Communications in Applied Analysis **10** (2006), no. 2, 133–148

THE GIBBS PHENOMENON FOR JACOBI EXPANSIONS

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Communicated by D.D. Bainov

ABSTRACT: The selection of the "best" Jacobi approximation of discontinuous functions is addressed. The selection criteria is the decreasing of the Gibbs constant and the increasing of the steepness.

AMS (MOS) Subject Classification: 65D10

1. INTRODUCTION

The Gibbs phenomenon occurs when a discontinuous function is approximated by polynomials or other families of smooth functions. Let us consider for illustration the case of the 2-periodic function defined on $[-1, 1]$ by $u(x) = x - \text{sign}\,(x)$ with the Fourier expansion

$$u = \sum_{k=-\infty}^{+\infty} \hat{u}_k e^{ikx}.$$

The truncated series

$$\pi_N u = \sum_{k=-N}^{N} \hat{u}_k e^{ikx}$$

for $N = 20$ and $N = 60$ is displayed in Figure 1: we notice oscillations, especially near the discontinuity. If the parameter N increases, the size of the oscillations decreases excepted near the discontinuity where $\mathcal{O}(1)$ oscillations (called overshoot or undershoot) remain. This is due to the lack of uniform convergence of the expansion or in other terms the slow decays of the Fourier

Received October 13, 2005

coefficients $\hat{u}_k$. We refer the reader to the books Gottlieb and Orszag [3] and Jerri [4]. The later is devoted to the Gibbs phenomenon in numerical approximations and contains a very rich bibliography on this topic.

In this work, we analyze the Gibbs phenomenon for non periodic function approximated by their Jacobi expansions. The main question we want to answer is the following: given a discontinuous function, let us say the Sign function, what is the best corresponding Jacobi expansion? Best means here:

- decreasing the overshoot/undershoot characterized by the Gibbs constant of the expansion,
- increasing the steepness of the approximation.

We introduce in Section 2 the Jacobi expansions of discontinuous functions. In Section 3, we analyze these expansions and try to optimize them with respect to the Jacobi parameter.

Numerous properties of the Jacobi polynomials can be found in many books on the topic, especially the excellent book by Szegö [5]. For sake of completeness, the remainder of this introductory section is devoted to recall some properties used here.

For $\lambda \in \mathbb{R}$, we denote $(C_n^{(\lambda)})_{n\geq 0}$ the family of polynomials defined by

$$C_0^{(\lambda)}(x) = 1, \quad C_1^{(\lambda)}(x) = \begin{cases} x & \text{for } \lambda = 0, \\ 2\lambda x & \text{for } \lambda \neq 0, \end{cases}$$

and for $n \geq 1$,

$$(n+1)C_{n+1}^{(\lambda)}(x) = 2(n+\lambda)xC_n^{(\lambda)}(x) - (n+2\lambda-1)C_{n-1}^{(\lambda)}(x). \qquad (1)$$

The $C_n^{(\lambda)}$ are called Gegenbauer (or ultraspherical) polynomials. We will call them, simply, the Jacobi polynomials[1]. The best known examples of Jacobi polynomials are the Legendre polynomials $P_n = C_n^{(1/2)}$, the Chebyshev polynomials (of the first kind) $T_n = \frac{n}{2}C_n^{(0)}$ also defined on $[-1,1]$ by $T_n(\cos\theta) = \cos(n\theta)$ and the Chebyshev polynomials of the second kind $U_n = C_n^{(1)}$ also defined on $[-1,1]$ by $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$.

From the recurrence relation (1), we compute $k_n^{(\lambda)}$ the coefficient of x^n in $C_n^{(\lambda)}(x)$: $k_0^{(0)} = 1$, $k_n^{(0)} = \frac{2^n}{n}$ $(n > 0)$, and for $\lambda \neq 0$

$$k_n^{(\lambda)} = \frac{2^n}{n!}(\lambda)_n \simeq \frac{2^n}{\Gamma(\lambda)}n^{\lambda-1}, \qquad n \to +\infty, \qquad (2)$$

by the Stirling asymptotic formula. We used here the Pochhammer symbol (or shifted factorial) defined for a real number z and an integer n by

$$(z)_n = \begin{cases} 1 & \text{for } n = 0, \\ z(z+1)\dots(z+n-1) & \text{for } n \geq 1. \end{cases}$$

[1]Jacobi polynomials depend on two parameters α and β. Here $\alpha = \beta = \lambda - 1/2$.

