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SMOOTHNESS OF THE TRANSITION DENSITIES CORRESPONDING TO CERTAIN STOCHASTIC EQUATIONS ON A HILBERT SPACE

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ABSTRACT: We consider the transition semigroup determined by a stochastic semilinear equation, where the nonlinear part of the drift is the gradient of a function. Under suitable conditions, we prove that the kernel of this semigroup, $p_t(x, y)$, studied in [Simão, *Semigroup Forum*, **71** (2005), no. 1, 49-72], is continuously differentiable on $H \times H$ and we give formulae for the derivatives of the functions $p_t(\cdot, y)$ and $p_t(x, \cdot)$.

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1. INTRODUCTION

In this paper, we consider the transition semigroup $P_t,\ t \geq 0$ corresponding to the stochastic evolution equation

$$dX(t) = [AX(t) + F(X(t)]dt + dW(t), \quad X(0) = x, \tag{1.1}$$

on a separable Hilbert space H, where W is a cylindrical Wiener process on H, A is a selfadjoint operator with a nuclear inverse, $F = DU$, where $U : H \to \mathbb{R}$ is measurable and Gâteaux differentiable on H, and DU has at most linear growth. In Simão [19] an explicit formula was obtained for the kernel $p_t(x, y)$ of P_t, with respect to the centered Gaussian measure on H with covariance $-\frac{1}{2}A^{-1}$, under the additional assumptions that U is in

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the domain of the Ornstein-Uhlenbeck generator, and U is the limit of a sequence of smooth cylindrical functions satisfying certain conditions. The aim of this paper is to prove that $p_t(\cdot,\cdot) : H \times H \to \mathbb{R}$ is smooth, and to give formulae for the derivatives of the functions $p_t(\cdot, y)$ and $p_t(x, \cdot)$.

The paper is organized as follows. In Section 2 we present our setting, introduce the notation and summarize some results to be used in the next section. In Section 3 the results of the paper are stated and proved. The main result is Proposition 3.6, where we provide conditions for the smoothness of $p_t(\cdot,\cdot)$ and give formulae for the derivatives of $p_t(\cdot, y)$ and $p_t(x, \cdot)$.

The regularity of transition probabilities and transition semigroups, in infinite dimentions, were investigated by several authors under different sets of conditions. For work related to this paper, we refer to Da Prato, Elworthy and Zabczyk [1], Da Prato and Zabczyk [6], Fuhrman [7], Fuhrman [8], Fuhrman [9], Gaveau and Moulinier [11], Peszat and Zabczyk [14], Simão [16], and Simão [18], and references therein.

2. NOTATION AND AUXILIARY RESULTS

Let H be a separable Hilbert space with norm $\|\cdot\|$ and inner product $< \cdot, \cdot >$, and let $A : D(A) \subset H \to H$ be a negative self-adjoint operator such that A^{-1} is nuclear. Then, there exists an orthonormal basis of H consisting of eigenvectors of A, $\{e_n\}_{n=1}^{\infty}$, with corresponding eigenvalues $(-\lambda_n)_{n\in\mathbb{N}}$ satisfying, $\lambda_n \geq \lambda_0 > 0$, $n \in \mathbb{N}$, and $\sum_{n=1}^{\infty} \lambda_n^{-1} < +\infty$. We denote by γ the Gaussian measure with mean 0 and covariance operator $-\frac{1}{2}A^{-1}$.

Let W be a cylindrical Wiener process on H with identity covariance, defined on a probability space $(\Omega, \mathcal{F}, P)$. The stochastic evolution equation on H,

$$dZ(t) = AZ(t)dt + dW(t), \quad X(s) = x, \quad 0 \leq s \leq t < \infty, \tag{2.1}$$

has a unique mild solution, $Z^{s,x}(t)$, which has almost surely continuous paths in H (cf. Da Prato and Zabczyk [4]).

Let R_t, $t \geq 0$, be the transition semigroup corresponding to equation (2.1),

$$R_t\varphi(x) = \mathbb{E}(\varphi(Z^x(t)),$$

where $\varphi : H \to \mathbb{R}$ is a bounded Borel function. The measure γ is an invariant measure for the semigroup R_t, $t \geq 0$, and therefore R_t, $t \geq 0$, has a unique extension to a strongly continuous semigroup of contractions on $L^2(\gamma)$.

We denote by $\mathcal{F}C_b^{\infty}(H)$ the space of smooth cylindrical functions

$$\mathcal{F}C_b^{\infty}(H) = \{f : H \to \mathbb{R} | f(x) = \varphi(< e_1, x >, \cdots, < e_n, x >), n \in \mathbb{N}, \varphi \in C_b^{\infty}(\mathbb{R}^n)\}.$$

The infinitesimal generator of the semigroup $R_t,\ t \geq 0$, $(L, \mathcal{D}\mathrm{om}(L))$, is the closure of the Ornstein-Uhlenbeck operator L_0 with domain $\mathcal{F}C_b^\infty$

$$L_0 f(x) = \frac{1}{2}\mathrm{Tr} D^2 f(x) + < x, ADf(x) >, \quad f \in \mathcal{F}C_b^\infty(H),$$

(cf. Da Prato and Zabczyk [5], Da Prato [2] and Da Prato [3]).

