(i)}(\xi,\eta) = \frac{1}{4\sqrt{2}}(2-\eta-\xi)^{\frac{1}{2}} f_n^{(i)}(\varrho(\xi,\eta), t(\xi,\eta)).$$

Therefore for fixed $n \in N$ solutions $(U^{(1)}, U^{(2)})$ of the integral equation system (3.13) - (3.15) depend on n and will be later marked by $(U_n^{(1)}, U_n^{(2)})$, which gives functions $(u_n^{(1)}, u_n^{(2)})$ by relation $\varrho^{\frac{1}{2}} u^{(i)}(\varrho, t) = U_n^{(i)}(\xi, \eta)$ (see (2.4)).

Furthermore we observe that classical solutions $(U_n^{(1)}, U_n^{(2)}) \in C^1(\bar{D}_\varepsilon^1)$, $U_{n,\xi\eta}^{(i)} \in C(\bar{D}_\varepsilon^1)$ of the integral equation system define functions $(u_n^{(1)}, u_n^{(2)})$ which are generalized solutions of Problem $P_{\alpha,1}$ in $\bar{G}_0 \setminus (0,0)$.

4. SOLUTIONS OF THE INTEGRAL EQUATION SYSTEM

Following Grammatikopoulos et al [11] we solve the integral equation system (3.13) - (3.15) by the successive approximation method. We define in $D_\varepsilon^{(1)}$ functions $(U_m^{(i)}, p_m^{(i)}, q_m^{(i)})$, $i = 1, 2$; $m \in N$, by the formulas

$$\begin{aligned}
&U_{m+1}^{(i)}(\xi_0,\eta_0) \\
&= \int_0^{\xi_0} \left(\int_{\xi_0}^{\eta_0} E_m^{(i)}(\xi,\eta)\, d\eta \right) d\xi + 2\int_0^{\xi_0} \left(\int_0^{\eta} E_m^{(i)}(\xi,\eta)\, d\xi \right) d\eta \\
&+ \int_0^{\xi_0} \alpha(1-\xi) U_m^{(i)}(\xi,\xi)\, d\xi, \qquad i = 1,2; \quad m = 0,1,2\ldots, \\
&p_{m+1}^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} E_m^{(i)}(\xi,\xi_0)\, d\xi + \int_{\xi_0}^{\eta_0} E_m^{(i)}(\xi_0,\eta)\, d\eta \\
&+\alpha(1-\xi_0) U_m^{(i)}(\xi_0,\xi_0), \qquad i = 1,2; \quad m = 0,1,2\ldots, \\
&q_{m+1}^{(i)}(\xi_0,\eta_0) = \int_0^{\xi_0} E_m^{(i)}(\xi,\eta_0)\, d\xi, \quad i = 1,2; \quad m = 0,1,2\ldots, \\
&U_0^{(i)}(\xi_0,\eta_0) = 0, \quad p_0^{(i)}(\xi_0,\eta_0) = 0, \quad q_0^{(i)}(\xi_0,\eta_0) = 0, \quad i = 1,2,
\end{aligned} \tag{4.1}$$

in $D_\varepsilon^{(1)}$, where

$$E_m^{(1)}(\xi,\eta) := [F^1 + A_1 p_m^{(1)} + B_1 q_m^{(1)} + C_1 U_m^{(1)} + D_1 U_m^{(2)}](\xi,\eta),$$

$$E_m^{(2)}(\xi,\eta) := [F^2 + A_2 p_m^{(2)} + B_2 q_m^{(2)} + C_2 U_m^{(2)} + D_2 U_m^{(1)}](\xi,\eta).$$

Lemma 4.1. *Let for* $(\xi_0, \eta_0) \in D_\varepsilon^{(1)} = \{(\xi, \eta) : 0 < \xi < \eta < 1, 0 < \xi < 1-\varepsilon\}$, $\varepsilon > 0$, *and* $\mu \in R_+$ *define*

$$I_\mu := \int_0^{\xi_0} \left(\int_{\xi_0}^{\eta_0} (2-\xi-\eta)^{-\mu-2} \, d\eta \right) d\xi \\ + 2 \int_0^{\xi_0} \left(\int_{\xi}^{\xi_0} (2-\xi-\eta)^{-\mu-2} \, d\eta \right) d\xi.$$

Then

$$I_\mu \leq \frac{1}{\mu(\mu+1)} (2-\xi_0-\eta_0)^{-\mu}. \tag{4.2}$$

Proof. The calculation shows

$$\begin{aligned} I_\mu &= \frac{1}{\mu+1} \Big[\int_0^{\xi_0} (2-\xi-\eta)^{-\mu-1} \Big|_{\eta=\xi_0}^{\eta=\eta_0} d\xi + 2 \int_0^{\xi_0} (2-\xi-\eta)^{-\mu-1} \Big|_{\eta=\xi}^{\eta=\xi_0} d\xi \Big] \\ &= \frac{1}{\mu+1} \int_0^{\xi_0} \left\{ (2-\xi-\eta_0)^{-\mu-1} + (2-\xi-\xi_0)^{-\mu-1} - 2(2-2\xi)^{-\mu-1} \right\} d\xi \\ &= \frac{1}{\mu(\mu+1)} \Big[(2-\xi_0-\eta_0)^{-\mu} + 2^{-\mu} - (2-\xi_0)^{-\mu} - (2-\eta_0)^{-\mu} \Big] \\ &\leq \frac{1}{\mu(\mu+1)} (2-\xi_0-\eta_0)^{-\mu}. \quad \square \end{aligned}$$

For the coefficients of (1.1) we assume Cd1 and using the notations

$$\sup_{\bar{\Omega}_0} \left\{ |b_1|, |b_2|, |b| \right\} \leq K_1, \ \sup_{\bar{\Omega}_0} |c| \leq K_0, \ \sup_{[0,1]} |\alpha(\varrho)| \leq K_\alpha. \tag{4.3}$$

Then, from (2.8) we obtain the following bounds

$$\begin{aligned} &|a_1| \leq 2K_1, \quad |a_2| \leq \frac{2K_1}{\varrho}, |A_1| = |A_2| \leq \frac{3K_1}{4}, \\ &|B_1| = |B_2| \leq \frac{3K_1}{4}, \quad |D_1| = |D_2| \leq \frac{nK_1}{2\varrho} = \frac{nK_1}{(2-\xi-\eta)}, \\ &|C_1| = |C_2| \leq \frac{\nu(\nu+1)}{(2-\xi-\eta)^2} + \frac{K_1}{2(2-\xi-\eta)} + \frac{K_0}{4}, \end{aligned} \tag{4.4}$$

where $\nu := n - \frac{1}{2}$.