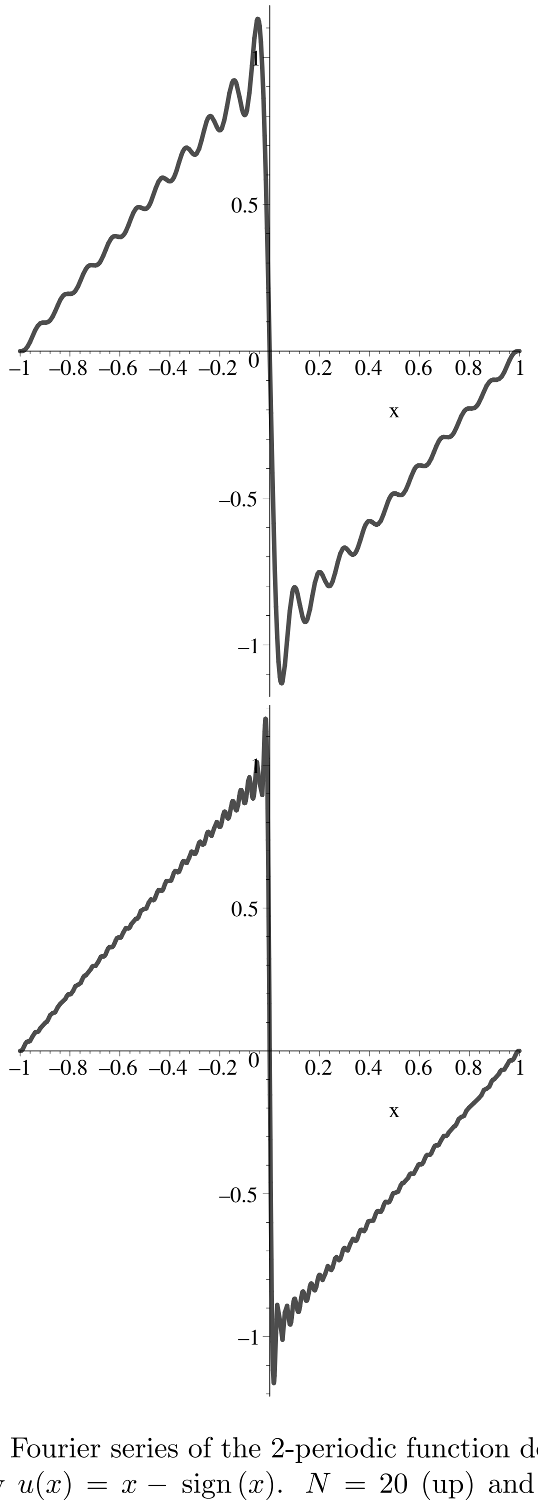

Figure 1. Fourier series of the 2-periodic function defined on $[-1, 1]$ by $u(x) = x - \operatorname{sign}(x)$. $N = 20$ (up) and $N = 60$ (down)

If $-z \notin \mathbb{N}$, we get the equality $(z)_n = \Gamma(z+n)/\Gamma(z)$, with Γ the Euler's Gamma function. The recurrence formula allows also the computation of $C_n^{(\lambda)}(1)$:

$$C_n^{(\lambda)}(1) = \frac{1}{n!}(2\lambda)_n \simeq \frac{1}{\Gamma(2\lambda)} n^{2\lambda-1}, \tag{3}$$

excepted for $\lambda = 0$ in which case $C_n^{(0)}(1) = \frac{2}{n}$ for $n \geq 1$ and $C_0^{(0)}(1) = 1$.

From now, we only consider the case $\lambda > -1/2$. Let ω_λ be the integrable function (called *weight*) defined on $I =]-1, 1[$ by $\omega(x) = (1-x^2)^{\lambda-1/2}$. The space $L^2_\lambda(I)$ of the square summable functions f in the sense that the integral $\int_{-1}^1 f^2(x)\omega_\lambda(x)dx$ is finite is a Hilbert space for the inner product $\langle f, g\rangle_\lambda = \int_{-1}^1 f(x)g(x)\omega_\lambda(x)dx$. The polynomials $(C_n^{(\lambda)})_{n\geq 0}$ are orthogonal with respect to this inner product and $(C_n^{(\lambda)}/\|C_n^{(\lambda)}\|_\lambda)_{n\geq 0}$ form a Hilbertian basis of $L^2_\lambda(I)$. We use here the standard notation $\|f\|_\lambda = \sqrt{\langle f, f\rangle_\lambda}$. The recurrence relation (1) gives also the norm of $C_n^{(\lambda)}$ (Szegö [5], equation 4.7.15): $\|C_0^{(0)}\|_0^2 = \pi$, $\|C_n^{(0)}\|_0^2 = \frac{2\pi}{n^2}$ for $n \geq 1$ and for $\lambda \neq 0$

$$\|C_n^{(\lambda)}\|_\lambda^2 = \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2} \simeq \frac{\sqrt{\pi}2^{1/2-\lambda}}{\Gamma(\lambda)} n^{\lambda-1}. \tag{4}$$

There are several formulas linking Jacobi polynomials associated to "close" values of λ, we will use this one for $\lambda \neq 0$ and $n \geq 1$ (see Szegö [5], equation 4.7.14)

$$[C_n^{(\lambda)}]' = 2\lambda C_{n-1}^{(\lambda+1)}. \tag{5}$$

For $\lambda = 0$, the relation is

$$[C_n^{(0)}]' = 2U_{n-1}.$$

The boundaries of the interval $x = \pm 1$ and its center play a crucial role. The values of $C_n^{(\lambda)}$ at $x = \pm 1$ are given by (3) and the values at the origin are $C_{2m+1}^{(\lambda)}(0) = 0$, $C_0^{(0)}(0) = 1$, $C_{2m}^{(0)}(0) = \frac{(-1)^m}{m}$ $(m > 0)$, and for $\lambda \neq 0$

$$C_{2m}^{(\lambda)}(0) = \frac{(-1)^m}{m!}(\lambda)_m \simeq \frac{(-1)^m}{\Gamma(\lambda)} m^{\lambda-1}. \tag{6}$$