We know from Gaveau [10] and Miyahara [13] that

$$(R_t\varphi)(x) = \int_H \varphi(y) q_t(x,y) d\gamma(y),$$

where

$$q_t(x,y) = \prod_{n\in\mathbb{N}} q_t^n(x_n, y_n),$$

$x_n =< x, e_n >,\ y_n =< y, e_n >$, and

$$q_t^n(x_n, y_n) = (1 - e^{-2\lambda_n t})^{-\frac{1}{2}} \exp\left(-\frac{(y_n - e^{-\lambda_n t} x_n)^2}{\lambda_n^{-1}(1 - e^{-2\lambda_n t})} + \frac{y_n^2}{\lambda_n^{-1}} \right). \tag{2.2}$$

Lemma 2.1. (Miyahara [13]) *For all $t \in \mathbb{R}_+$, $q_t(x,y)$ is continuous on $H \times H$.*

Set

$$I_s(x,z) = Q_s^{-1} S_s (z - S_s x),\ \ \forall x,\ z \in H, s \in (0,t], \tag{2.3}$$

where S_t denotes the C_0 semigroup on H generated by A and $Q_t = \int_0^t S_{2u} du$.

The following result is an easy consequence of (2.2) and Lemma 2.1.

Lemma 2.2. *For each $t \in \mathbb{R}_+$, the function, $q_t(x,y)$ is continuously differentiable on $H \times H$ and we have*

$$\frac{\partial}{\partial x} q_t(x,y) = q_t(x,y) I_t(x,y),$$

and

$$\frac{\partial}{\partial y} q_t(x,y) = q_t(x,y) I_t(y,x).$$

Fix $y \in H$, and $T \in \mathbb{R}_+$. Consider the stochastic differential equation

$$\begin{aligned} d\widehat{X}(t) &= [A\widehat{X}(t) + D_1 \log q_{T-t}(\widehat{X}(t), y)]dt + dW(t),\ \ t \in (s,T), \\ \widehat{X}(s) &= x, \end{aligned} \tag{2.4}$$

where D_1 denotes the derivative with respect to the first variable and $s \in [0, T)$. This equation has a unique mild solution on the interval $[s, T)$, given by

$$\widehat{X}(t) = \sum_{i=1}^{\infty} \widehat{X}_n(t) e_n, \ t \in [s, T),$$

where

$$\begin{aligned} \widehat{X}_n(t) &= e^{-\lambda_n(t-s)} \frac{1 - e^{-2\lambda_n(T-t)}}{1 - e^{-2\lambda_n(T-s)}} x_n + e^{\lambda_n(T-t)} \frac{e^{-2\lambda_n(T-t)} - e^{-2\lambda_n(T-s)}}{1 - e^{-2\lambda_n(T-s)}} y_n \\ &+ \frac{1 - e^{-2\lambda_n(T-t)}}{e^{-\lambda_n(T-t)}} \int_s^t \frac{e^{-\lambda_n(T-u)}}{1 - e^{-2\lambda_n(T-u)}} dw_n(u), \end{aligned}$$

and $w_n(t) =< W(t), e_n >$. Equation (2.4) defines a nonhomogeneous Markov process with transition densities with respect to γ given by

$$p(s, x; t, z) = \frac{q_{t-s}(x, z) q_{T-t}(z, y)}{q_{T-s}(x, y)}$$

(cf. Simão [15] and Simão [17]). From Lemma 2.2, we obtain the following lemma.

Lemma 2.3. *For each $t \in \mathbb{R}_+$, $z \in H$, the function $H \to \mathbb{R}$, $x \to p(0, x; t, z)$ is continuously differentiable on H, and*

$$\frac{\partial}{\partial x} p(0, x; t, z) = p(0, x; t, z)(I_t(x, z) - I_T(x, y)).$$

Consider the process

$$X_{T,y}^{s,x}(t) = \widehat{X}(t), \ \ t \in (s, T), \ \ X_{T,y}^{s,x}(s) = x, \ \ X_{T,y}^{s,x}(T) = y. \qquad (2.5)$$

It was proved in Simão [17], that $X_{T,y}^{s,x}(t)$ has a version with continuous paths in H. The process $X_{T,y}^{s,x}(t)$ is the Ornstein-Uhlenbeck process given by the equation (2.2) conditioned to go from x at time $t = s$ to y at time $t = T$. Therefore, for every bounded measurable function $\phi : C([s, T], H) \to \mathbb{R}$, we have

$$\mathbb{E}(\phi(Z^{s,x}) | Z^{s,x}(t) = y) = \mathbb{E}(\phi(X_{T,y}^{s,x})).$$

3. MAIN RESULT AND ITS PROOF

Let $U : H \to \mathbb{R}$ be Gâteaux differentiable on H, and such that $\|DU(x)\| \leq C(1 + \|x\|)$, for all $x \in H$, for some constant C. Under these conditions, the equation

$$dX(t) = [AX(t) + F(X(t)]dt + dW(t), \quad X(0) = x,$$

has a weakly unique martingale solution (cf. Da Prato and Zabczyk [4]). Let $P_t,\ t \geq 0$, be the corresponding transition semigroup,

$$P_t\varphi(x) = \mathbb{E}(\varphi(X^x(t))),$$

where $\varphi : H \to \mathbb{R}$ is a bounded Borel function.