According to (3.12)

$$E_m^{(i)}(\xi, \eta) := \left\{ F^i + A_i p_m^{(i)} + B_i q_m^{(i)} + C_i U_m^{(i)} + D_i U_m^{(\tau_i)} \right\} (\xi, \eta),$$

with $\tau_1 = 2,\ \tau_2 = 1$ and thus we have

$$|(E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)| \le \left\{ \frac{\nu(\nu+1)}{(2-\xi-\eta)^2} + \frac{K_1}{2(2-\xi-\eta)} + \frac{K_0}{4} \right\} |U_m^{(i)} - U_{m-1}^{(i)}| + \frac{(\nu+1/2)K_1}{(2-\xi-\eta)} |U_m^{(\tau_i)} - U_{m-1}^{(\tau_i)}| + \frac{3K_1}{4}|p_m^{(i)} - p_{m-1}^{(i)}| + \frac{3K_1}{4}|q_m^{(i)} - q_{m-1}^{(i)}| \quad (4.5)$$

for $i = 1, 2$.

Lemma 4.2. *Let the condition Cd1 be fulfiled and there exists a constant $A > 0$, such that*

$$\begin{aligned} |(U_m^{(i)} - U_{m-1}^{(i)})(\xi_0,\eta_0)| &\le A(2-\xi_0-\eta_0)^{-\mu}, \\ |(p_m^{(i)} - p_{m-1}^{(i)})(\xi_0,\eta_0)| &\le \mu A(2-\xi_0-\eta_0)^{-\mu-1}, \\ |(q_m^{(i)} - q_{m-1}^{(i)})(\xi_0,\eta_0)| &\le \mu A(2-\xi_0-\eta_0)^{-\mu-1}, \end{aligned} \quad (4.6)$$

where $\mu \in R_+,\ \mu > \nu = n - 1/2,\ m \in N$. If the parameter δ_ν is such, that

$$(\mu-\nu)(\mu+\nu+1) \ge \delta_\nu \mu(\mu+1) + (3\mu+2\nu+2)K_1 + 2(\mu+2)K_\alpha + K_0\,, \quad (4.7)$$

then for $m \in N, i = 1, 2$ we have

$$\begin{aligned} |(U_{m+1}^{(i)} - U_m^{(i)})(\xi_0,\eta_0)| &\le A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu}, \\ |(p_{m+1}^{(i)} - p_m^{(i)})(\xi_0,\eta_0)| &\le \mu A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu-1}, \\ |(q_{m+1}^{(i)} - q_m^{(i)})(\xi_0,\eta_0)| &\le \mu A(1-\delta_\nu)(2-\xi_0-\eta_0)^{-\mu-1}. \end{aligned}$$

Proof. Step 1. From (4.5) and (4.6) we obtain

$$\begin{aligned} |(E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)| &\le A(2-\xi-\eta)^{-\mu-2} \Big\{ \nu(\nu+1) + \frac{3\mu K_1}{2}(2-\xi-\eta) \\ &+ \frac{K_1}{2}(2-\xi-\eta) + \frac{K_0}{4}(2-\xi-\eta)^2 + (\nu+\frac{1}{2})K_1(2-\xi-\eta) \Big\} \\ &\le A\left\{\nu(\nu+1) + (3\mu+2\nu+2)K_1 + K_0\right\}(2-\xi-\eta)^{-\mu-2}. \end{aligned}$$

From (4.1) we have

$$(U_{m+1}^{(i)} - U_m^{(i)})(\xi_0,\eta_0) = \int_0^{\xi_0} \left(\int_{\xi_0}^{\eta_0} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\, d\eta \right) d\xi + 2\int_0^{\xi_0} \left(\int_0^{\eta} (E_m^{(i)} - E_{m-1}^{(i)})(\xi,\eta)\, d\xi \right) d\eta + \int_0^{\xi_0} \alpha(1-\xi)(U_m^{(i)} - U_{m-1}^{(i)})(\xi,\xi)\, d\xi,$$

$i = 1, 2; m = 0, 1, \ldots,$ and now using Lemma 4.1 we calculate

$$
\begin{aligned}
&|(U^{(i)}_{m+1} - U^{(i)}_m)(\xi_0, \eta_0)| \\
&\le A\left\{\nu(\nu+1) + (3\mu + 2\nu + 2)K_1 + K_0\right\} I_\mu + AK_\alpha \int_0^{\xi_0} (2 - 2\xi)^{-\mu}\, d\xi \\
&\le A(2 - \xi_0 - \eta_0)^{-\mu} \\
&\quad \times \left\{ \frac{\nu(\nu+1) + (3\mu + 2\nu + 2)K_1 + K_0}{\mu(\mu+1)} \right\} + \frac{AK_\alpha}{\mu}(2 - 2\xi_0)^{-\mu} \\
&\le A(2 - \xi_0 - \eta_0)^{-\mu} \\
&\quad \times \left\{ 1 - \frac{(\mu - \nu)(\mu + \nu + 1) - [(3\mu + 2\nu + 2)K_1 + (\mu+1)K_\alpha + K_0]}{\mu(\mu+1)} \right\} \\
&\le A(1 - \delta_\nu)(2 - \xi_0 - \eta_0)^{-\mu}.
\end{aligned}
\tag{4.8}
$$

Step 2. Using (4.1), the estimate (4.5) and the assumption (4.6) gives

$$
\begin{aligned}
&|(p^{(i)}_{m+1} - p^{(i)}_m)(\xi_0, \eta_0)| \le A\left\{\nu(\nu+1) + (3\mu + 2\nu + 2)K_1 + K_0\right\} \\
&\quad \times \int_0^{\eta_0} (2 - \xi - \xi_0)^{-\mu-2}\, d\xi + A\alpha(1 - \xi_0)(2 - 2\xi_0)^{-\mu} \\
&\le \mu A(2 - \xi_0 - \eta_0)^{-\mu-1} \\
&\quad \times \left\{ 1 - \frac{(\mu - \nu)(\mu + \nu + 1) - [(3\mu + 2\nu + 2)K_1 + 2(\mu+1)K_\alpha + K_0]}{\mu(\mu+1)} \right\} \\
&= \mu A(1 - \delta_\nu)(2 - \xi_0 - \eta_0)^{-\mu-1}.
\end{aligned}
$$