From the relation (5) we get the derivatives at $x = 0$: $[C_{2m}^{(\lambda)}]'(0) = 0$ and

$$[C_{2m+1}^{(\lambda)}]'(0) = \begin{cases} 2(-1)^m, & \text{if } \lambda = 0, \\ \\ 2\dfrac{(-1)^m}{m!}(\lambda)_{m+1} \simeq 2\dfrac{(-1)^m}{\Gamma(\lambda)} m^\lambda, & \text{if } \lambda \neq 0. \end{cases} \tag{7}$$

The polynomial $C_n^{(\lambda)}$ takes its maximal value at ± 1 or near 0 depending on λ: for $\lambda \in]-\frac{1}{2}, 0[$, the maximum occurs in the center of the interval and for $\lambda > 0$, the maximum is reached at the boundaries ± 1 (see Theorem 7.33.1 of Szegö [5])

$$|C_n^{(\lambda)}(x)| \leq |C_n^{(\lambda)}(\pm 1)| \simeq \frac{1}{\Gamma(2\lambda)} n^{2\lambda - 1}. \tag{8}$$

In the simple case $\lambda = 0$ the following upper bound holds:

$$\forall n \neq 0, \quad |C_n^{(0)}(x)| \leq \frac{2}{n}.$$

The polynomial $C_n^{(\lambda)}$ is a solution of the differential equation

$$\Big(\omega_{\lambda+1}[C_n^{(\lambda)}]'\Big)' + n(n+2\lambda)\omega_\lambda C_n^{(\lambda)} = 0. \tag{9}$$

Finally, we recall a key ingredient in remarkable identities involving orthogonal polynomials, namely the Christoffel-Darboux formula: for $n \in \mathbb{N}$ and (x, y) two distinct points in $[-1, 1]$ (Theorem 3.2.2 of Szegö [5])

$$\sum_{m=0}^{n} \frac{C_m^{(\lambda)}(x) C_m^{(\lambda)}(y)}{\|C_m^{(\lambda)}\|_\lambda^2} = \frac{k_n^{(\lambda)}}{k_{n+1}^{(\lambda)} \|C_n^{(\lambda)}\|_\lambda^2} \; \frac{C_{n+1}^{(\lambda)}(x) C_n^{(\lambda)}(y) - C_{n+1}^{(\lambda)}(y) C_n^{(\lambda)}(x)}{x - y}, \tag{10}$$

where $k_n^{(\lambda)}$ is the coefficient of x^n in $C_n^{(\lambda)}(x)$.

Concerning the zeros of the Jacobi polynomials we recall that for all $n \geq 1$, the polynomial $C_n^{(\lambda)}$ has n distinct zeros in $]-1, 1[$:

$$-1 < \xi_{n,n}^{(\lambda)} < \ldots < \xi_{1,n}^{(\lambda)} < 1.$$

Writing $\xi_{k,n}^{(\lambda)} = \cos \theta_{k,n}^{(\lambda)}$, the symmetry of the Jacobi polynomials implies $\theta_{k,n}^{(\lambda)} = \pi - \theta_{n+1-k,n}^{(\lambda)}$. In the special cases $\lambda = 0$ and $\lambda = 1$, the angles $\theta_{k,n}^{(\lambda)}$ are explicitly known:

$$\theta_{k,n}^{(0)} = \frac{2k-1}{2n}\pi, \quad \theta_{k,n}^{(1)} = \frac{k}{n+1}\pi, \qquad k = 1, \cdots, n.$$

In the general case, the angles are quasi-uniformly distributed on the unit circle: the distance between two consecutive angles is near constant and decays like π/n (see Theorem 6.21.3 of Szegö [5])

$$\theta_{k,n}^{(0)} \;\leq\; \theta_{k,n}^{(\lambda)} \;\leq\; \theta_{k,n}^{(1)}, \quad k = 1, \cdots, [n/2]. \tag{11}$$

2. EXPANSIONS INTO JACOBI SERIES

We associated to a function u (at least formally) its Jacobi coefficients

$$\hat{u}_k^{(\lambda)} = \frac{1}{\|C_k^{(\lambda)}\|_\lambda^2} \langle u, C_k^{(\lambda)} \rangle_\lambda$$

and the Jacobi series

$$\mathcal{J}^{(\lambda)}(u) = \sum_{k=0}^{\infty} \hat{u}_k^{(\lambda)} C_k^{(\lambda)}.$$

It is quite natural to approximate this infinite series by the N-truncated one:

$$\pi_N^{(\lambda)} u = \sum_{k=0}^{N} \hat{u}_k^{(\lambda)} C_k^{(\lambda)}.$$

We are interested in this work by discontinuous functions u that belong to $L_\lambda^2(I)$, hence expandable into Jacobi series

$$u = \sum_{k=0}^{\infty} \hat{u}_k^{(\lambda)} C_k^{(\lambda)}.$$

2.1. EXPANSION OF THE SIGN FUNCTION

Let us first explain why this function plays a special role. Assume that a function u is regular on both sides of a discontinuity $x_s \in]-1,1[$

$$u(x) = \begin{cases} u_-(x) & \text{for } x < x_s \,, \\ u_+(x) & \text{for } x > x_s \,, \end{cases}$$

with u_- and u_+ two regular functions such that $u_-(x_s) \neq u_+(x_s)$. We know that the decreasing of the Jacobi coefficients $\hat{u}_k^{(\lambda)}$ is limited by the decreasing of the leading term (use the differential equation (9))

$$\frac{\omega_{\lambda+1}(x_s)(C_k^{(\lambda)})'(x_s)}{k(k+2\lambda)} [u]_{x_s}, \tag{12}$$

with $[u]_{x_s} = u_+(x_s) - u_-(x_s)$ the jump of the function u at x_s. The expression (12) is precisely the Jacobi coefficient $\hat{S}_k^{\alpha}$ of the piecewise constant function

$$S(x) = \begin{cases} u_-(x_s) & \text{for } x < x_s \\ u_+(x_x) & \text{for } x > x_s. \end{cases}$$