We introduce some further assumptions on the function U.

Hypothesis 3.1. 1) $U : H \to \mathbb{R}$ is Gâteaux differentiable on H;

2) there exists a constant C such that $\|DU(x)\| \leq C(1 + \|x\|)$, for all $x \in H$;

3) $U \in \mathcal{D}\mathrm{om}(L)$;

4) there exists a sequence of functions $U_n \in \mathcal{F}C_b^\infty(H)$ such that:

(i) $U_n(x) \to U(x)$ for all $x \in H$; $\|DU_n - DU\| \to 0$ and $LU_n \to LU$, in $L^2(H, \gamma)$, as $n \to \infty$;

(ii) there exist constants $C_1 > 0$ and $\alpha \geq 1$ such that for all $n \in \mathbb{N},\ x \in H$, $\|DU_n(x)\| \leq C_1(1 + \|x\|^\alpha)$;

(iii) there exist constants $C_2 > 0$ and $\beta \geq 2$ such that for all $n \in \mathbb{N},\ x \in H$, $|LU_n(x)| \leq C_2(1 + \|x\|^\beta)$.

Hypothesis 3.2. $U \in \mathcal{D}\mathrm{om}(L)$, and the equivalence class of functions $LU + \frac{1}{2}\|DU\|^2 \in L^2(H, \gamma)$ has a representative $V : H \to \mathbb{R}$ which is bounded from below.

Remarks 3.3. 1) Hypothesis 3.2 was introduced by Jona-Lasinio and Sénéor [12], and was also used in Fuhrman [8].

2) If $U : H \to R$ is Gâteaux differentiable, $U \in \mathcal{D}\mathrm{om}(L)$, and the following conditions hold, then U satisfies Hypothesis 3.1 (cf. Simão [19]):

(i) $\mathrm{Im}(DU) \subset \mathcal{D}\mathrm{om}(A)$, and there exists $C_1 > 0$ such that $\|ADU(x)\| \leq C_1(1 + \|x\|)$ for all $x \in H$;

(ii) there exists $C_2 > 0$ such that for all $n \in \mathbb{N}$,

$$\left| \sum_{i=1}^{n} (\partial_i^2 U(x) + \lambda_i x_i \partial_i U(x)) \right| \leq C_2(1 + \|x\|^2),$$

γ-almost everywhere, where ∂_i denotes the closure in $L^2(H, \gamma)$ of the operator $\partial_i f(x) =< Df, e_i >,\ f \in \mathcal{F}C_b^\infty(H)$.

It was proved in Simão [19] that if U satisfies Hypotheses 3.1 and 3.2, then

$$P_t\varphi(x) = \int_H \varphi(y) p_t(x, y) d\gamma(y),$$

where

$$p_t(x,y) = q_t(x,y)e^{U(y)}e^{-U(x)}\mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{0,x}(s))ds\Big\}\right),$$

where $V : H \to \mathbb{R}$ denotes a representative of $L_0 U + \frac{1}{2}\|DU\|^2 \in L^2(H,\gamma)$ bounded from below.

Lemma 3.4. *Let $V : H \to \mathbb{R}$ be a measurable function, bounded from below and such that for some $\beta \geq 2$, $|V(x)| \leq K(1+\|x\|^\beta)$, for γ-almost all $x \in H$. Then, for all $t \in \mathbb{R}_+$,*

$$\mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{0,x}(u))du\Big\}\right)$$

is a continuous function of (x,y) on $H \times H$.

Proof. See Simão [19]. □

From Lemmas 2.1 and 3.4 we obtain the following.

Lemma 3.5. *If U satisfies Hypotheses 3.1 and 3.2, then for each $t \in \mathbb{R}_+$, the function $p_t(x,y)$ is continuous on $H \times H$.*

The main result of this paper is the following.

Proposition 3.6. *Assume that the operator $(-A)^{-\frac{1}{2}}$ is nuclear, and that U satisfies Hypotheses 3.1 and 3.2. Let $V : H \to \mathbb{R}$ be a representative of $L_0 U + \frac{1}{2}\|DU\|^2$, such that $V(x) \geq -C$ for all $x \in H$, for some constant $C \geq 0$. Then, for each $t \in \mathbb{R}_+$, the functions $H \to \mathbb{R}$, $\phi_1 = p_t(\cdot, y)$ and $\phi_2 = p_t(x,\cdot)$, are Gâteaux differentiable and we have*

$$\begin{aligned} D\phi_1(x) = {} & \Big(I_t(x,y) - DU(x)\Big)p_t(x,y) \\ & + e^{U(x)}\int_0^t\int_H (V+C)(z)e^{Cs}e^{U(z)}\Big(I_s(x,z) - I_t(x,y)\Big) \\ & \qquad \times q_s(x,z)p_{t-s}(z,y)d\gamma(z)ds, \end{aligned}$$

and

$$\begin{aligned} D\phi_2(y) = {} & \Big(I_t(y,x) + DU(y)\Big)p_t(x,y) \\ & + e^{U(y)}\int_0^t\int_H (V+C)(z)e^{Cs}e^{-U(z)}\Big(I_s(y,z) - I_t(y,x)\Big) \\ & \qquad q_s(y,z)p_{t-s}(x,z)d\gamma(z)ds, \end{aligned}$$

where $I_s(\cdot,\cdot)$ is defined by (2.3).