Step 3. Obviously considerations as in Steps 1 and 2 lead to

$$
|(q^{(i)}_{m+1} - q^{(i)}_m)(\xi_0, \eta_0)| \le \mu A(1 - \delta_\nu)(2 - \xi_0 - \eta_0)^{-\mu-1}. \qquad \square
$$

Lemma 4.3. *Let now $\nu = n - 1/2, n \in N$ be fixed. If the parameter μ is enough large, $\mu > \nu$, then*

$$
(\mu - \nu)(\mu + \nu + 1) - [(3\mu + 2\nu + 2)K_1 + 2(\mu + 1)K_\alpha + K_0] > 0 \tag{4.9}
$$

and we can choose the parameter $\delta_\nu > 0$, such that the condition (4.7) to be fulfiled.

Proof. We set $\lambda := \mu - \nu$ in (4.9) and get the equivalent inequality for $\lambda > 0:$

$$
\lambda^2 + \lambda(2\nu + 1 - 3K_1 - 2K_\alpha) - [(5\nu + 2)K_1 + 2(\nu + 1)K_\alpha + K_0] > 0 \tag{4.10}
$$

Since

$$\triangle := (2\nu + 1 - 3K_1 - 2K_\alpha)^2 + 4\left[(5\nu + 2)K_1 + 2(\nu + 1)K_\alpha + K_0\right] > 0,$$

the condition (4.9) is equivalent to

$$\lambda > \frac{1}{2}\left\{\sqrt{\triangle} - (2\nu + 1 - 3K_1 - 2K_\alpha)\right\}$$
$$= 2\frac{(5\nu + 2)K_1 + 2(\nu + 1)K_\alpha + K_0}{\sqrt{\triangle} + (2\nu + 1 - 3K_1 - 2K_\alpha)}. \quad (4.11)$$

The proof is completed. □

We now fix $n \in N$ in (2.3) and try to solve the Problem $P_{\alpha,2}$ in $\bar{D}^{(1)}_\varepsilon$, $\varepsilon > 0$, i.e. the system (2.6) with boundary conditions (2.9) where in (2.6) we have

$$F^{(i)}_n = \frac{1}{4\sqrt{2}}(2 - \xi - \eta)^{\frac{1}{2}} f^{(i)}_n(\xi, \eta)$$

and $U^{(i)}_n(\xi, \eta) = \varrho^{\frac{1}{2}} u^{(i)}_n(\varrho, t)$, $i = 1, 2$. Now, we solve this Problem $P_{\alpha,2}$ in $\bar{D}^{(1)}_\varepsilon$, $\varepsilon > 0$, using the successive approximations method (4.1) for the integral equation system (3.13) - (3.15).

Theorem 4.4. *Let $n \in N$ be fixed. Assume:*

(i) $a_1 = b_1 cos\varphi + b_2 sin\varphi$, $a_2 = \varrho^{-1}(b_2 cos\varphi - b_1 sin\varphi)$, b, c are functions of (ϱ, t), $\alpha = \alpha(\varrho)$;

(ii) b_1, b_2, b, $c \in C(\bar{\Omega}_0)$, $\alpha(\varrho) \in C^1([0, 1])$, $f^{(i)}_n \in C(\bar{\Omega}_0)$, $i = 1, 2$;

(iii) the parameter $\mu = \mu_n$ is such large, that

$$(\mu - \nu)(\mu + \nu + 1) > \left[(3\mu + 2\nu + 2)K_1 + 2(\mu + 1)K_\alpha + K_0\right] \quad (4.12)$$

(see Lemma 4.3).

Then there exists a classical solution $(U^{(1)}_n, U^{(2)}_n) \in C^1(\bar{D}^{(1)}_\varepsilon)$, $U^{(i)}_{n,\xi_0\eta_0}(\xi_0, \eta_0) \in C(\bar{D}^{(1)}_\varepsilon)$ of Problem $P_{\alpha,2}$ and the estimates hold:

$$\left|U^{(i)}_n(\xi, \eta)\right| \leq A_\mu \delta^{-1}_\nu (2 - \xi - \eta)^{-\mu}, \quad (4.13)$$
$$\left|U^{(i)}_{n,\xi}(\xi, \eta)\right| \leq \mu A_\mu \delta^{-1}_\nu (2 - \xi - \eta)^{-\mu-1}, \quad (4.14)$$
$$\left|U^{(i)}_{n,\eta}(\xi, \eta)\right| \leq \mu A_\mu \delta^{-1}_\nu (2 - \xi - \eta)^{-\mu-1}, \quad (4.15)$$

where

$$A_\mu := \frac{1}{\mu(\mu + 1)} \max_{\bar{G}_0} \left|\frac{1}{4\sqrt{2}}(2\rho)^{\mu+\frac{5}{2}} f^{(i)}_n(\rho, t)\right|, \quad (4.16)$$

and

$$\delta_\nu := \frac{1}{\mu(\mu+1)} \times \{(\mu-\nu)(\mu+\nu+1) - [(3\mu+2\nu+2)K_1 + 2(\mu+1)K_\alpha + K_0]\}\,. \quad (4.17)$$

Proof. A calculation shows 1) $m = 0$

$$U^{(i)}_{n,0}(\xi,\eta) = p^{(i)}_{n,0}(\xi,\eta) = q^{(i)}_{n,0}(\xi,\eta) \equiv 0 \quad \text{in} \quad \bar{D}^{(1)}_\varepsilon,$$

$$E^{(i)}_{n,0}(\xi,\eta) = F^{(i)}_n(\xi,\eta) = \frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\frac{1}{2}} f^{(i)}_n(\xi,\eta).$$

