

**SECOND ORDER CARATHÉODORY AND  
DISCONTINUOUS DIFFERENTIAL  
EQUATIONS IN BANACH ALGEBRAS**

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**ABSTRACT:** In this paper an existence theorem for the second order differential equations in Banach algebras is proved under the mixed generalized Lipschitz and Carathéodory conditions. The existence of extremal solutions is also proved under certain monotonicity conditions.

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**1. INTRODUCTION**

The study of nonlinear differential and integral equations in Banach algebras is initiated in the works of Dhage [1] and Dhage and O'Regan [6] and discussed the existence theory for first order differential equations only. The main tool used in the study is hybrid fixed point theorem of Dhage [4] type. Like nonlinear perturbed differential equations, the solutions of such differential equations are also obtained under the mixed Lipschitz and compactness type conditions. In recent years, the topic of nonlinear differential equations in Banach algebras is received the attention of several authors and at present, considerable literature available is this direction. In this paper, we deal with the second order ordinary differential equations in Banach algebras and discuss the existence results under mixed Lipschitz and Carathéodory conditions. We will employ the fixed point theorems of Dhage [2], Dhage [3], Dhage [5] for proving our main existence results. We claim that the nonlinear differential equation as well as the existence results of this are new to the literature on the theory of ordinary differential equations.

Let  $\mathbb{R}$  denote the real line. Given a closed and bounded interval  $J = [0, T]$  in  $\mathbb{R}$ , consider the second order differential equation (in short DE)

$$\left. \begin{aligned} \frac{d^2}{dt^2} \left[ \frac{x(t)}{f(t, x(t))} \right] &= g(t, x(t)) \quad \text{a.e. } t \in J \\ x(0) = x_0, x'(0) &= x_1, \end{aligned} \right\} \quad (1.1)$$

where  $f : I \times \mathbb{R} \rightarrow \mathbb{R} - \{0\}$  and  $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ .

By a solution of DE (1.1) we mean a function  $x \in AC^1(J, \mathbb{R})$  that satisfies:

- (i) the function  $t \mapsto \left( \frac{x}{f(t, x)} \right)'$  is absolutely continuous on  $J$  for all  $x \in \mathbb{R}$ , and
- (ii)  $x$  satisfies the equations in (1.1),

where  $AC^1(J, \mathbb{R})$  is the space of continuous functions whose first derivative exists and is absolutely continuous real-valued functions on  $J$ .

Our method of study is to convert the DE (1.1) into an equivalent integral equation and to apply the fixed point theorems of Dhage [2], Dhage [3], Dhage [5] under suitable conditions on the nonlinearities  $f$  and  $g$ . In the following section we shall give some preliminaries needed in the sequel.

## 2. AUXILIARY RESULTS

Let  $X$  be a Banach algebra with norm  $\|\cdot\|$ . A mapping  $A : X \rightarrow X$  is called  $\mathcal{D}$ -Lipschitz if there exists a continuous nondecreasing function  $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  satisfying

$$\|Ax - Ay\| \leq \psi(\|x - y\|) \quad (2.1)$$

for all  $x, y \in X$  with  $\psi(0) = 0$ . In the special case when  $\psi(r) = \alpha r$  ( $\alpha > 0$ ),  $A$  is called a  $\mathcal{D}$ -Lipchitz with a Lipschitz constant  $\alpha$ . In particular, if  $\alpha < 1$ ,  $A$  is called a  $\mathcal{D}$ -contraction with a contraction constant  $\alpha$ . Further, if  $\psi(r) < r$  for all  $r > 0$ , then  $A$  is called a nonlinear  $\mathcal{D}$ -contraction on  $X$ . Sometimes we call the function  $\psi$  a  $\mathcal{D}$ -function (for convenience) for the  $\mathcal{D}$ -Lipschitz mappings  $T$  on  $X$ .

An operator  $T : X \rightarrow X$  is called compact if  $\overline{T(S)}$  is a compact subset of  $X$  for any  $S \subset X$ . Similarly  $T : X \rightarrow X$  is called totally bounded if  $T$  maps a bounded subset of  $X$  into the relatively compact subset of  $X$ . Finally  $T : X \rightarrow X$  is called completely continuous operator if it is continuous and totally bounded operator on  $X$ . It is clear that every compact operator is totally bounded, but the converse may not be true. The nonlinear alternative of Schaefer type recently proved by Dhage [4] is embodied in the following theorem.

**Theorem 2.1.** (Dhage [4]) *Let  $X$  be a Banach algebra and let  $A, B : X \rightarrow X$  be two operators satisfying:*

- (a)  $A$  is Lipschitz with a Lipschitz constant  $\alpha$ ,

- (b)  $B$  is compact and continuous, and
- (c)  $\alpha M < 1$ , where  $M = \|B(X)\| := \sup\{\|Bx\| : x \in X\}$ .

Then either:

- (i) the equation  $\lambda[Ax Bx] = x$  has a solution for  $\lambda = 1$ , or
- (ii) the set  $\mathcal{E} = \{u \in X \mid \lambda[Au Bx] = u, 0 < \lambda < 1\}$  is unbounded.