If in addition U is Fréchet differentiable, then the functions ϕ_1 and ϕ_2, are Fréchet differentiable; moreover, if $DU : H \to L(H, \mathbb{R})$ is continuous, then for each $t \in \mathbb{R}_+$, the function $p_t(\cdot,\cdot) : H \times H \to \mathbb{R}$, is continuously differentiable.

We will prove the Proposition by means of several lemmas.

Lemma 3.7. *Let $V : H \to \mathbb{R}$ be a Borel function, such that $V(x) \geq 0$ and $|V(x)| \leq K(1 + \|x\|^{\beta})$, for all $x \in H$, for some constants $K > 0$ and $\beta \geq 2$. Then, we have*

$$\begin{aligned} &\mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{0,x}(u))du\Big\}\right) \\ = \;& 1 - \int_0^t \int_H V(z)\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\right)p(0,x;s,z)d\gamma(z)ds. \end{aligned}$$

Proof. Step 1. We first prove the formula under the assumption that V is bounded. For $t \in \mathbb{R}_+,\ s \in [0,t],\ z \in H$, we have

$$\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\right) = \sum_{k=0}^{\infty} \frac{1}{k!}\,\mathbb{E}\left(\Big[-\int_s^t V(X_{t,y}^{s,z}(u))du\Big]^k\right).$$

Define, for each $k \in \mathbb{N}$,

$$T_k(s,z;t,y) = \frac{1}{k!}\mathbb{E}\left(\Big[\int_s^t V(X_{t,y}^{s,z}(u))du\Big]^k\right).$$

Then,

$$\mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{s,z}(u))du\Big\}\right) = \sum_{k=0}^{\infty} (-1)^k\, T_k(s,z;t,y).$$

It was proved in Simão [19], by adapting a proof in Van Casteren [20], that

$$T_{k+1}(s,z;t,y) = \int_s^t \int_H V(h)T_k(r,h;t,y)p(s,z;r,h)d\gamma(h)dr.$$

Using this we obtain

$$\begin{aligned}
&\mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{0,x}(u))du\Big\}\right) = 1 + \sum_{k=0}^{\infty}(-1)^{k+1}\, T_{k+1}(0,x;t,y) \\
&= 1 + \sum_{k=0}^{\infty}(-1)^{k+1}\int_0^t\int_H V(z)T_k(s,z;t,y)p(0,x;s,z)d\gamma(z)ds \\
&= 1 - \int_0^t\int_H V(z)\left[\sum_{k=0}^{\infty}(-1)^k T_k(s,z;t,y)\right]p(0,x;s,z)d\gamma(z)ds \\
&= 1 - \int_0^t\int_H V(z)\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\right)p(0,x;s,z)d\gamma(z)ds.
\end{aligned}$$

Step 2. For each $k \in \mathbb{N}$ and $x \in H$, let $V_k(x) = \min(V(x), k)$. Then V_k is bounded and so by Step 1

$$\begin{aligned}
&\mathbb{E}\left(\exp\Big\{-\int_0^t V_k(X_{t,y}^{0,x}(u))du\Big\}\right) \\
&\qquad = 1 - \int_0^t\int_H V_k(z)\mathbb{E}\left(\exp\Big\{-\int_s^t V_k(X_{t,y}^{s,z}(u))du\Big\}\right) \\
&\qquad\qquad \times p(0,x;s,z)d\gamma(z)ds. \qquad (3.1)
\end{aligned}$$

Fix $s \in [0,t)$ and $z \in H$. Since $V \geq 0$, we have

$$\begin{aligned}
&\left|\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\right) - \mathbb{E}\left(\exp\Big\{-\int_s^t V_k(X_{t,y}^{s,z}(u))du\Big\}\right)\right| \\
&\qquad \leq \int_s^t \mathbb{E}\Big(|V(X_{t,y}^{s,z}(u)) - V_k(X_{t,y}^{s,z}(u))|\Big)du. \qquad (3.2)
\end{aligned}$$

On the other hand

$$|V(X_{t,y}^{s,z}(u)(\omega)) - V_k(X_{t,y}^{s,z}(u)(\omega))| \to 0,$$

as $k \to \infty$, ($P\times$Lebesgue)-almost everywhere on $\Omega \times [0,t]$, and

$$|V(X_{t,y}^{s,z}(u)) - V_k(X_{t,y}^{s,z}(u))| \leq 2K(1 + \|X_{t,y}^{s,z}(u)\|^\beta)\ 2K(1 + \|X_{t,y}^{s,z}\|_{C([0,t],H)}).$$

The function on the right hand side of this inequality is P-integrable on Ω by Fernique's Theorem, therefore, we can deduce that

$$\mathbb{E}\left(\int_0^t |V(X_{t,y}^{s,z}(u)) - V_k(X_{t,y}^{s,z}(u))|du\right) \to 0,$$

as $k \to \infty$. This, together with (3.2) implies that for fixed $s \in [0,t)$ and $z \in H$, we have

$$\mathbb{E}\left(\exp\Big\{-\int_0^t V_k(X_{t,y}^{s,z}(u))du\Big\}\right) \to \mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{s,z}(u))du\Big\}\right) \quad (3.3)$$

as $k \to \infty$.