1) $m = 1$

$$\begin{aligned}
&\left(U^{(i)}_{n,1} - U^{(i)}_{n,0}\right)(\xi_0,\eta_0) \\
&\quad = \int_0^{\xi_0}\left(\int_{\xi_0}^{\eta_0} E^{(i)}_{n,0}(\xi,\eta)\,d\eta\right)d\xi + 2\int_0^{\xi_0}\left(\int_0^{\eta} E^{(i)}_{n,0}(\xi,\eta)\,d\xi\right)d\eta \\
&\quad = \int_0^{\xi_0}\left(\int_{\xi_0}^{\eta_0}(2-\xi-\eta)^{-\mu-2}\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\mu+\frac{5}{2}} f^{(i)}_n(\xi,\eta)\,d\eta\right)d\xi \\
&\quad + 2\int_0^{\xi_0}\left(\int_0^{\eta}(2-\xi-\eta)^{-\mu-2}\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\mu+\frac{5}{2}} f^{(i)}_n(\xi,\eta)\,d\xi\right)d\eta. \quad (4.18)
\end{aligned}$$

With (4.16) and Lemma 4.1 we have

$$|\left(U^{(i)}_{n,1} - U^{(i)}_{n,0}\right)(\xi_0,\eta_0)| \le A_\mu(2-\xi_0-\eta_0)^{-\mu}, \quad (4.19)$$

where

$$A_\mu := \frac{1}{\mu(\mu+1)}\max_{\bar{D}^{(1)}_0}|\frac{1}{4\sqrt{2}}(2-\xi-\eta)^{\mu+\frac{5}{2}} f^{(i)}_n(\xi,\eta)| \equiv \frac{1}{\mu(\mu+1)}\max_{\bar{G}_0}|\frac{1}{4\sqrt{2}}(2\rho)^{\mu+\frac{5}{2}} f^{(i)}_n(\rho,t)|,$$

Likewise we obtain

$$\left(p^{(i)}_{n,1} - p^{(i)}_{n,0}\right)(\xi_0,\eta_0) = \int_0^{\xi_0}\frac{1}{4\sqrt{2}}(2-\xi-\xi_0)^{\frac{1}{2}} f^{(i)}_n(\xi,\xi_0)\,d\xi + \int_{\xi_0}^{\eta_0}\frac{1}{4\sqrt{2}}(2-\xi_0-\eta)^{\frac{1}{2}} f^{(i)}_n(\xi_0,\eta)\,d\eta.$$

and with (4.16) and integration

$$|\left(p^{(i)}_{n,1} - p^{(i)}_{n,0}\right)(\xi_0,\eta_0)| \le \mu A_\mu(2-\xi_0-\eta_0)^{-\mu-1}, \quad (4.20)$$

respectively

$$|\left(q^{(i)}_{n,1}-q^{(i)}_{n,0}\right)(\xi_0,\eta_0)|\le \mu A_\mu(2-\xi_0-\eta_0)^{-\mu-1}. \tag{4.21}$$

Now with Lemma 4.2 and induction, there exist a sequences $\left\{U^{(i)}_{n,m}\right\}$, $\left\{p^{(i)}_{n,m}\right\}$ and $\left\{q^{(i)}_{n,m}\right\}$, $m\in N$, of continuous functions and the estimates

$$\begin{aligned}
&|\left(U^{(i)}_{n,m+1}-U^{(i)}_{n,m}\right)(\xi_0,\eta_0)|\le A_\mu(1-\delta_\nu)^m(2-\xi_0-\eta_0)^{-\mu},\\
&|\left(p^{(i)}_{n,m+1}-p^{(i)}_{n,m}\right)(\xi_0,\eta_0)|\le \mu A_\mu(1-\delta_\nu)^m(2-\xi_0-\eta_0)^{-\mu-1},\\
&|\left(q^{(i)}_{n,m+1}-q^{(i)}_{n,m}\right)(\xi_0,\eta_0)|\le \mu A_\mu(1-\delta_\nu)^m(2-\xi_0-\eta_0)^{-\mu-1}
\end{aligned} \tag{4.22}$$

hold for $m\in N$ and n fixed. According to the condition (4.12), $\delta_\nu>0$.

The functions $\{U^{(i)}_{n,m},p^{(i)}_{n,m},q^{(i)}_{n,m}\}_{m=0}^{\infty}$, n fixed, belong to $C(\bar{D}^{(1)}_\varepsilon)$ and we have uniform convergent to functions $\{U^{(i)}_n,p^{(i)}_n,q^{(i)}_n\}\in C(\bar{D}^{(1)}_\varepsilon)$, as $m\to\infty$ and

$$\left|U^{(i)}_n(\xi_0,\eta_0)\right|\le \Big|\sum_{m=0}^{\infty}(U^{(i)}_{n,m+1}-U^{(i)}_{n,m})(\xi_0,\eta_0)\Big|\le A_\mu\delta_\nu^{-1}(2-\xi_0-\eta_0)^{-\mu},$$

$$\begin{aligned}
&\Big|\frac{\partial U^{(i)}_n(\xi_0,\eta_0)}{\partial\xi_0}\Big|\\
&\qquad\le\Big|\sum_{m=0}^{\infty}(p^{(i)}_{n,m+1}-p^{(i)}_{n,m})(\xi_0,\eta_0)\Big|\le \mu A_\mu\delta_\nu^{-1}(2-\xi_0-\eta_0)^{-\mu-1},
\end{aligned} \tag{4.23}$$

$$\Big|\frac{\partial U^{(i)}_n(\xi_0,\eta_0)}{\partial\eta_0}\Big|\le\Big|\sum_{m=0}^{\infty}(q^{(i)}_{n,m+1}-q^{(i)}_{n,m})(\xi_0,\eta_0)\Big|\le \mu A_\mu\delta_\nu^{-1}(2-\xi_0-\eta_0)^{-\mu-1}.$$

Finally, $U^{(i)}_{n,\xi_0\eta_0}\in C(\bar{D}^{(1)}_\varepsilon)$ follows from (3.13). □

Remark 4.5. Actually, the parameter μ shows the possible singularity of the generalized solution at the point O. According to this, the best choice of it $\mu=\mu_n$ would be closely to the $\inf\mu$ of those values of μ, which satisfy (4.12). The parameter δ_ν, which shows the growth of the constant in the a priori estimate (4.13), one finds from the condition (4.17).