A non-empty closed set  $K$  in a Banach algebra  $X$  is called a cone if (i)  $K + K \subseteq K$ , (ii)  $\lambda K \subseteq K$  for  $\lambda \in \mathbb{R}, \lambda \geq 0$ , and (iii)  $\{-K\} \cap K = 0$ , where  $0$  is the zero element of  $X$ . A cone  $K$  is called to be positive if (iv)  $K \circ K \subseteq K$ , where "o" is a multiplication composition in  $X$ . We introduce an order relation  $\leq$  in  $X$  as follows. Let  $x, y \in X$ . Then  $x \leq y$  if and only if  $y - x \in K$ . A cone  $K$  is called to be normal if the norm  $\|\cdot\|$  is monotone increasing on  $K$ . It is known that if the cone  $K$  is normal in  $X$ , then every order-bounded set in  $X$  is norm-bounded. The details of cones and their properties appear in Guo and Lakshmikantham [10].

**Lemma 2.1.** (Dhage [3]) *Let  $K$  be a positive cone in a real Banach algebra  $X$  and let  $u_1, u_2, v_1, v_2 \in K$  be such that  $u_1 \leq v_1$  and  $u_2 \leq v_2$ . Then  $u_1 u_2 \leq v_1 v_2$ .*

For any  $a, b \in X, a \leq b$ , the order interval  $[a, b]$  is a set in  $X$  given by

$$[a, b] = \{x \in X : a \leq x \leq b\}.$$

We use the following fixed point theorem of Dhage [2], Dhage [3], Dhage [5] for proving the existence of extremal solutions for the DE (1.1) under certain monotonicity conditions.

**Theorem 2.2.** (Dhage [2]) *Let  $K$  be a cone in a Banach algebra  $X$  and let  $a, b \in X$ . Suppose that  $A, B : [a, b] \rightarrow K$  are two operators such that:*

- (a)  $A$  is Lipschitz with a Lipschitz constant  $\alpha$ ,
- (b)  $B$  is completely continuous,
- (c)  $Ax Bx \in [a, b]$  for each  $x \in [a, b]$ , and
- (d)  $A$  and  $B$  are nondecreasing.

*Further, if the cone  $K$  is positive and normal, then the operator equation  $Ax Bx = x$  has a least and a greatest positive solution in  $[a, b]$ , whenever  $\alpha M < 1$ , where  $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$ .*

**Theorem 2.3.** (Dhage [5]) *Let  $K$  be a cone in a Banach algebra  $X$  and let  $a, b \in X$ . Suppose that  $A, B : [a, b] \rightarrow K$  are two operators such that*

- (a)  $A$  is completely continuous,
- (b)  $B$  is totally bounded,
- (c)  $Ax By \in [a, b]$  for all  $x, y \in [a, b]$ , and
- (d)  $A$  and  $B$  are nondecreasing.

*Further if the cone  $K$  is positive and normal, then the operator equation  $Ax Bx = x$  has a least and a greatest positive solution in  $[a, b]$ .*

**Theorem 2.4.** (Dhage [5]) *Let  $K$  be a cone in a Banach algebra  $X$  and let  $a, b \in X$ . Suppose that  $A, B : [a, b] \rightarrow K$  are two operators such that:*

- (a)  *$A$  is Lipschitz with a Lipschitz constant  $\alpha$ ,*
- (b)  *$B$  is totally bounded,*
- (c)  *$AxBy \in [a, b]$  for each  $x, y \in [a, b]$ , and*
- (d)  *$A$  and  $B$  are nondecreasing.*

*Further, if the cone  $K$  is positive and normal, then the operator equation  $Ax Bx = x$  has one least and one greatest positive solution in  $[a, b]$ , whenever  $\alpha M < 1$ , where  $M = \|B([a, b])\| := \sup\{\|Bx\| : x \in [a, b]\}$ .*

**Remark 2.1.** Note that hypothesis (c) of Theorems 2.2, 2.3, and 2.4 holds if the operators  $A$  and  $B$  are positive monotone increasing and there exist elements  $a$  and  $b$  in  $X$  such that  $a \leq Aa Ba$  and  $Ab Bb \leq b$ .

In the following sections we prove the main existence results of this paper.

### 3. EXISTENCE THEORY

Let  $M(J, \mathbb{R})$  and  $B(J, \mathbb{R})$  respectively denote the spaces of measurable and bounded real-valued functions on  $J$ . Let  $C(J, \mathbb{R})$ , denote the space of all continuous real-valued functions on  $J$ . Define a norm  $\|\cdot\|$  in  $C(J, \mathbb{R})$  by

$$\|x\| = \sup_{t \in J} |x(t)|.$$

Clearly  $C(J, \mathbb{R})$  becomes a Banach algebra with this norm. By  $L^1(J, \mathbb{R})$  we denote the set of Lebesgue integrable functions on  $J$  and the norm  $\|\cdot\|_{L^1}$  in  $L^1(J, \mathbb{R})$  is defined by

$$\|x\|_{L^1} = \int_0^T |x(t)| ds.$$

We need the following lemma in the sequel.