On the other hand,

$$|V_k(z)\mathbb{E}\left(\exp\Big\{-\int_s^t V_k(X_{t,y}^{s,z}(u))du\Big\}\right)p(0,x;s,z)| \leq K(1+\|z\|^\beta)p(0,x;s,z),$$

and we have

$$\int_0^t\int_H (1+\|z\|^\beta)p(0,x;s,z)d\gamma(z)ds = \int_0^t \Big(1+\mathbb{E}(\|X_{t,y}^{0,x}(s)\|^\beta)\Big)ds \leq t\Big(1+\mathbb{E}(\|X_{t,y}^{0,x}\|_{C([0,t],H)}^\beta)\Big) < +\infty.$$

This together with (3.3) implies that we can pass to the limit in equation (3.1), hence the formula holds for V. □

Lemma 3.8. *Let $V : H \to \mathbb{R}$ be a Borel measurable function, such that $V(x) > 0$, and $|V(x)| \leq K(1+\|x\|^\beta)$, for all $x \in H$, for some constants K and $\beta \geq 2$. Then, for each $t \in \mathbb{R}_+$, the function $\psi : H \times H \to \mathbb{R}$,*

$$\psi(x,y) = \mathbb{E}\left(\exp\Big\{-\int_0^t V(X_{t,y}^{0,x}(u))du\Big\}\right),$$

is continuously differentiable, and we have

$$\frac{\partial}{\partial x}\psi(x,y) = \int_0^t\int_H V(z)\,\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\right)\Big(I_s(x,z)-I_t(x,y)\Big) \times p(0,x;s,z)\,d\gamma(z)ds, \quad (3.4)$$

and

$$\frac{\partial}{\partial y}\psi(x,y) = \int_0^t\int_H V(z)\,\mathbb{E}\left(\exp\Big\{-\int_s^t V(X_{t,x}^{s,z}(u))du\Big\}\right)\Big(I_s(y,z)-I_t(y,x)\Big) \times p(0,y;s,z)\,d\gamma(z)ds. \quad (3.5)$$

Proof. Step 1. Fix $t \in \mathbb{R}_+, s \in (0,t)$. Set

$$F_s(x,y,z) = V(z)\mathbb{E}\bigg(\exp\Big\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Big\}\bigg)\frac{q_s(x,z)q_{t-s}(z,y)}{q_t(x,y)}, \qquad \forall x,\ z \in H.$$

From Lemma 2.2 it follows that for fixed $y, z \in H$ the function $H \to \mathbb{R}$, $x \to F_s(x,y,z)$, is Fréchet differentiable and

$$\frac{\partial}{\partial x}F_s(x,y,z) = F_s(x,y,z)(I_s(x,z) - I_t(x,y)).$$

Moreover, $\frac{\partial}{\partial x}F_s(\cdot,\cdot,z)$ is a continuous function on $H \times H$, by Lemma 2.1 and Lemma 3.4. We have

$$\Big\|\frac{\partial}{\partial x}F_s(x,,y,z)\Big\|_{L(H,\mathbb{R})} \leq KC_{s,t}(1+\|z\|^\beta)(\|x\|+\|y\|+\|z\|)\frac{q_s(x,z)q_{t-s}(z,y)}{q_t(x,y)}, \tag{3.6}$$

where $C_{s,t}$ is a constant which depends on s and t. Fix $x_0, y_0 \in H$. An easy calculation shows that

$$q_s(x,z) \leq q_s(x_0,z)\exp\left\{2\|x-x_0\|\sum_{n=1}^{\infty} e^{-\lambda_n s}\frac{|z_n - e^{-\lambda_n s}x_{0,n}|}{\lambda_n^{-1}(1-e^{-2\lambda_n s})}\right\}$$

$$\leq q_s(x_0,z)\exp\left\{2\|x-x_0\|\Big(\|z\|+\|x_0\|\Big)\sum_{n=1}^{\infty}\frac{\lambda_n e^{-\lambda_n s}}{1-e^{-2\lambda_n s}}\right\},$$

$$q_{t-s}(z,y) \leq q_{t-s}(z,y_0)\exp\left\{2\|y-y_0\|\sum_{n=1}^{\infty} e^{-\lambda_n (t-s)}\frac{|z_n - e^{-\lambda_n (t-s)}x_{0,n}|}{\lambda_n^{-1}(1-e^{-2\lambda_n (t-s)})}\right\}$$

$$\leq q_{t-s}(z,y_0)\exp\left\{2\|y-y_0\|\Big(\|z\|+\|x_0\|\Big)\sum_{n=1}^{\infty}\frac{\lambda_n e^{-\lambda_n (t-s)}}{1-e^{-2\lambda_n (t-s)}}\right\},$$

$$\frac{1}{q_t(x,y)} \le \frac{1}{q_t(x_0,y_0)}$$

$$\times \exp\left\{ \Big(\|x-x_0\|^2 + \|y-y_0\|^2 + \|x-x_0\|\|y-y_0\| \Big) \sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n t}}{1-e^{-2\lambda_n t}} \right\}$$

$$\times \exp\left\{ 2\Big[(\|x-x_0\| + \|y-y_0\|)(\|x_0\| + \|y_0\|) \Big] \sum_{n=1}^{\infty} \frac{\lambda_n e^{-\lambda_n t}}{1-e^{-2\lambda_n t}} \right\}.$$