Let now the function $f\in C(\bar{\Omega}_0)$ is a trigonometric polynomial

$$f(\varrho,\varphi,t)=\sum_{n=0}^{l}\left\{f^{(1)}_n(\varrho,t)\cos n\varphi+f^{(2)}_n(\varrho,t)\sin n\varphi\right\}. \tag{4.24}$$

Theorem 4.6. *Let the conditions (i) - (iii) of Theorem 4.4 be fulfiled. If the function $f \in C(\bar{\Omega}_0)$ is of the form (4.24), then there exists one and only one generalized solution $u \in C^1(\bar{\Omega}_0 \backslash O)$ of Problem P_α and the a priori estimates*

$$|u(x,t)| \leq \sum_{n=0}^{l} c_n \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-\alpha_n}$$

$$\sum_{|\beta|=1} |D^\beta u(x,t)| \leq \sum_{n=0}^{l} c_n \alpha_n \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-\alpha_n - 1}$$

hold, where $c_n = \{\mu(\mu+1)\delta_\nu\}^{-1}$ and the orders of singularity $\alpha_n > n$ are such large that

$$\alpha_n^2 \;>\; n^2 + (3\alpha_n + 2n - 1/2)K_1 + 2(\alpha_n + 1/2)K_\alpha + K_0. \tag{4.25}$$

Proof. Note that for a fixed n in case Theorem 4.4 holds, we get from $\varrho^{\frac{1}{2}} u_n^{(i)}(\varrho,t) = U_n^{(i)}(\xi,\eta)$, $i = 1,2$, generalized solutions $(u_n^{(1)}(\varrho,t), u_n^{(2)}(\varrho,t))$ of Problem $\boldsymbol{P_{\alpha,1}}$, smooth in $\bar{G}_0 \setminus (0,0)$ and generalized solutions $u(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi + u_n^{(2)}(\varrho,t)\sin n\varphi$ of Problem P_α , which are smooth in $\bar{\Omega}_0 \backslash O$. The only one difference is that now $\alpha_n = \mu_n + 1/2$ and we must replace the condition (4.12) by (4.25). □

We will treat now the general case of Problem $\boldsymbol{P_\alpha}$ for $f \in C(\bar{\Omega}_0)$, when the trigonometric polynomial (4.24) is replaced by

$$f(\varrho,\varphi,t) = \sum_{n=0}^{\infty} \left\{ f_n^{(1)}(\varrho,t)\cos n\varphi + f_n^{(2)}(\varrho,t)\sin n\varphi \right\}. \tag{4.26}$$

Theorem 4.7. *Let the conditions (i) - (iii) of Theorem 4.4 be fulfiled and the function $f \in C(\bar{\Omega}_0)$ be of the form (4.26). Let the Fourier coefficients $f_n^{(1)}, f_n^{(2)}$ be such that*

$$\sum_{n=1}^{\infty} \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} z^n \;<\; \infty, \qquad \forall z \;>\; 0. \tag{4.27}$$

Then there exists one and only one generalized solution $u \in C^1(\bar{\Omega}_0 \backslash O)$ of Problem P_α and the a priori estimates

$$|u(x,t)| \leq \sum_{n=0}^{\infty} c_n \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-\alpha_n},$$

$$\sum_{|\beta|=1} |D^\beta u(x,t)| \leq \sum_{n=0}^{\infty} c_n \alpha_n \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-\alpha_n - 1}$$

hold, where the orders of singularity $\alpha_n = n + \frac{5}{2}K_1 + K_\alpha + \varepsilon$ $(\varepsilon > 0)$ *and* $c_n = 1/n$*, for each enough large number* $n \in N$.

Proof. In view of Theorem 4.6 it is enough to check only the values of α_n and c_n. Using Lemma (4.3) and computing the condition (4.25) with the above values of $\alpha_n = \mu_n + 1/2$, we see that it is equivalent to

$$\mu(\mu+1)\delta_\nu := 2n\varepsilon + \left(\frac{5}{2}K_1 + K_\alpha + \varepsilon\right)^2 - \frac{1}{2}\left(15K_1 + 6K_\alpha + 6\varepsilon - 1\right)K_1 - (5K_1 + 2K_\alpha + 2\varepsilon + 1)K_\alpha - K_0 > 0,$$

for $n \geq N_1(K_1, K_\alpha, K_0, \varepsilon)$ and then $\mu(\mu+1)\delta_\nu \geq n\varepsilon$ for $n \geq N_2(K_1, K_\alpha, K_0, \varepsilon)$. □

Remark 4.8. According to the Proof of the above Theorem 4.7, one can choose even $\varepsilon = \varepsilon_n \to 0$, if ε_n are such that $n\varepsilon_n \to \infty$.

Remark 4.9. Compare with the exact results of singularity of order $\alpha_n = n$ for the Problem $P2$ for the wave equation without lower order terms Popivanov and Popov [27], we see that the order of singularity here $\alpha_n = n + \frac{5}{2}K_1 + K_\alpha$, for large $n \in N$, is possible higher with the fixed value $(\frac{5}{2}K_1 + K_\alpha)$, which does not depend on n and K_0.

Remark 4.10. Let mention here the difference between the condition (4.27) and the analogous one for existence of generalized solution in the case, when f is not trigonometric polynomial :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2d)} \exp\left(\frac{8n(n+3d)}{\varepsilon^2}\right) \times \left(\|f_n^{(1)}\|_{C^0} + \|f_n^{(2)}\|_{C^0}\right) < \infty$$

(see Grammatikopoulos et al [11]).

5. EXAMPLES

Example 5.1. We consider Problem $P2$ for the wave equation, i.e.

$$P2 \begin{cases} \Box u := u_{x_1x_1} + u_{x_2x_2} - u_{tt} = \dfrac{1}{\varrho}(\varrho u_\varrho)_\varrho + \dfrac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} = f \quad \text{in } \Omega_0, \\ u|_{\Sigma_1} = 0, \quad u_t|_{\Sigma_0} = 0. \end{cases} \tag{5.1}$$

Thus we have $b_1 = b_2 = b = c \equiv 0$; $a_1 = a_2 \equiv 0$, $\alpha(\varrho) \equiv 0$. First of all, for

$$f = f_n^{(1)}(\varrho, t)\cos n\varphi + f_n^{(2)}(\varrho, t)\sin n\varphi, \qquad n \in \mathbb{N},$$

we ask for generalized solutions in a form

$$u(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t)\cos n\varphi + u_n^{(2)}(\varrho, t)\sin n\varphi.$$