This is why we study the special case of the Sign function as a prototype of discontinuous functions. The Jacobi coefficients are then $\hat{S}_{2k}^{(\lambda)} = 0$ for all integers k, $\hat{S}^{(0)}_{2k+1} = (-1)^k \frac{2}{\pi}$ and for all $\lambda \neq 0$

$$\hat{S}^{(\lambda)}_{2k+1} = (-1)^k \frac{2^{2\lambda+1}}{\pi} \Gamma(\lambda) \frac{(2k)!}{k!} \frac{\Gamma(k+\lambda+1)}{\Gamma(2k+2\lambda+1)} \frac{2k+1+\lambda}{2k+1+2\lambda}. \tag{13}$$

Here are some special Jacobi expansions:

- for $\lambda = 0$, we get the Chebyshev expansion

$$\mathcal{J}^{(0)}(S) = \sum_{k=0}^{\infty} (-1)^k \frac{2}{\pi} C^{(\lambda)}_{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} T_{2k+1},$$

- for $\lambda = 1/2$, we get the Legendre expansion

$$\mathcal{J}^{(1/2)}(S) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}} \frac{4n+3}{(n+1)!} \frac{(2n)!}{n!} P_{2n+1},$$

- for $\lambda = 1$, we have the Chebyshev expansion of the second kind

$$\mathcal{J}^{(1)}(S) = \frac{8}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(n+1)}{(2n+1)(2n+3)} U_{2n+1}.$$

3. ANALYSIS OF THE EXPANSION

The goal here is to select the "best" Jacobi expansion of the Sign function. Excepted at $x = 0$, we impose the requirement that $\mathcal{J}^{(\lambda)}(S)(x)$ converges toward $S(x)$. In the neighborhood of $x = 0$, the selection criteria for the attenuation of the Gibbs phenomenon is twofold:

- the derivative of S at $x = 0$ are infinite, so we seek an approximation with large steepness (defined as the derivative of $\pi^{(\lambda)}_{2N+1}(S)$ at $x = 0$),
- the reduction of the Gibbs constant defined as the limit (as $N \to +\infty$) of the size of the overshoot/undershoot.

We first derive a new expression of $\pi^{(\lambda)}_{2N+1}(S)]'$.

Proposition 1. *For $\lambda > -1/2$ and $x \in [-1, 1] \setminus \{0\}$,*

$$[\pi^{(\lambda)}_{2N+1}(S)]'(x) = d_{N,\lambda} \frac{C^{(\lambda+1)}_{2N+1}(x)}{x}, \tag{14}$$

with

$$d_{N,\lambda} = \frac{2N+1}{2N+1+\lambda} \frac{1}{\|C^{(\lambda+1)}_{2N}\|^2_{\lambda+1}} C^{(\lambda+1)}_{2N}(0). \tag{15}$$

Proposition 1 gives an integral representation of $\pi_{2N+1}^{(\lambda)}(S)(x)$

$$\pi_{2N+1}^{(\lambda)}(S)(x) = d_{N,\lambda} \int_0^x \frac{C_{2N+1}^{(\lambda+1)}(y)}{y} dy. \tag{16}$$

Making $x \to 0$ in (14), we get

$$[\pi_{2N+1}^{(\lambda)}(S)]'(0) = d_{N,\lambda}[C_{2N+1}^{(\lambda+1)}]'(0),$$

with the derivative evaluated by (7).

Proof of Proposition 1. Using the differential equation (9) and (5), we express the Jacobi coefficients of S as

$$\hat{S}_{2k+1}^{(\lambda)} = \frac{4\lambda}{(2k+1)(2k+1+2\lambda)} \frac{1}{\|C_{2k+1}^{(\lambda)}\|_\lambda^2} C_{2k}^{(\lambda+1)}(0).$$

From the identity (see (4))

$$4\lambda^2 \frac{\|C_{2k}^{(\lambda+1)}\|_{\lambda+1}^2}{\|C_{2k+1}^{(\lambda)}\|_\lambda^2} = (2k+1)(2k+1+2\lambda)$$

we deduce for all $\lambda \neq 0$

$$\pi_{2N+1}^{(\lambda)}(S) = \frac{1}{\lambda} \sum_{k=0}^{N} \frac{1}{\|C_{2k}^{(\lambda+1)}\|_{\lambda+1}^2} C_{2k}^{(\lambda+1)}(0) C_{2k+1}^{(\lambda)}.$$