**Lemma 3.1.** *If  $h \in L^1(J, \mathbb{R})$ , then  $x$  is a solution of the DE*

$$\left. \begin{aligned} \frac{d^2}{dt^2} \left[ \frac{x(t)}{f(t, x(t))} \right] &= h(t) \quad \text{a.e. } t \in J \\ x(0) = x_0, \quad x'(0) &= x_1, \end{aligned} \right\} \tag{3.1}$$

*if and only it is solution of the integral equation*

$$x(t) = [f(t, x(t))] \left( \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)h(s) ds \right), \tag{3.2}$$

*where*

$$c = \frac{f(0, x_0)x_1 - x_0f_1(0, x_0) - x_0x_1f_2(0, x_0)}{[f(0, x_0)]^2},$$

*and  $\frac{\partial f}{\partial t} = f_1(t, x)$ ,  $\frac{\partial f}{\partial x} = f_2(t, x)$ .*

We seek the solutions of the FIE (3.2) in the space  $C(J, \mathbb{R})$ . We need the following definition in the sequel.

**Definition 3.1.** A mapping  $\beta : J \times \mathbb{R} \rightarrow \mathbb{R}$  is said to be *Carathéodory* if:

- (i)  $t \mapsto \beta(t, x)$  is measurable for each  $x \in \mathbb{R}$ , and
- (ii)  $x \mapsto \beta(t, x)$  is continuous almost everywhere for  $t \in J$ .

Again a Carathéodory function  $\beta(t, x)$  is called  $L^1$ -Carathéodory if:

- (iii) for each real number  $r > 0$  there exists a function  $h_r \in L^1(J, \mathbb{R})$  such that

$$|\beta(t, x)| \leq h_r(t), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$  with  $|x| \leq r$ .

Finally a Carathéodory function  $\beta(t, x)$  is called  $L^1_X$ -Carathéodory if:

- (iv) there exists a function  $h \in L^1(J, \mathbb{R})$  such that

$$|\beta(t, x)| \leq h(t), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$ .

For convenience, the function  $h$  is referred to as a bound function of  $\beta$ .

We will need the following hypotheses in the sequel:

- (H<sub>1</sub>) The function  $f : J \times \mathbb{R} \rightarrow \mathbb{R}$  is continuous and there exists a function  $k \in B(J, \mathbb{R})$  such that  $k(t) > 0$ , a.e.  $t \in J$  and

$$|f(t, x) - f(t, y)| \leq k(t) |x - y|, \quad \text{a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ .

- (H<sub>2</sub>) The function  $g$  is  $L^1_X$ -Carathéodory with bound function  $h$ .

- (H<sub>3</sub>) There exists a continuous and nondecreasing function  $\Omega : [0, \infty) \rightarrow (0, \infty)$  and a function  $\gamma \in L^1(J, \mathbb{R})$  such that  $\gamma(t) > 0$ , a.e.  $t \in J$  satisfying

$$|g(t, x)| \leq \gamma(t)\Omega(|x|), \quad \text{a.e. } t \in J,$$

for all  $x \in \mathbb{R}$ .

**Theorem 3.1.** Assume that the hypotheses (H<sub>1</sub>)-(H<sub>3</sub>) hold. Suppose that

$$\int_{C_1}^{\infty} \frac{ds}{\Omega(s)} > C_2 \|\gamma\|_{L^1}, \quad (3.3)$$

where

$$C_1 = \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T \right)}{1 - \|k\| \left[ \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|h\|_{L^1} \right]},$$

$$C_2 = \frac{FT}{1 - \|k\| \left[ \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|h\|_{L^1} \right]},$$

$\|k\| \left[ \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|h\|_{L^1} \right] < 1$ ,  $F = \max_{t \in J} |f(t, 0)|$ , and  $\|k\| = \max_{t \in J} |k(t)|$ .

Then the DE (1.1) has a solution on  $J$ .

**Proof.** Set  $X = C(J, \mathbb{R})$ . Define the two mappings  $A$  and  $B$  on  $X$  by

$$Ax(t) = f(t, x(t)), \quad t \in J, \quad (3.4)$$

and

$$Bx(t) = \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, x(s)) ds, \quad t \in J. \quad (3.5)$$

Obviously  $A$  and  $B$  define the operators  $A, B : X \rightarrow X$ . Then the FIE (3.2) is equivalent to the operator equation

$$x(t) = Ax(t) Bx(t), \quad t \in J. \quad (3.6)$$

We shall show that the operators  $A$  and  $B$  satisfy all the hypotheses of Theorem 2.1.