From (4.3) we see $K_1 = K_0 = K_\alpha \equiv 0$. Assumption (3) in Theorem 4.4 holds for $\mu > \nu = n - \frac{1}{2}$, such that we choose $\mu - \nu = 1/n$ and thus by (4.17)

$$\mu(\mu+1)\delta_\nu = (\mu-\nu)(\mu+\nu+1) = \frac{2n^2+1}{n^2}. \tag{5.2}$$

According to Theorem 4.6

$$|u_n^{(i)}(\varrho,t)| \le \left\{\max_{\bar{G}_0} |\varrho^n f_n^{(i)}(\varrho,t)|\right\} \varrho^{-n-\frac{1}{n}} \tag{5.3}$$

With this estimate and Theorem 4.7 we formulate

Theorem 5.1. *Suppose that in the problem* (5.1) *with*

$$f(x,t) = \sum_{n=0}^{\infty} \left\{ f_n^{(1)}(|x|,t)\cos n\varphi + f_n^{(2)}(|x|,t)\sin n\varphi \right\}$$

there is a sequence $\{a_n\}_{n=1}^{\infty}$ *with* $n^{-1}a_n \to +\infty$, *as* $n \to \infty$, *such that for the Fourier coefficients* $f_n^{(1)}$ *and* $f_n^{(2)}$ *of the function* $f(x,t)$

$$\sum_{n=0}^{\infty} \max_{\bar{G}_0} \left\{ \varrho^n [|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)|] \right\} \exp(a_n) < \infty. \tag{5.4}$$

Then there exists one and only one generalized solution $u(x,t)$ *of the Problem* (5.1), *for which the a priori estimates*

$$\begin{aligned} |u(x,t)| &\le \sum_{n=0}^{\infty} \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-n-\frac{1}{n}}, \\ \sum_{|\beta|=1} |D^\beta u(x,t)| & \\ &\le \sum_{n=0}^{\infty} n \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-n-1-\frac{1}{n}} \end{aligned} \tag{5.5}$$

hold.

Proof. Obviously, the condition (5.4) implies the convergence of the series (4.27) in this case. Thus, from the above calculations and Theorem 4.7 the estimates (5.5) follow immediately. □

Example 5.2. In Ω_0 we consider the Problem P_α with $\alpha = 0$:

$$\begin{aligned} &u_{x_1x_1} + u_{x_2x_2} - u_{tt} + cu = f \qquad \text{in } \Omega_0, \\ &u|_{\Sigma_1} = 0, \quad u_t|_{\Sigma_0} = 0. \end{aligned} \tag{5.6}$$

where $c = c(|x|,t)$.

Remark 5.2. Note that an equation

$$L[\tilde{u}] = \Box\tilde{u} + b_1\tilde{u}_{x_1} + b_2\tilde{u}_{x_2} + b_3\tilde{u}_t + \tilde{c}\tilde{u} = f,$$

where b_1, b_2, b_3 are constants and $\tilde{c} = \tilde{c}(|x|, t)$ is a continuous function, can be transform to an equation (5.6) by $u = \tilde{u}\exp\{\frac{b_1}{2}x_1 + \frac{b_2}{2}x_2 - \frac{b}{2}t\}$ with $c = \tilde{c} - \frac{1}{4}(b_1^2 + b_2^2 - b^2)$.

First of all we are looking for a solution of Problem (5.6) for the function $f(x, t)$ of the form

$$f = f_n^{(1)}(\varrho, t)\cos n\varphi + f_n^{(2)}(\varrho, t)\sin n\varphi,$$

in which case we ask for generalized solutions

$$u(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t)\cos n\varphi + u_n^{(2)}(\varrho, t)\sin n\varphi$$

of (5.6) in $\bar{\Omega}_0\backslash(0, 0, 0)$. According to notations (4.3), now $K_1 = K_\alpha \equiv 0$, $K_0 = \max|c|$. Condition (4.25) in Theorem 4.6 becomes

$$\alpha_n^2 > n^2 + K_0,$$

which is satisfied if we choose $\alpha_n := n + K_0/2n$. With this choice of α_n we have the following requirement for the value (4.17) of δ_ν :

$$\mu(\mu + 1)\delta_\nu := (\alpha_n - n)(\alpha_n + n) - K_0 = \frac{K_0}{2n}(2n + \frac{K_0}{2n}) - K_0 = \left(\frac{K_0}{2n}\right)^2.$$

Thus, according to Theorem 4.6 it follows

$$|u_n^{(i)}(\varrho, t)| \leq \frac{4n^2}{K_0^2}\left\{\max_{\bar{G}_0}|\varrho^n f_n^{(i)}(\varrho, t)|\right\}\varrho^{-n-\frac{K_0}{2n}} \tag{5.7}$$

Theorem 5.3. *If for the problem* (5.6) $f_n^{(i)}(\varrho, t), c(\varrho, t) \in C(\bar{\Omega}_0)$, *then there exists a generalized solution*

$$u(\varrho, \varphi, t) = u_n^{(1)}(\varrho, t)\cos n\varphi + u_n^{(2)}(\varrho, t)\sin n\varphi$$

and the estimate (5.7) *holds.*

From Theorem 5.3 and Theorem 4.7 we obtain

Theorem 5.4. *Suppose that in the problem* (5.6) *with*

$$f(x, t) = \sum_{n=0}^{\infty}\left\{f_n^{(1)}(|x|, t)\cos n\varphi + f_n^{(2)}(|x|, t)\sin n\varphi\right\},$$

there is a sequence $\{a_n\}_{n=1}^{\infty}$ *with* $n^{-1}a_n \to +\infty$, *as* $n \to \infty$, *such that*

$$\sum_{n=0}^{\infty}\max_{\bar{G}_0}\left\{(\varrho^n[|f_n^{(1)}(\varrho, t)| + |f_n^{(2)}(\varrho, t)|]\right\}\exp(a_n) < \infty.$$

Then there exists one and only one generalized solution $u(x,t)$ of the Problem (5.6), for which the a priori estimates

$$|u(x,t)| \le C_0 \sum_{n=0}^{\infty} n^2 \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-n-\frac{K_0}{n}}$$

$$\sum_{|\beta|=1} |D^\beta u(x,t)| \tag{5.8}$$

$$\le C_0 \sum_{n=0}^{\infty} n^3 \left\{ \max_{\bar{G}_0} \varrho^n \left[|f_n^{(1)}(\varrho,t)| + |f_n^{(2)}(\varrho,t)| \right] \right\} |x|^{-n-1-\frac{K_0}{n}}$$

hold, where $C_0 = 4K_0^{-2}$.