These inequalities imply that for all $x, y \in H$ such that $\|x-x_0\|+\|y-y_0\| < R$, we have

$$\frac{q_s(x,z)q_{t-s}(z,y)}{q_t(x,y)} \le \frac{q_s(x_0,z)q_{t-s}(z,y_0)}{q_t(x_0,y_0)} \exp\{C_{R,t,s}(1+\|x_0\|+\|y_0\|)\} \exp\{C_{R,t,s}\,\|z\|\}, \quad (3.7)$$

where $C_{R,t,s}$ is a constant depending only on R, t and $s \in (0,t)$. It follows from (3.6) and (3.7) that, for all $x, y \in H$ such that $\|x-x_0\|+\|y-y_0\| < R$, we have

$$\Big\| \frac{\partial}{\partial x} F_s(x,y,z) \Big\|_{L(H,\mathbb{R})} \le K C_s (1+\|z\|^{\beta})(2R + \|x_0\| + \|y_0\| + \|z\|)$$

$$\times \exp\{C_{R,t,s}(1+\|x_0\|+\|y_0\|)\} \exp\{C_{R,t,s}\,\|z\|\} \frac{q_s(x_0,z)q_{t-s}(z,y_0)}{q_t(x_0,y_0)}$$

$$\le C_{K,s,\beta} \exp\{C'_{R,t,s}(1+\|x_0\|+\|y\|_0)\}$$

$$\times \exp\{C'_{R,t,s}\|z\|\} \frac{q_s(x_0,z)q_{t-s}(z,y_0)}{q_t(x_0,y_0)}, \quad (3.8)$$

where $C_{K,s,\beta}$ and $C'_{R,t,s}$ are constants, depending on K, s, β, and R, t, s, respectively. The right hand side of the last inequality in (3.8) is a γ-integrable function of z, because

$$\int_H \exp\{C'_{R,t,s}\|z\|\} \frac{q_s(x_0,z)q_{t-s}(z,y_0)}{q_t(x_0,y_0)} d\gamma(z)$$

$$= \mathbb{E}\left(\exp\left\{ C'_{R,t,s}\|X^{0,x_0}_{t,y_0}(s)\| \right\} \right) < +\infty,$$

by Fernique's Theorem. Therefore, we can conclude that for each $y \in H$ the function $G_s(\cdot, y) : H \to \mathbb{R}$,

$$G_s(x,y) = \int_H F_s(x,y,z)d\gamma(z)$$
$$= \int_H V(z)\mathbb{E}\biggl(\exp\Bigl\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Bigr\}\biggr)p(0,x;s,z)d\gamma(z),$$

is Fréchet differentiable, with derivative given by

$$\frac{\partial}{\partial x}G_s(x,y) = \int_H V(z)\mathbb{E}\biggl(\exp\Bigl\{-\int_s^t V(X_{t,y}^{s,z}(u))du\Bigr\}\biggr)$$
$$\times (I_s(x,z) - I_t(x,y))p(0,x;s,z)d\gamma(z), \quad (3.9)$$

and moreover, that the mapping $\frac{\partial}{\partial x}G_s : H \times H \to L(H,\mathbb{R})$ is continuous.

Step 2. Fix $t \in \mathbb{R}_+$. We will show that, for each $y \in H$, the function $H \to \mathbb{R}$, $x \to \int_0^t G_s(x,y)ds$, is Fréchet differentiable, that

$$\frac{\partial}{\partial x}\int_0^t G_s(x,y)ds = \int_0^t \frac{\partial}{\partial x}G_s(x,y)ds, \quad (3.10)$$

and that the mapping $H \times H \to L(H,\mathbb{R})$, $(x,y) \to \frac{\partial}{\partial x}\int_0^t G_s(x,y)ds$ is continuous.

To this end it is enough to prove that given $(x_0, y_0) \in H \times H$ and $R > 0$, there exists a function $f \in L^1([0,t],\mathbb{R})$ such that

$$\left\| \frac{\partial}{\partial x}G_s(x,y) \right\|_{L(H,\mathbb{R})} \le f(s),$$

for all $s \in (0,t)$ and for all $x, y \in H$ such that $\|x - x_0\| + \|y - y_0\| < R$.