We first show that  $A$  is a Lipschitz on  $X$ . Let  $x, y \in X$ . Then by  $(H_1)$ ,

$$\begin{aligned} |Ax(t) - Ay(t)| &\leq |f(t, x(t)) - f(t, y(t))| \\ &\leq k(t) |x(t) - y(t)| \\ &\leq \|k\| \|x - y\| \end{aligned}$$

for all  $t \in J$ . Taking the supremum over  $t$  we obtain

$$\|Ax - Ay\| \leq \|k\| \|x - y\|$$

for all  $x, y \in X$ . So  $A$  is a Lipschitz on  $X$  with Lipschitz constant  $\|k\|$ . Next we show that  $B$  is completely continuous on  $X$ . Using the standard arguments as in Granas et al [9], it is shown that  $B$  is a continuous operator on  $X$ . Let  $S$  be a bounded set in  $X$ . We shall show that  $B(X)$  is a uniformly bounded and equicontinuous set in  $X$ . Since  $g(t, x(t))$  is  $L_X^1$ -Carathéodory, we have

$$\begin{aligned} |Bx(t)| &\leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|t + \int_0^t |t-s| |g(s, x(s))| ds \\ &\leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \int_0^t h(s) ds \\ &\leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \|h\|_{L^1}. \end{aligned}$$

Taking the supremum over  $t$ , we obtain  $\|Bx\| \leq M$  for all  $x \in S$ , where  $M = \left| \frac{x_0}{f(0, x_0)} \right| + \|h\|_{L^1} T$ . This shows that  $B(X)$  is a uniformly bounded set in  $X$ . Now we show that  $B(X)$  is an equicontinuous set. Let  $t, \tau \in J$ . Then for any  $x \in X$  we have by (3.4),

$$\begin{aligned} &|Bx(t) - Bx(\tau)| \\ &\leq |c||t - \tau| + \left| \int_0^t (t-s)g(s, x(s)) ds - \int_0^\tau (\tau-s)g(s, x(s)) ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq |c||t - \tau| + \left| \int_0^t (t - s)g(s, x(s)) ds - \int_0^t (\tau - s)g(s, x(s)) ds \right| \\
&\quad + \left| \int_0^t (\tau - s)g(s, x(s)) ds - \int_0^\tau (\tau - s)g(s, x(s)) ds \right| \\
&\leq |c||t - \tau| + \left| \int_0^t (t - \tau)g(s, x(s)) ds \right| + \left| \int_\tau^t (\tau - s)g(s, x(s)) ds \right| \\
&\leq |c||t - \tau| + \int_0^T |t - \tau||g(s, x(s))| ds + T \left| \int_\tau^t |g(s, x(s))| ds \right| \\
&\leq |c||t - \tau| + \int_0^T |t - \tau|h(s) ds + T \left| \int_\tau^t h(s) ds \right| \\
&\leq |t - \tau| (|c| + \|h\|_{L^1}) + |p(t) - p(\tau)|,
\end{aligned}$$

where  $p(t) = T \int_0^t h(s) ds$ . Therefore,

$$|Bx(t) - Bx(\tau)| \rightarrow 0 \quad \text{as } t \rightarrow \tau.$$

Hence  $B(X)$  is an equicontinuous set and consequently  $B(X)$  is relatively compact by Arzelà-Ascoli Theorem. As a result  $B$  is a compact and continuous operator on  $X$ . Thus all the conditions of Theorem 2.1 are satisfied and a direct application of it yields that either the conclusion (i) or the conclusion (ii) holds. We show that the conclusion (ii) is not possible. Let  $x \in X$  be any solution to DE (1.1). Then we have, for any  $\lambda \in (0, 1)$ ,

$$x(t) = \lambda [f(t, x(t))] \left( \left| \frac{x_0}{f(0, x_0)} \right| + ct + \int_0^t (t - s)g(s, x(s)) ds \right)$$

for  $t \in J$ . Therefore,

$$\begin{aligned}
|x(t)| &\leq \lambda |f(s, x(t))| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|t + \left| \int_0^t (t - s)g(s, x(s)) ds \right| \right) \\
&\leq \lambda \left( |f(s, x(t)) - f(t, 0)| + |f(t, 0)| \right) \\
&\quad \times \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + \int_0^t |t - s||g(s, x(s))| ds \right) \\
&\leq [k(t) |x(t)| + F] \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + \int_0^t |t - s||g(s, x(s))| ds \right) \\
&\leq k(t) |x(t)| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + \int_0^t |t - s||g(s, x(s))| ds \right) \\
&\quad + F \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + \int_0^t |t - s||g(s, x(s))| ds \right) \\
&\leq \|k\| |x(t)| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + \|h\|_{L^1} \right) + F \left| \frac{x_0}{f(0, x_0)} \right| + |c|T
\end{aligned}$$

$$+ FT \int_0^t \gamma(s) \Omega(|x(t)|) ds. \quad (3.7)$$

Put  $u(t) = \sup_{s \in [0, t]} |x(s)|$ , for  $t \in J$ . Then we have  $|x(t)| \leq u(t)$  for all  $t \in J$ , and so, there is a point  $t^* \in [0, t]$  such that  $u(t) = |x(t^*)|$ . From (3.7) it follows that