It follows from (5) that the quantity

$$\frac{1}{2}[\pi_{2N+1}^{(\lambda)}(S)]'(x) = \sum_{k=0}^{N} \frac{1}{\|C_{2k}^{(\lambda+1)}\|_{\lambda+1}^2} C_{2k}^{(\lambda+1)}(0) C_{2k}^{(\lambda+1)}(x)$$

is a Christoffel-Darboux sum. Hence using (10) and (2) we obtain (14) with $d_{N,\lambda}$ defined in (15). In the case $\lambda = 0$, we get $d_{N,0} = 2(-1)^N/\pi$ and

$$[\pi_{2N+1}^{(0)}(S)]'(\cos\theta) = (-1)^N \frac{4}{\pi} \frac{\sin[(N+1)2\theta]}{\sin 2\theta}, \tag{17}$$

that ends the proof. □

We now derive the asymptotic behavior for $d_{N,\lambda}$ as $N \to +\infty$. By using relations (4) and (6) we have

$$\frac{C_{2N}^{(\lambda+1)}(0)}{\|C_{2N}^{(\lambda+1)}\|_{\lambda+1}^2} = \frac{(2N)!(2N+\lambda+1)\Gamma(\lambda+1)}{\pi 2^{-3-2\lambda}} \frac{(-1)^N}{N!} \frac{\Gamma(N+\lambda+1)}{\Gamma(2N+2\lambda+2)}.$$

Inserting this expression in (15) and using the Legendre duplication formula,[2] we get

$$d_{N,\lambda} = \frac{(-1)^N}{\sqrt{\pi}} \frac{(2N)!}{N!2^{2N}}(2N+1)\,\frac{\Gamma(\lambda+1)}{\Gamma(N+\lambda+3/2)} \simeq (-1)^N \frac{2\Gamma(\lambda+1)}{\pi N^\lambda}. \quad (18)$$

3.1. CONVERGENCE

The series obviously converges for $x = 0$. Let us first check the convergence at the boundaries $x = \pm 1$. According to (3) and (13), the general term of the alternating numerical series $\mathcal{J}^{(\lambda)}(S)(1)$ is equivalent to

$$\frac{2}{\sqrt{\pi}} \frac{1}{\Gamma(\lambda+1/2)} k^{\lambda-1}, \quad (19)$$

the series converges if and only if $\lambda < 1$. This condition will be be assumed thereafter. We know from (19) that the size of the oscillations in the vicinity of $x = 1$ depend on the Jacobi parameter λ. This is observed on Figure 2 (right). Figure 2 (left) displays the behavior of $\mathcal{J}^{(\lambda)}(S)$ near $x = 0$: the size of the oscillations seems much less dependent on the Jacobi parameter. These observations will be proven in Theorem 1.

We consider now the convergence for $x \in]\alpha, 1]$ with $\alpha \in]0, 1[$. The error can be split into two terms:

$$\pi^{(\lambda)}_{2N+1}(S)(x) - S(x) \\ = \left\{\pi^{(\lambda)}_{2N+1}(S)(x) - \pi^{(\lambda)}_{2N+1}(S)(1)\right\} + \left\{\pi^{(\lambda)}_{2N+1}(S)(1) - S(x)\right\}.$$

The last term goes to zero and, by Proposition 1, the first term equals

$$d_{N,\lambda} \int_x^1 \frac{C^{(\lambda+1)}_{2N+1}(y)}{y} dy$$

and the following upper bound holds

$$|\pi^{(\lambda)}_{2N+1}(S)(x) - \pi^{(\lambda)}_{2N+1}(S)(1)| \le \frac{1-\alpha}{\alpha}\, |d_{N,\lambda}| \max_{\alpha \le y \le 1} |C^{(\lambda+1)}_{2N+1}(y)|.$$

Since $\lambda + 1$ is positive, we obtain by (8) the bound

$$|\pi^{(\lambda)}_{2N+1}(S)(x) - \pi^{(\lambda)}_{2N+1}(S)(1)| \le \frac{1-\alpha}{\alpha}\, |d_{N,\lambda}| \frac{1}{\Gamma(2\lambda+2)} (2N)^{2\lambda-1}.$$

Combined with (18) this bound becomes

$$|\pi^{(\lambda)}_{2N+1}(S)(x) - \pi^{(\lambda)}_{2N+1}(S)(1)| \le \frac{1-\alpha}{\alpha} \frac{2^{2\lambda}\Gamma(\lambda+1)}{\pi\Gamma(2\lambda+2)} N^{\lambda-1}.$$

Hence $\pi^{(\lambda)}_{2N+1}(S)(x)$ converges to $S(x)$ since $\lambda < 1$.

[2] $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+1/2)$.