From (3.9) we obtain the estimate

$$\begin{aligned}
\left\| \frac{\partial}{\partial x} G_s(x,y) \right\|_{L(H,\mathbb{R})} &\leq K \int_H (1+\|z\|^\beta) \Big(\|I_s(x,z)\| + \|I_t(x,y)\| \Big) \\
&\qquad \times p(0,x;s,z) d\gamma(z) \\
\leq K \int_H (1+\|z\|^\beta) \Big(\|Q_s^{-1} S_s(z - S_s x)\| &+ \|Q_t^{-1} S_t\|_{L(H)} \|z\| + \|Q_t^{-1} S_{2t}\|_{L(H)} \|x\| \Big) \\
&\qquad \times p(0,x;s,z) d\gamma(z) \\
&= K\,\mathbb{E}\Bigg(\Big(1 + \|X_{t,y}^{0,x}(s)\|^\beta\Big) \|Q_s^{-1} S_s(X_{t,y}^{0,x}(s) - S_s x)\| \Bigg) \\
&\quad + K\|Q_t^{-1} S_t\|_{L(H)} \mathbb{E}\Bigg(\Big(1 + \|X_{t,y}^{0,x}(s)\|^\beta\Big) \|X_{t,y}^{0,x}(s)\| \Bigg) \\
&\quad + K\|Q_t^{-1} S_{2t}\|_{L(H)} \|x\| \mathbb{E}\Bigg(1 + \|X_{t,y}^{0,x}(s)\|^\beta \Bigg) \\
&\leq 2K \left[\mathbb{E}\Big(1 + \|X_{t,y}^{0,x}(s)\|^{2\beta}\Big) \right]^{\frac{1}{2}} \left[\mathbb{E}\Big(\|Q_s^{-1} S_s(X_{t,y}^{0,x}(s) - S_s x)\|^2 \Big) \right]^{\frac{1}{2}} \\
&\qquad + C_{K,t}(1+\|x\|)\mathbb{E}\Big(1 + \|X_{t,y}^{0,x}(s)\|^{\beta+1}\Big), \quad (3.11)
\end{aligned}$$

where $C_{K,t}$ is a constant depending on K and t.

An easy calculation will show that

$$\begin{aligned}
< Q_s^{-1} S_s(X_{t,y}^{0,x}(s) - S_s x), e_n > &= \frac{2\lambda_n e^{-2\lambda_n t}}{1 - e^{-2\lambda_n t}} (y_n - x_n) \\
&+ \frac{2\lambda_n e^{-2\lambda_n s + \lambda_n t} (1 - e^{-2\lambda_n (t-s)})^2}{(1 - e^{-2\lambda_n s}) e^{-\lambda_n (t-s)}} \int_0^s \frac{e^{-\lambda_n (t-u)}}{1 - e^{-2\lambda_n (t-u)}} dw_n(u),
\end{aligned}$$

and from this we obtain

$$\left[\mathbb{E}\Big(\|Q_s^{-1}S_s(X_{t,y}^{0,x}(s)-S_sx)\|^2\Big)\right]^{\frac{1}{2}} \le C_t\|y-x\|$$

$$+\left[\sum_{n=1}^{\infty}\frac{8\lambda_n^2 e^{-4\lambda_n s+2\lambda_n t}(1-e^{-2\lambda_n(t-s)})^4}{(1-e^{-2\lambda_n s})^2 e^{-2\lambda_n(t-s)}}\mathbb{E}\left(\left|\int_0^s \frac{e^{-\lambda_n(t-u)}}{1-e^{-2\lambda_n(t-u)}}dw_n(u)\right|^2\right)\right]^{\frac{1}{2}}$$

$$= C_t\|y-x\| + 2\sqrt{2}\left[\sum_{n=1}^{\infty}\frac{\lambda_n e^{-2\lambda_n s}(1-e^{-2\lambda_n(t-s)})}{(1-e^{-2\lambda_n s})(1-e^{-2\lambda_n t})}\right]^{\frac{1}{2}}$$

$$\le C_t\|y-x\| + 2\sqrt{2}\sum_{n=1}^{\infty}\frac{\lambda_n^{\frac{1}{2}}e^{-\lambda_n s}}{(1-e^{-2\lambda_n s})^{\frac{1}{2}}},$$

where C_t is a constant depending only on t. This together with (3.11) implies that

$$\begin{aligned}\left\|\frac{\partial}{\partial x}G_s(x,y)\right\|_{L(H,\mathbb{R})} &\le C'_{K,t}\left(1+\left[\mathbb{E}\Big\|X_{t,y}^{0,x}(s)\|^{2\beta}\Big)\right]^{\frac{1}{2}}\right)\\ &\quad\times\left(\|y-x\|+\sum_{i=1}^{\infty}\frac{\lambda_n^{\frac{1}{2}}e^{-\lambda_n s}}{(1-e^{-2\lambda_n s})^{\frac{1}{2}}}\right)\\ &\quad + C'_{K,t}(1+\|x\|)\left(1+\mathbb{E}\|X_{t,y}^{0,x}(s)\|^{\beta+1}\right)\Bigg),\end{aligned} \tag{3.12}$$

where $C'_{K,t}$ is a constant depending on K and t.