Example 5.3. For the equation

$$Lu = \frac{1}{\varrho}(\varrho u_\varrho)_\varrho + \frac{1}{\varrho^2}u_{\varphi\varphi} - u_{tt} + a_1 u_\varrho + b u_t + cu = f(\varrho,\varphi,t) \text{ in } \Omega_0, \tag{5.9}$$

we consider the boundary conditions of Problem P_α, i.e. (1.4), and prove the following result.

Theorem 5.5. *Let $\alpha(\varrho) \ge 0$; a_1, b, $c \in C^1(\bar{\Omega}_0 \backslash O)$, depend only on $(|x|,t)$ and*

$$\begin{gathered} a_1(|x|,t) \ge |b|(|x|,t), \quad |x|[a_1(|x|,t) - 2|x|c(|x|,t)] \ge c_0 > 0, \\ (|x|,t) \in \Omega_0, \end{gathered} \tag{5.10}$$

with some positive constant c_0. Then for arbitrary $n \in N$, $n \ge 4$, there exists a function $f_n(x,t) \in C^{n-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0)$, for which the corresponding generalized solution $u_n(x,t)$ of Problem P_α belongs to $C^2(\bar{\Omega}_0 \backslash O)$ and satisfies the estimate

$$|u_n(x,t)|_{t=|x|} \ge \frac{1}{2}|u_n(2x,0)| + |x|^{-(n+\sigma)}|\cos n(\arctan \frac{x_2}{x_1})|, \qquad 0 < |x| < 1/4, \tag{5.11}$$

with some positive constant σ.

Proof. Let the number $n \in N$, $n \ge 4$, is arbitrary and fixed. Denote $\sigma = \left(\sqrt{n^2 + c_0/2} - n\right) > 0$, i.e. $2\sigma(2n+\sigma) = c_0$. Now, consider the special case of Problem P_α:

$$Lu = f_n \equiv \varrho^{-n-\sigma}(\varrho^2 - t^2)^{n+\sigma-1/2} \cos n\varphi \quad \text{in } \Omega_0. \tag{5.12}$$

Observe also that

$$f_n(x_1,x_2,t) = (x_1^2 + x_2^2)^{-n-\sigma/2}(x_1^2 + x_2^2 - t^2)^{n+\sigma-1/2} \operatorname{Re}(x_1 + ix_2)^n$$

and obviously $f_n \in C^{n+[\sigma]-2}(\bar{\Omega}_0) \cap C^\infty(\Omega_0)$. Theorem 5.1 from Grammatikopoulos et al [11] states that the Problem P_α for equation (5.12) has at most one generalized solution. On the other hand, from Theorem 5.2 from Grammatikopoulos et al [11] it is known that, for the above right-hand side, there exists a generalized solution in Ω_0 of the form

$$u_n(\varrho,\varphi,t) = u_n^{(1)}(\varrho,t)\cos n\varphi \in C^2(\bar{\Omega}_0\backslash O).$$

As in Section 2, by setting $u_n^{(2)}(\varrho,t) = \varrho^{\frac{1}{2}} u_n^{(1)}(\varrho,t)$ and substituting

$$\xi = 1-\varrho-t, \quad \eta = 1-\varrho+t, \tag{5.13}$$

the equation (5.12), with boundary conditions (1.4), in view of

$$U(\xi,\eta) = u_n^{(2)}(\varrho(\xi,\eta),t(\xi,\eta)), \tag{5.14}$$

becomes a Darboux-Goursat Problem $P_{\alpha,3}$:

$$U_{\xi\eta} - AU_\xi - BU_\eta - CU = F(\xi,\eta), \tag{5.15}$$

$$U(0,\eta) = 0, \quad (U_\eta - U_\xi)(\xi,\xi) + \alpha(1-\xi)U(\xi,\xi) = 0. \tag{5.16}$$

Remind, that the coefficients of (5.15) are defined as follows:

$$\begin{aligned} A &= \frac{1}{4}(a_1+b) \geq 0, \quad B = \frac{1}{4}(a_1-b) \geq 0, \\ C(\xi,\eta) &= \frac{1}{4}\Big(\frac{4n^2-1}{(2-\eta-\xi)^2} + \frac{a_1(\xi,\eta)}{2-\eta-\xi} - c(\xi,\eta)\Big) \geq 0, \end{aligned} \tag{5.17}$$

$$F(\xi,\eta) = 2^{n+\sigma-\frac{5}{2}}\Big[\frac{(1-\xi)(1-\eta)}{2-\eta-\xi}\Big]^{n+\sigma-\frac{1}{2}} \in C^{n+[\sigma]-1}(\bar{D}_\varepsilon^{(1)}),$$

$$F(\xi,\eta) \geq 0, \tag{5.18}$$

where $D_\varepsilon^{(1)} := \{(\xi,\eta) : 0<\xi<\eta<1, 0<\xi<1-\varepsilon\}$, $\varepsilon > 0$ and we preserve the same notations for a_1, b and c in the new coordinates (ξ,η). Next, in view of Theorem 3.2 and Lemma 3.1 from Grammatikopoulos et al [11], we formulate the following result.