$$\begin{aligned} u(t) = |x(t^*)| &\leq \|k\| |x(t^*)| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \|h\|_{L^1} \right) \\ &+ F \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \int_0^{t^*} \gamma(s) \Omega(|x(t)|) ds \right) \\ &\leq \|k\| u(t) \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \|h\|_{L^1} \right) \\ &+ F \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \int_0^t \gamma(s) \Omega(u(s)) ds \right) \\ &= C_1 + C_2 \int_0^t \gamma(s) \Omega(u(s)) ds, \quad (3.8) \end{aligned}$$

where

$$C_1 = \frac{F \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T \right)}{1 - \|k\| \left[ \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \|h\|_{L^1} \right]},$$

and

$$C_2 = \frac{FT}{1 - \|k\| \left[ \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \|h\|_{L^1} \right]}.$$

Let

$$w(t) = C_1 + C_2 \int_0^t \gamma(s) \Omega(u(s)) ds.$$

Then  $u(t) \leq w(t)$  and a direct differentiation of  $w(t)$  yields

$$\left. \begin{aligned} w'(t) &\leq C_2 \gamma(t) \Omega(w(t)), \\ w(0) &= C_1, \end{aligned} \right\} \quad (3.9)$$

that is

$$\int_0^t \frac{w'(s)}{\Omega(w(s))} ds \leq C_2 \int_0^t \gamma(s) ds \leq C_2 \|\gamma\|_{L^1}$$

A change of variables in the above integral gives that

$$\int_{C_1}^{w(t)} \frac{ds}{\Omega(s)} \leq C_2 \|\gamma\|_{L^1} < \int_{C_1}^{\infty} \frac{ds}{\Omega(s)}.$$

Now an application of mean value theorem yields that there is a constant  $M > 0$  such that  $w(t) \leq M$  for all  $t \in J$ . This further implies that

$$|x(t)| \leq u(t) \leq w(t) \leq M.$$



for all  $t \in J$ . Thus the conclusion (ii) of Corollary 2.1 does not hold. Therefore the operator equation  $AxBx = x$  and consequently the DE (1.1) has a solution on  $J$ . This completes the proof.  $\square$

#### 4. EXISTENCE OF EXTREMAL SOLUTIONS

We equip the space  $C(J, \mathbb{R})$  with the order relation  $\leq$  with the help of the cone defined by

$$K = \{x \in C(J, \mathbb{R}) : x(t) \geq 0, \forall t \in J\}. \quad (4.1)$$

It is well known that the cone  $K$  is positive and normal in  $C(J, \mathbb{R})$ . We need the following definitions in the sequel.

**Definition 4.1.** A function  $u \in AC^1(J, \mathbb{R})$  is called a lower solution of the DE (1.1) on  $J$  if

$$\frac{d^2}{dt^2} \left[ \frac{u(t)}{f(t, u(t))} \right] \leq g(t, u(t)), \quad \text{a.e. } t \in J, \quad \text{and } u(0) \leq x_0, \quad u'(0) \leq x_1.$$

Again a function  $v \in AC^1(J, \mathbb{R})$  is called an upper solution of the DE (1.1) on  $J$  if

$$\frac{d^2}{dt^2} \left[ \frac{v(t)}{f(t, v(t))} \right] \geq g(t, v(t)), \quad \text{a.e. } t \in J, \quad \text{and } v(0) \geq x_0, \quad v'(0) \geq x_1.$$

**Definition 4.2.** A solution  $x_M$  of the DE (1.1) is said to be maximal if for any other solution  $x$  to DE(1.1) one has  $x(t) \leq x_M(t)$ , for all  $t \in J$ . Again a solution  $x_m$  of the DE (1.1) is said to be minimal if  $x_m(t) \leq x(t)$ , for all  $t \in J$ , where  $x$  is any solution of the DE (1.1) on  $J$ .

#### 4.1. CARATHÉODORY CASE

We consider the following set of assumptions:

(B<sub>0</sub>)  $f : J \times \mathbb{R} \rightarrow \mathbb{R}^+ - \{0\}$ ,  $g : J \times \mathbb{R} \rightarrow \mathbb{R}^+$ , and

$$\frac{x_0}{f(0, x_0)} + \frac{f(0, x_0)x_1 - x_0f_1(0, x_0) - x_0x_1f_2(0, x_0)}{[f(0, x_0)]^2}t \geq 0$$

for all  $t \in J$ .

(B<sub>1</sub>)  $g$  is Carathéodory.

(B<sub>2</sub>) The functions  $f(t, x)$  and  $g(t, x)$  are nondecreasing in  $x$  and  $y$  almost everywhere for  $t \in J$ .

(B<sub>3</sub>) The DE (1.1) has a lower solution  $u$  and an upper solution  $v$  on  $J$  with  $u \leq v$ .

(B<sub>4</sub>) The function  $\ell : J \rightarrow \mathbb{R}$  defined by

$$\ell(t) = |g(t, u(t))| + |g(t, v(t))|, \quad t \in J$$

is Lebesgue integrable.