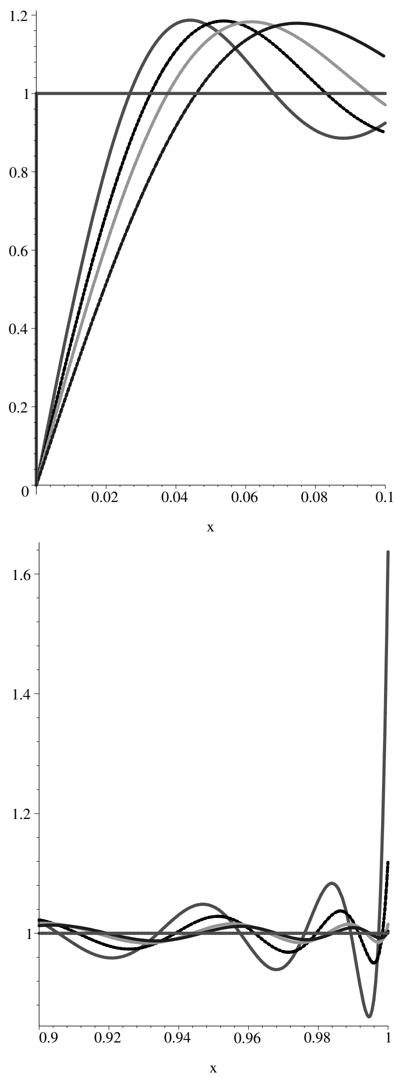

Figure 2. Jacobi series of the Sign function ($N = 20$) for $\lambda = 0$ (dots), $\lambda = 10$ (dashes), $\lambda = 20$ (dots-dashes), $\lambda = 40$ (solid line). Zoom in the vicinity of $x = 0$. Down: Jacobi series of the Sign function ($N = 20$) for $\lambda = -1/4$ (dots), $\lambda = 0$ (dashes), $\lambda = 1/2$ (dots-dashes), $\lambda = 1$ (solid line). Zoom in the vicinity of $x = 1$

3.2. THE STEEPNESS

In the neighborhood of $x = 0$, we shall compare the derivative of $\pi_{2N+1}^{(\lambda)}(S)(0)$ (the steepness) as a function of λ.

Proposition 2. (Steepness) *For N fixed*

$$\lambda_1 > \lambda_2 > -1/2 \Longrightarrow \left(\pi^{(\lambda_1)}_{2N+1}(S)\right)'(0) > \left(\pi^{(\lambda_2)}_{2N+1}(S)\right)'(0).$$

For $\lambda > -1/2$ fixed and $N \to +\infty$:

$$\left(\pi^{(\lambda)}_{2N+1}(S)\right)'(0) \simeq \left(\pi^{(0)}_{2N+1}(S)\right)'(0) = \frac{4}{\pi}(N+1).$$

This proposition tells us that for a fixed N, we can improve the steepness of the approximation by increasing λ. There is no value of λ that gives the optimal steepness; we have to choose λ as large as possible. But, this causes troubles since for $\lambda > 1$ the series does not converge at the boundaries. On the other hand, all the Jacobi approximants give, asymptotically, the same steepness.

Proof of Proposition 2. For fixed N, we define the function

$$f_N : \lambda \longmapsto \left(\pi^{(\lambda)}_{2N+1}(S)\right)'(0).$$

In the simple case $\lambda = 0$, equation (17) gives

$$f_N(0) = \frac{4}{\pi}(N+1).$$

For $\lambda \neq 0$, we know from Proposition 1 that $f_N(\lambda) = d_{N,\lambda}[C^{(\lambda+1)}_{2N+1}]'(0)$. Using (7) and the definition of $d_{N,\lambda}$, we get

$$f_N(\lambda) = \frac{2(2N+1)}{\sqrt{\pi}} \frac{(2N)!}{(N!)^2 2^{2N}} \frac{\Gamma(N+\lambda+2)}{\Gamma(N+\lambda+3/2)}.$$

Hence $f_N(\lambda)$ is the product of two terms: the first one

$$\frac{2(2N+1)}{\sqrt{\pi}} \frac{(2N)!}{(N!)^2 2^{2N}} \simeq 4\frac{\sqrt{N}}{\pi}, \tag{20}$$

is independent of λ, the second

$$\frac{\Gamma(z+2)}{\Gamma(z+3/2)}$$

is an increasing function of $z = N + \lambda$. This proves the first part of the Proposition, the second part comes from (20) and the equivalence $\frac{\Gamma(z+2)}{\Gamma(z+3/2)} \simeq \sqrt{z}$ as $z \to +\infty$. □

3.3. THE GIBBS CONSTANT

For a fixed N, we define the function g_N that associate to λ the smallest positive zero of $[\pi_{2N+1}^{(\lambda)}(S)]'$. The problem is to find λ, if it exists, solution of the problem

$$\inf_{\lambda>-1/2} \quad \mathcal{G}_\lambda,$$

where the Gibbs constant $\mathcal{G}_\lambda$ is defined as the limit of $\pi_{2N+1}^{(\lambda)}(S)(g_N(\lambda))$ for $N \to +\infty$. The main result of the paper is the following.

Theorem 1. *For all $\lambda > -1/2$,*

$$\mathcal{G}_\lambda = \mathcal{G}_0 = \frac{2}{\pi}\int_0^\pi \frac{\sin u}{u} du.$$

Theorem 1 tells us that the Gibbs constant is the same for all λ. To prove this we need the two following lemmas the proofs of which are given at the end of the section.

Lemma 1. *For n even, $n \to +\infty$:*

$$(\forall\lambda \geq -1/2) \qquad \lim_{n\ even \to +\infty} n(\frac{\pi}{2} - \theta_{[n/2],n}^{(\lambda)}) = \frac{\pi}{2}.$$

For n odd, $n \to +\infty$:

$$(\forall\lambda \geq -1/2) \qquad \lim_{n\ odd \to +\infty} n(\frac{\pi}{2} - \theta_{[n/2],n}^{(\lambda)}) = \pi.$$

Note that $x_{[n/2],n}^{(\lambda)} = \cos\theta_{[n/2],n}^{(\lambda)}$ is the smallest positive zero of $C_n^{(\lambda)}$.