Fix $x_0, y_0 \in H$ and $R > 0$. It is easy to check that

$$\|X_{t,y}^{0,x}(s) - X_{t,y_0}^{0,x_0}(s)\| \le \sqrt{2}(\|x-x_0\|+\|y-y_0\|),$$

therefore, for all x, y satisfying $\|x-x_0\|+\|y-y_0\| < R$, we have

$$\|X_{t,y}^{0,x}(s)\| \le C_R(1+\|X_{t,y_0}^{0,x_0}(s)\|),$$

for a constant C_R depending on R. This inequality and (3.12), imply that for all x, y such that $\|x - x_0\| + \|y - y_0\| < R$, we have,

$$\begin{aligned}\left\|\frac{\partial}{\partial x} G_s(x,y)\right\|_{L(H,\mathbb{R})} &\leq C_{t,K,R,\beta}\left(1 + \left[\mathbb{E}\Big(\|X^{0,x_0}_{t,y_0}(s)\|^{2\beta}\Big)\right]^{\frac{1}{2}}\right) \\ &\quad \times \left(1 + \|y_0 - x_0\| + \sum_{n=1}^{\infty} \frac{\lambda_n^{\frac{1}{2}} e^{-\lambda_n s}}{(1 - e^{-2\lambda_n s})^{\frac{1}{2}}}\right) \\ &\quad + C_{t,K,R,\beta}(1 + \|x_0\|)\left(1 + \mathbb{E}\Big(\|X^{0,x_0}_{t,y_0}(s)\|^{\beta+1}\Big)\right), \qquad (3.13)\end{aligned}$$

where $C_{t,K,R,\beta}$ is a constant depending on t, K R and β.

For all $s \in [0,t]$ and for all $p \geq 1$,

$$\mathbb{E}\left(\|X^{0,x_0}_{t,y_0}(s)\|^p\right) \leq \mathbb{E}\left(\|X^{0,x_0}_{t,y_0}\|^p_{C([0,t],H)}\right) \leq +\infty,$$

by Fernique's Theorem, therefore (3.13) gives

$$\left\|\frac{\partial}{\partial x} G_s(x,y)\right\|_{L(H,\mathbb{R})} \leq C_{t,K,R,\beta,x_0,y_0}\left(1 + \sum_{n=1}^{\infty} \frac{\lambda_n^{\frac{1}{2}} e^{-\lambda_n s}}{(1 - e^{-2\lambda_n s})^{\frac{1}{2}}}\right), \qquad (3.14)$$

where $C_{t,K,R,\beta}, x_0, y_0$ is a constant depending on t, K R, β, x_0 and y_0.

We have

$$\int_0^t \left(\sum_{n=1}^{\infty} \frac{\lambda_n^{\frac{1}{2}} e^{-\lambda_n s}}{(1 - e^{-2\lambda_n s})^{\frac{1}{2}}}\right) ds \leq 2 \sum_{n=1}^{\infty} \lambda_n^{-\frac{1}{2}} < +\infty,$$

therefore, the right hand side of (3.14) is an integrable finction of s on $[0,t]$. This completes the proof of Step 2.

Step 3. From Steps 1 and 2 and Lemma 3.7, it follows that

$$\psi(x,y) = \mathbb{E}\left(\exp\Big\{-\int_0^t V(X^{0,x}_{t,y}(u))du\Big\}\right), \quad x, y \in H,$$

is Fréchet differentiable as a function of x, $\dfrac{\partial}{\partial x}\psi(x,y)$ is given by (3.4) and the mapping $H \times H \to L(H,\mathbb{R})$, $(x,y) \to \dfrac{\partial}{\partial x}\psi(x,y)$ is continuous.

Since the processes $X^{0,x}_{t,y}(u)$ and $X^{0,y}_{t,x}(t-u)$ have the same law, we have for all $x, y \in H$

$$\begin{aligned}\mathbb{E}\left(\exp\left\{-\int_0^t V(X^{0,x}_{t,y}(u))du\right\}\right) &= \mathbb{E}\left(\exp\left\{-\int_0^t V(X^{0,y}_{t,x}(t-u))du\right\}\right)\\ &= \mathbb{E}\left(\exp\left\{-\int_0^t V(X^{0,y}_{t,x}(u))du\right\}\right),\end{aligned}$$

and this implies that $\psi(x,y)$ is Fréchet differentiable as a function of y, $\frac{\partial}{\partial y}\psi(x,y)$ is given by (3.5) and the mapping

$$H \times H \to L(H,\mathbb{R}), \quad (x,y) \to \frac{\partial}{\partial x}\psi(x,y)$$

is continuous.

We can then deduce that the function $\psi : H \times H \to \mathbb{R}$, is continuously differentiable, and this completes the proof of the lemma. □

Proof of Proposition 3.6. The result follows from Lemmas 2.2 and 3.8. The formulas for $D\phi_1(x)$ $D\phi_2(x)$ are obtained using (3.4) and (3.5) and the fact that

$$\mathbb{E}\left(\exp\left\{-\int_s^t V(X^{s,z}_{t,y}(u))du\right\}\right) = \mathbb{E}\left(\exp\left\{-\int_0^{t-s} V(X^{0,z}_{t-s,y}(u))du\right\}\right). \quad \square$$

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