**Remark 4.1.** Assume that (B<sub>2</sub>)-(B<sub>4</sub>) hold. Then the function  $t \mapsto g(t, x(t))$  is Lebesgue integrable on  $J$  and

$$|g(t, x(t))| \leq \ell(t), \quad \text{a.e. } t \in J,$$

for all  $x \in [u, v]$ .

**Theorem 4.1.** *Suppose that the assumptions (H<sub>1</sub>)-(H<sub>3</sub>) and (B<sub>0</sub>)-(B<sub>4</sub>) hold. Further if  $\|k\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1} \right) < 1$ , and  $\ell$  is given in Remark 4.1, then DE (1.1) has a minimal and a maximal positive solution on  $J$ .*

**Proof.** Now DE (1.1) is equivalent to FIE (3.2) on  $J$ . Let  $X = C(J, \mathbb{R})$ . Define two operators  $A$  and  $B$  on  $X$  by (3.4) and (3.5) respectively. Then FIE (3.2) is transformed into an operator equation  $Ax(t)Bx(t) = x(t)$  in a Banach algebra  $X$ . Notice that (B<sub>1</sub>) implies  $A, B : [u, v] \rightarrow K$ . Since the cone  $K$  in  $X$  is normal,  $[u, v]$  is a norm bounded set in  $X$ . Now it is shown, as in the proof of Theorem 3.1, that  $A$  is a Lipschitz with a Lipschitz constant  $\|\alpha\|$  and  $B$  is completely continuous operator on  $[u, v]$ . Again the hypothesis (B<sub>2</sub>) implies that  $A$  and  $B$  are nondecreasing on  $[u, v]$ . To see this, let  $x, y \in [u, v]$  be such that  $x \leq y$ . Then by (B<sub>2</sub>),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all  $t \in J$ . Similarly, we have

$$\begin{aligned} Bx(t) &= \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, x(s)) ds \\ &\leq \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, x(s)) ds \\ &= By(t) \end{aligned}$$

for all  $t \in J$ . So  $A$  and  $B$  are nondecreasing operators on  $[u, v]$ . Again Lemma 4.1 and hypothesis (B<sub>3</sub>) together imply that

$$\begin{aligned} u(t) &\leq [f(t, u(t))] \left( \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, u(s)) ds \right) \\ &\leq [f(t, x(t))] \left( \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, x(s)) ds \right) \\ &\leq [f(t, v(t))] \left( \frac{x_0}{f(0, x_0)} + ct + \int_0^t (t-s)g(s, v(s)) ds \right) \\ &\leq v(t), \end{aligned}$$

for all  $t \in J$  and  $x \in [u, v]$ . As a result  $u(t) \leq Ax(t)Bx(t) \leq v(t)$ , for all  $t \in J$  and  $x \in [u, v]$ . Hence  $Ax Bx \in [u, v]$  for all  $x \in [u, v]$ .

Again

$$\begin{aligned} M &= \|B([u, v])\| \\ &= \sup\{\|Bx\| : x \in [u, v]\} \\ &\leq \sup\left\{\left|\frac{x_0}{f(0, x_0)}\right| + |c|T + T \sup_{t \in J} \int_0^t |g(s, x(s))| ds \mid x \in [u, v]\right\} \\ &\leq \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + T \int_0^T \ell(s) ds \\ &= \left|\frac{x_0}{f(0, x_0)}\right| + |c|T + T\|\ell\|_{L^1}. \end{aligned}$$

Since  $\alpha M \leq \|k\|(\left|\frac{x_0}{f(0, x_0)}\right| + |c|T + T\|\ell\|_{L^1}) < 1$ , we apply Theorem 2.2 to the operator equation  $Ax Bx = x$  to yield that the DE (1.1) has a minimal and a maximal positive solution on  $J$ . This completes the proof.  $\square$

## 4.2. DISCONTINUOUS CASE

We need the following definition in the sequel.

**Definition 4.3.** A mapping  $\beta : J \times \mathbb{R} \rightarrow \mathbb{R}$  is said to be *Chandrabhan* if:

- (i)  $t \mapsto \beta(t, x)$  is measurable for each  $x \in \mathbb{R}$ , and
- (ii)  $x \mapsto \beta(t, x)$  is nondecreasing almost everywhere for  $t \in J$ .

Again a Chandrabhan function  $\beta(t, x)$  is called  $L^1$ -Chandrabhan if:

- (iii) for each real number  $r > 0$  there exists a function  $h_r \in L^1(J, \mathbb{R})$  such that

$$|\beta(t, x)| \leq h_r(t), \quad \text{a.e. } t \in J$$

for all  $x \in \mathbb{R}$  with  $|x| \leq r$ .

Finally, a Chandrabhan function  $\beta(t, x)$  is called  $L^1_X$ -Chandrabhan if:

- (iv) there exists a function  $h \in L^1(J, \mathbb{R})$  such that

$$|\beta(t, x)| \leq h(t), \quad \text{a.e. } t \in I$$

for all  $x \in \mathbb{R}$ .