Lemma 2. *For $\lambda > -1/2$ and $u \in \mathbb{R}$,*

$$\lim_{n\to+\infty} (-1)^n n^{1-\lambda} C_{2n+1}^{(\lambda)}(\sin\frac{u}{2n}) = \frac{1}{\Gamma(\lambda)}\sin u.$$

Remark 1. Comments on Lemma 2. Note that the case $\lambda = 0$ is easily deduced from the lemma using the fact that $C_n^{(0)} = \lim_{\lambda\to 0}\frac{1}{\lambda}C_n^{(\lambda)}$. We get in this case

$$(\forall u \in \mathbb{R}) \qquad \lim_{n\to+\infty}(-1)^n T_{2n+1}(\sin\frac{u}{2n}) = \sin u.$$

However this is is a trivial identity. The identity obtained for $\lambda = 1/2$ (Legendre approximation) is much more interesting:

$$(\forall u \in \mathbb{R}) \qquad \lim_{n\to+\infty}(-1)^n\sqrt{\frac{\pi}{n}}P_{2n+1}(\sin\frac{u}{2n}) = \sin u.$$

Finally, notice that Lemma 2 has to be compared to the Mehler-Heine formulas (see for example Theorem 8.1.1 of Szegö [5])

$$\lim_{n\to+\infty} n^{1-2\lambda}C_n^{(\lambda)}(\cos\frac{u}{n}) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)}(\frac{u}{2})^{1/2-\lambda}J_{\lambda-1/2}(u),$$

with J_α the Bessel functions.

Proof of Theorem 1. We deduce from (14) that $g_N(\lambda)$ is the smallest strictly positive zero of $C_{2N+1}^{(\lambda+1)}$ and

$$\pi_{2N+1}^{(\lambda)}(S)(g_N(\lambda)) = d_{N,\lambda} \int_0^{g_N(\lambda)} \frac{C_{2N+1}^{(\lambda+1)}(y)}{y} dy.$$

For $\lambda = 0$, the computations are explicit: using (17), we get

$$\pi_{2N+1}^{(0)}(S)(x) = (-1)^N \frac{4}{\pi} \int_0^x \frac{\sin[(N+1)2\theta]}{\sin 2\theta} dt \quad (t = \cos\theta).$$

On the other hand we know that $g_N(0) = \cos(\frac{N}{N+1}\frac{\pi}{2})$ hence

$$\pi_{2N+1}^{(0)}(S)(g_N(0)) = (-1)^N \frac{2}{\pi} \int_{\frac{N}{N+1}\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin[(N+1)2\theta]}{\cos(\theta)} d\theta$$

and by the change of variable $\theta \to \pi/2 - \theta$

$$\pi_{2N+1}^{(0)}(S)(g_N(0))$$
$$= \frac{2}{\pi} \int_0^{\frac{1}{N+1}\frac{\pi}{2}} \frac{\sin[(N+1)2\theta]}{\sin(\theta)} d\theta = \frac{2}{\pi} \int_0^{\pi} \frac{\sin u}{u} \left[\frac{\frac{u}{2(N+1)}}{\sin \frac{u}{2(N+1)}} \right] du.$$

For large N, the expression delimited by the brackets is equivalent to 1 and we can make the approximation

$$\pi_{2N+1}^{(0)}(S)(g_N(0)) \simeq \frac{2}{\pi} \operatorname{Si}(\pi).$$

with $\operatorname{Si}(x) = \int_0^x \frac{\sin z}{z} dz$ the Sine integral. Hence the Gibbs constant is

$$\mathcal{G}_0 = \frac{2}{\pi} \operatorname{Si}(\pi) \simeq 1.178979744.$$

In the general case $\lambda \in [0, 1]$, we define $g_N(\lambda) = \cos[\theta_{N,2N+1}^{(\lambda+1)}]$ and write

$$\begin{aligned} \int_0^{g_N(\lambda)} \frac{C_{2N+1}^{(\lambda+1)}(y)}{y} dy &= \int_0^{\frac{\pi}{2} - \theta_{N,2N+1}^{(\lambda+1)}} \frac{C_{2N+1}^{(\lambda+1)}(\sin\theta)}{\sin\theta} \cos\theta \, d\theta \\ &= \int_0^{\varphi_N^\lambda} \frac{C_{2N+1}^{(\lambda+1)}(\sin \frac{u}{2(N+1)})}{2(N+1)\sin \frac{u}{2(N+1)}} \cos(\frac{u}{2(N+1)}) du. \end{aligned}$$

By Lemma 1, the upper bound

$$\varphi_N^\lambda = 2(N+1)(\frac{\pi}{2} - \theta_{N,2N+1}^{(\lambda+1)})$$

goes to π, the cosine term is equivalent to 1 and the denominator is equivalent to u. Combining the equivalence (18) and Lemma 2 we get

$$\lim_{N\to+\infty} d_{N,\lambda} C^{(\lambda+1)}_{2N+1}(\sin \frac{u}{2(N+1)}) = \frac{2}{\pi}\sin u,$$

that proves the Theorem. □

The remainder of this section is devoted to the proof of the Lemma 1 and Lemma 2.