Proposition 5.6. *There exists a classical solution* $U(\xi,\eta) \in C^2(\bar{D}_0 \setminus (1,1))$ *for the problem* (5.15), (5.16) *for which*

$$U(\xi,\eta) \geq 0, \quad U_\xi(\xi,\eta) \geq 0, \quad U_\eta(\xi,\eta) \geq 0 \quad \text{in } \bar{D}_\varepsilon^{(1)}.$$

Set

$$K = \int_{D_{\frac{1}{2}}^{(1)}} F^2(\xi,\eta)\, d\xi d\eta > 0. \tag{5.19}$$

Then for $0 < \varepsilon < 1/2$ it follows from (5.15) that

$$0 < K \le \int_{D_\varepsilon^{(1)}} F^2(\xi,\eta)\, d\xi d\eta = \int_{D_\varepsilon^{(1)}} (U_{\xi\eta}F)(\xi,\eta) d\xi\, d\eta - \int_{D_\varepsilon^{(1)}} [(AU_\xi + BU_\eta + CU)F](\xi,\eta)\, d\xi d\eta. \quad (5.20)$$

By (5.18), it is obvious that $F(\xi,1) = 0$. So, in view of (5.16), (5.20) becomes

$$0 < K \le -\int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta - \int_0^{1-\varepsilon} [U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\eta(\xi,\xi)]\, d\xi + \int_{D_\varepsilon^{(1)}} \{(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta)\}\, d\xi\, d\eta. \quad (5.21)$$

An elementary calculation shows that

$$\begin{aligned} F_\xi(\xi,\eta) &= -(n+\sigma-\frac{1}{2})\frac{1-\eta}{(1-\xi)(2-\xi-\eta)}F(\xi,\eta) \le 0,\\ F_\eta(\xi,\eta) &= -(n+\sigma-\frac{1}{2})\frac{1-\xi}{(1-\eta)(2-\xi-\eta)}F(\xi,\eta) \le 0,\\ F_\xi(\xi,\xi) &= F_\eta(\xi,\xi) = -\frac{1}{8}(n+\sigma-\frac{1}{2})(1-\xi)^{n+\sigma-\frac{3}{2}} \end{aligned} \quad (5.22)$$

and

$$F_{\xi\eta}(\xi,\eta) - \frac{4(n+\sigma)^2-1}{4(2-\eta-\xi)^2}F(\xi,\eta) = 0. \quad (5.23)$$

So, because of (5.10), (5.17) and Proposition 5.6,

$$(F_{\xi\eta} - CF)U - F(AU_\xi + BU_\eta) = -\left\{\frac{a_1(\xi,\eta)}{2-\eta-\xi} - c(\xi,\eta) - \frac{4\sigma(2n+\sigma)}{(2-\xi-\eta)^2}\right\}\frac{FU}{4} - F(AU_\xi + BU_\eta) \le 0.$$

Thus, we find

$$0 < K \le -\int_0^{1-\varepsilon} \{U_\xi(\xi,\xi)F(\xi,\xi) + U(\xi,\xi)F_\eta(\xi,\xi)\}\, d\xi - \int_{1-\varepsilon}^{1} U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\, d\eta, \quad (5.24)$$

where, as it is easy to check from (5.22)

$$F_\eta(\xi,\xi) = \frac{1}{2}[F(\xi,\xi)]_\xi. \quad (5.25)$$

The function $U(\xi,\eta)$ is a classical solution of (5.15), (5.16) in $\bar{D}_\varepsilon$, $\varepsilon\in(0,1)$ with

$$U_\xi(\xi,\xi)=\frac{1}{2}[U(\xi,\xi)]_\xi+\frac{1}{2}\alpha(1-\xi)U(\xi,\xi). \tag{5.26}$$

If we substitute (5.25) and (5.26) into (5.24), we get

$$\begin{aligned}K\le&-\frac{1}{2}\int_0^{1-\varepsilon}\{F(\xi,\xi)U(\xi,\xi)\}_\xi\,d\xi\\&-\frac{1}{2}\int_0^{1-\varepsilon}\alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\,d\xi-\int_{1-\varepsilon}^{1}U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\,d\eta\\&=-\frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)-\frac{1}{2}\int_0^{1-\varepsilon}\alpha(1-\xi)U(\xi,\xi)F(\xi,\xi)\,d\xi\\&\qquad-\int_{1-\varepsilon}^{1}U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\,d\eta.\end{aligned}\tag{5.27}$$

Next, in view of Proposition 5.6 and the properties of the function $F(\xi,\eta)$, we find

$$U(\xi,\eta)\ge0,\quad U_\eta(\xi,\eta)\ge0,\quad \alpha(\xi)\ge0,\quad F(\xi,\eta)\ge0,$$
$$F_\eta(\xi,\eta)\le0\qquad\text{in }\bar{D}_\varepsilon^{(1)},$$

which together with (5.27) and because of $F(1-\varepsilon,1)=0$ implies

$$\begin{aligned}K&\le-\int_{1-\varepsilon}^{1}U(1-\varepsilon,\eta)F_\eta(1-\varepsilon,\eta)\,d\eta-\frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\&=\int_{1-\varepsilon}^{1}U(1-\varepsilon,\eta)|F_\eta(1-\varepsilon,\eta)|\,d\eta-\frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\&\le\int_{1-\varepsilon}^{1}U(1-\varepsilon,1)|F_\eta(1-\varepsilon,\eta)|\,d\eta-\frac{1}{2}(FU)(1-\varepsilon,1-\varepsilon)\\&=\left\{U(1-\varepsilon,1)-\frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right\}F(1-\varepsilon,1-\varepsilon).\end{aligned}$$

Since $F(1-\varepsilon,1-\varepsilon)=\frac{1}{4}\varepsilon^{n+\sigma-\frac{1}{2}}$, we have

$$0<K\le\frac{1}{4}\left\{U(1-\varepsilon,1)-\frac{1}{2}U(1-\varepsilon,1-\varepsilon)\right\}\varepsilon^{n+\sigma-\frac{1}{2}}.$$

For $\xi=1-\varepsilon$, $\eta=1$ we have $\varrho=t=\varepsilon/2$ and so

$$0<4K\varepsilon^{\frac{1}{2}-n-\sigma}\le u_n^{(2)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)-\frac{1}{2}u_n^{(2)}(\varepsilon,0). \tag{5.28}$$

Finally, the inverse transformation gives

$$u_n^{(1)}\left(\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right)\ge\frac{1}{2}u_n^{(1)}(\varepsilon,0)+\tilde{C}_1\varepsilon^{-(n+\sigma)}\ge\tilde{C}_1\varepsilon^{-(n+\sigma)},\quad 0<\varepsilon<\frac{1}{2},$$

where $\tilde{C}_1 = 2^{\frac{5}{2}}K$. Multiplying the function u_n by $\tilde{C}_1^{-1}$, we see that (5.11) holds for the solution of equation $Lu = \tilde{C}_1^{-1} f_n$ with boundary conditions of Problem P_α. The proof is complete. □

Remark 5.7. Note that the second of conditions (5.10) can be satisfied only if the coefficients a_1 or c grow up like ϱ^{-1} and ϱ^{-2} respectively.

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