For convenience, the function  $h$  is referred to as a *bound function* of  $\beta$ .

We consider the following hypotheses in the sequel:

- (C<sub>1</sub>) The function  $f$  is continuous on  $J \times \mathbb{R}$ .
- (C<sub>2</sub>) There is a function  $k \in B(J, \mathbb{R})$  such that  $k(t) > 0$ , a.e.  $t \in I$  and

$$|f(t, x) - f(t, y)| \leq k(t) |x - y|, \quad \text{a.e. } t \in J$$

for all  $x, y \in \mathbb{R}$ .

(C<sub>3</sub>) The function  $f(t, x)$  is nondecreasing in  $x$  almost everywhere for  $t \in J$ .

(C<sub>4</sub>) The function  $g$  is Chandrabhan.

**Theorem 4.2.** *Suppose that the assumptions (B<sub>0</sub>), (B<sub>3</sub>)-(B<sub>4</sub>) and (C<sub>1</sub>)-(C<sub>4</sub>) hold. Then DE (1.1) has a minimal and a maximal positive solution on  $J$ .*

**Proof.** Now DE (1.1) is equivalent to FIE (3.2) on  $J$ . Let  $X = C(J, \mathbb{R})$ . Define two operators  $A$  and  $B$  on  $X$  by (3.4) and (3.5) respectively. Then FIE (3.2) is transformed into an operator equation  $Ax(t)Bx(t) = x(t)$  in a Banach algebra  $X$ . Notice that (B<sub>0</sub>) implies  $A, B : [u, v] \rightarrow K$ . Since the cone  $K$  in  $X$  is normal,  $[u, v]$  is a norm bounded set in  $X$ .

**Step I.** Next we show that  $A$  is completely continuous on  $[a, b]$ . Now the cone  $K$  in  $X$  is normal, so the order interval  $[a, b]$  is norm-bounded. Hence there exists a constant  $r > 0$  such that  $\|x\| \leq r$  for all  $x \in [a, b]$ . As  $f$  is continuous on compact  $J \times [-r, r]$ , it attains its maximum, say  $M$ . Therefore for any subset  $S$  of  $[a, b]$  we have

$$\begin{aligned} \|A(S)\|_{\mathcal{P}} &= \sup\{\|Ax\| : x \in S\} \\ &= \sup\left\{\sup_{t \in J} |f(t, x(t))| : x \in S\right\} \\ &\leq \sup\left\{\sup_{t \in J} |f(t, x)| : x \in [-r, r]\right\} \\ &\leq M. \end{aligned}$$

This shows that  $A(S)$  is a uniformly bounded subset of  $X$ .

Next we note that the function  $f(t, x)$  is uniformly continuous on  $[0, T] \times [-r, r]$ . Therefore for any  $t, \tau \in [0, T]$  we have

$$|f(t, x) - f(\tau, x)| \rightarrow 0 \text{ as } t \rightarrow \tau$$

for all  $x \in [-r, r]$ . Similarly for any  $x, y \in [-r, r]$

$$|f(t, x) - f(t, y)| \rightarrow 0 \text{ as } x \rightarrow y$$

for all  $t \in [0, T]$ . Hence any  $t, \tau \in [0, T]$  and for any  $x \in S$  one has

$$\begin{aligned} |Ax(t) - Ax(\tau)| &= |f(t, x(t)) - f(\tau, x(\tau))| \\ &\leq |f(t, x(t)) - f(\tau, x(t))| + |f(\tau, x(t)) - f(\tau, x(\tau))| \\ &\rightarrow 0 \text{ as } t \rightarrow \tau. \end{aligned}$$

This shows that  $A(S)$  is an equi-continuous set in  $X$ . Now an application of Arzelà-Ascoli Theorem yields that  $A$  is a completely continuous operator on  $[a, b]$ .

**Step II.** Next we show that  $B$  is totally bounded operator on  $[a, b]$ . To finish, we shall show that  $B(S)$  is uniformly bounded and equi-continuous set

in  $X$  for any subset  $S$  of  $[a, b]$ . Since the cone  $K$  in  $X$  is normal, the order interval  $[a, b]$  is norm-bounded. Let  $y \in B(S)$  be arbitrary. Then,

$$y(t) = \frac{x_0}{f(0, x_0)} + \int_0^t (t-s)g(s, x(s)) ds$$

for some  $x \in S$ . By hypothesis  $(B_2)$  one has

$$\begin{aligned} |y(t)| &= \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \int_0^t |g(s, x(s))| ds \\ &\leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T \int_0^t \ell(s) ds \\ &\leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1}. \end{aligned}$$