Proof of Lemma 1. In the case $\lambda \in [0,1]$, we know from (11) that for all $k \le [n/2]$

$$\theta^{(0)}_{k,n} \le \theta^{(\lambda)}_{k,n} \le \theta^{(1)}_{k,n}.$$

From which we deduce the bounds

$$\frac{n-2k+1}{2(n+1)}\pi \le \frac{\pi}{2} - \theta^{(\lambda)}_{k,n} \le \frac{n-2k+1}{2n}\pi.$$

For $k = [n/2]$, $j = n-2k+1$ belongs to the set $\{1,2\}$ depending on the parity of n and we get the result by passing to the limit. In the general case $\lambda > -1/2$ we exploit a result in Elbert and Laforgia [2] about the zeros of ultraspherical polynomials. The Corollary 3.2 of Elbert and Laforgia [2] gives the approximation

$$\xi^{(\lambda)}_{[n/2],n} \simeq \frac{n+1-2[n/2]}{n}\frac{\pi}{2} \simeq \cos[\frac{\pi}{2} - \frac{n+1-2[n/2]}{n}\frac{\pi}{2}].$$

From which we deduce

$$\frac{\pi}{2} - \theta^{(\lambda)}_{[n/2],n} \simeq \frac{n+1-2[n/2]}{n}\frac{\pi}{2},$$

then the lemma follows. □

Proof of Lemma 2. We use a hypergeometric representation of $C^{(\lambda)}_{2n+1}$. For integers p and q and real numbers

$$x_1,\dots,x_p, \quad y_1,\dots,y_q, \quad {}_pF_q(x_1,\dots,x_p;y_1,\dots,y_q;.)$$

denotes the hypergeometric function defined for $z \in \mathbb{C}$ by (see for example Askey [1])

$${}_pF_q(x_1,\dots,x_p;y_1,\dots,y_q;z) = \sum_{k\ge 0} \frac{(x_1)_k\dots(x_p)_k}{(y_1)_k\dots(y_q)_k}\frac{z^k}{k!}.$$

A simple example is given by the Sine function

$$\sin u = \sum_{k\ge 0}(-1)^k \frac{u^{2k+1}}{(2k+1)!} = u\ {}_0F_1(.;3/2;-u^2/4).$$

There exist several representations of $C_n^{(\lambda)}$ in terms of hypergeometric series, among them (Szegö [5], equation 4.7.30)

$$\begin{aligned} C_{2n+1}^{(\lambda)}(x) &= (-1)^n \frac{2}{n!}\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda)} x\, {}_2F_1(-n, n+\lambda+1; 3/2; x^2). \\ &= (-1)^n \frac{2}{n!}\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda)} x \sum_{k=0}^{n} \frac{(-n)_k (n+\lambda+1)_k}{(3/2)_k}\frac{1}{k!} x^{2k}. \end{aligned}$$

Observing that $(-1)^k(-n)_k = (n!)/(n-k)!$ and $\Gamma(n+\lambda+1)(n+\lambda+1)_k = \Gamma(n+\lambda+k+1)$, we write

$$C_{2n+1}^{(\lambda)}(\sin\frac{u}{2n}) = (-1)^n \frac{2}{\Gamma(\lambda)} \sin\frac{u}{2n} \sum_{k=0}^{n} \frac{(-1)^k A_{n,k}}{(3/2)_k}\frac{1}{k!}(\sin\frac{u}{2n})^{2k},$$

with $A_{n,k} = \frac{\Gamma(n+\lambda+k+1)}{(n-k)!}$. For a fixed k, we have the equivalence

$$A_{n,k}(\sin\frac{u}{2n})^{2k} \simeq n^{\lambda+2k}(\frac{u}{2n})^{2k} = n^{\lambda}(\frac{u^2}{4})^k$$

and we can write

$$\begin{aligned} &\Gamma(\lambda)(-1)^n n^{1-\lambda} C_{2n+1}^{(\lambda)}(\sin\frac{u}{2n}) \\ &\qquad = [2n \sin\frac{u}{2n}] \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(3/2)_k}[n^{-\lambda}A_{n,k}(\sin\frac{u}{2n})^{2k}]. \end{aligned}$$

The first expression delimited by brackets goes to u and the second one is equivalent to $(\frac{u^2}{4})^k$. Since the general term of the sum is bounded by

$$\frac{1}{k!}\frac{1}{(3/2)_k}(\frac{u^2}{4})^k,$$

we can pass to limit as $n \to +\infty$ to get

$$\begin{aligned} \lim_{n\to+\infty} \Gamma(\lambda)(-1)^n n^{1-\lambda} C_{2n+1}^{(\lambda)}(\sin\frac{u}{2n}) &= u \lim_{n\to+\infty} \sum_{k=0}^{n} \frac{(-1)^k}{(3/2)_k}\frac{1}{k!}(\frac{u^2}{4})^k \\ &= u\, {}_0F_1(.; 3/2; -u^2/4) \\ &= \sin u, \end{aligned}$$

that proves the lemma. □

4. CONCLUSION

We have solved the problem of optimizing the Jacobi expansions of a discontinuous function, namely the Sign function. The criteria are the decreasing of the Gibbs constant and the increasing of the steepness of the N truncated expansion. We prove that all the Jacobi expansions have asymptotically the same steepness. For a fixed N, the steepness increases with λ. The critical value being $\lambda = 1$, beyond which the series diverges at the endpoints. Concerning the Gibbs constant, we prove that all the Jacobi expansions have the same Gibbs constant.

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