Taking the supremum over  $t$ ,

$$\|y\| \leq \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1},$$

which shows that  $B(S)$  is a uniformly bounded set in  $X$ . Similarly let  $t, \tau \in J$ . Then for any  $y \in B(S)$ ,

$$\begin{aligned} |y(t) - y(\tau)| &\leq |c||t - \tau| + \left| \int_0^t (t-s)g(s, x(s)) ds - \int_0^\tau (\tau-s)g(s, x(s)) ds \right| \\ &\leq |c||t - \tau| + \left| \int_0^t (t-s)g(s, x(s)) ds - \int_0^t (\tau-s)g(s, x(s)) ds \right| \\ &\quad + \left| \int_0^t (\tau-s)g(s, x(s)) ds - \int_0^\tau (\tau-s)g(s, x(s)) ds \right| \\ &\leq |c||t - \tau| + \left| \int_0^t (t-\tau)g(s, x(s)) ds \right| + \left| \int_\tau^t (\tau-s)g(s, x(s)) ds \right| \\ &\leq |c||t - \tau| + \int_0^T |t-\tau||g(s, x(s))| ds + T \left| \int_\tau^t |g(s, x(s))| ds \right| \\ &\leq |c||t - \tau| + \int_0^T |t-\tau|\ell(s) ds + T \left| \int_\tau^t \ell(s) ds \right| \\ &\leq |t - \tau| (\|\ell\|_{L^1} + |c|) + |p(t) - p(\tau)|, \end{aligned}$$

where  $p(t) = T \int_0^t \ell(s) ds$ . Since the function  $p$  is continuous on compact interval  $J$ , it is uniformly continuous, and therefore we have

$$|y(t) - y(\tau)| \rightarrow 0 \quad \text{as } t \rightarrow \tau,$$

for all  $y \in B(S)$ . Hence  $B(S)$  is an equi-continuous set in  $X$ . Thus  $B$  is totally bounded in view of Arzelà-Ascoli Theorem.

Thus all the conditions of Theorem 2.3 are satisfied and hence an application of it yields that the DE (1.1) has a maximal and a minimal positive solution on  $J$ .  $\square$

**Theorem 4.3.** *Suppose that the assumptions  $(B_0), (B_3)$  and  $(C_1)-(C_4)$  hold. Further if*

$$\|k\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1} \right) < 1,$$

and  $\ell$  is given in Remark 4.1, then DE (1.1) has a minimal and a maximal positive solution on  $J$ .

**Proof.** Now DE (1.1) is equivalent to FIE (3.2) on  $J$ . Let  $X = C(J, \mathbb{R})$ . Define two operators  $A$  and  $B$  on  $X$  by (3.4) and (3.5), respectively. Then FIE (3.2) is transformed into an operator equation  $Ax(t)Bx(t) = x(t)$  in a Banach algebra  $X$ . Notice that  $(B_0)$  implies  $A, B : [u, v] \rightarrow K$ . Since the cone  $K$  in  $X$  is normal,  $[u, v]$  is a norm bounded set in  $X$ . Now it can be shown as in the proofs of Theorem 3.1 and Theorem 2.4 that the operator  $A$  is a Lipschitz with a Lipschitz constant  $\alpha = \|k\|$  and  $B$  is totally bounded with  $M = \|B([u, v])\| = \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1}$ , respectively. Since  $\alpha M = \|k\| \left( \left| \frac{x_0}{f(0, x_0)} \right| + |c|T + T\|\ell\|_{L^1} \right) < 1$ , the desired conclusion follows by an application of Theorem 2.4. This completes the proof.  $\square$

## 5. AN EXAMPLE

Given the closed and bounded interval  $J = [0, 1]$  in  $\mathbb{R}$ , consider the non-linear DE

$$\left. \begin{aligned} \frac{d^2}{dt^2} \left[ \frac{x(t)}{f(t, x(t))} \right] &= \frac{p(t)x(t)}{1 + |x(t)|}, \quad \text{a.e. } t \in J \\ x(0) = 0, \quad x'(0) &= 1, \end{aligned} \right\} \quad (5.1)$$

where  $p \in L^1(J, \mathbb{R})$  and the function  $f : J \times \mathbb{R} - \{0\} \rightarrow \mathbb{R}$  is defined by

$$f(t, x) = 1 + \alpha t|x|,$$

for  $\alpha > 0$ . Obviously  $f : J \times \mathbb{R} \rightarrow \mathbb{R}^+ - \{0\}$ . Define a function  $g : J \times \mathbb{R} \rightarrow \mathbb{R}$  by

$$g(t, x) = \frac{p(t)x}{1 + |x|}.$$

It is easy to verify that  $f$  is continuous and Lipschitz on  $J \times \mathbb{R}$  with a Lipschitz constant  $\alpha$ . Further  $g(t, x)$  is  $L^1_X$ -Carathéodory with the bound function  $h(t) = p(t)$  on  $J$ . Therefore if  $\alpha(1 + \|p\|_{L^1}) < 1$ , then by Theorem 3.1, the DE (5.1) has a solution on  $J$ , because the function  $\Omega$  satisfies condition (3.3) with  $\gamma(t) = p(t)$  for all  $t \in J$  and  $\Omega(r) = r$  for all  $r \in \mathbb{R}^+$ .

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