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POSITIVE SOLUTIONS FOR REGULAR AND SINGULAR FOURTH-ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT: By applying well-known fixed point theorems in cones, we study the existence of positive solutions for fourth-order boundary value problem $x^{(4)}(t) + \beta x''(t) = f(t, x),\ 0 < t < 1$ with $x(0) = x(1) = x''(0) = x''(1) = 0$, where $0 < \beta < \pi^2$. We discuss both the singular case and the regular case.

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1. INTRODUCTION

In this paper we study the existence of positive solutions of the fourth-order boundary value problem

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$$\begin{cases} x^{(4)}(t) + \beta x''(t) = f(t,x), \quad 0 < t < 1\,, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{1.1}$$

where $0 < \beta < \pi^2$. By applying well-known fixed point theorems in cones, we will establish two existence results for (1.1) in the regular case and one result in the singular case.

Fourth-order boundary value problems have proved to be important in applications. For example, the deformations of an elastic beam in the equilibrium state Reiss et al [16], whose two ends are simply supported, can be described by

$$\begin{cases} x^{(4)}(t) = f(t,x,x''), \quad 0 < t < 1\,, \\ x(0) = x(1) = x''(0) = x''(1) = 0. \end{cases} \tag{1.2}$$

In recent years, there have been many results on the existence of positive solutions of such problems. We refer the reader to Agarwal [1], De Coster et al [5], Del Pino and Manasevich [6], Jiang et al [11], Li [14], Ma [15], Zhang and Kong [18].

In this paper we apply two well-known fixed point theorems in cones (see Theorem 2.1 and Theorem 3.2) to the boundary value problem (1.1) and we show that (1.1) has at least one positive solution under some suitable conditions. We discuss both regular and singular cases.

As applications of our new results for the regular case, we consider the existence of positive solutions of

$$\begin{cases} x^{(4)}(t) + \beta x''(t) = h(t)g(x), \quad 0 < t < 1\,, \\ x(0) = x(1) = x''(0) = x''(1) = 0. \end{cases} \tag{1.3}$$

In addition, we characterize the eigenvalues for the nonlinear problem

$$\begin{cases} x^{(4)}(t) + \beta x''(t) = \lambda h(t)g(x), \quad 0 \le t \le 1\,, \\ x(0) = x(1) = x''(0) = x''(1) = 0, \end{cases} \tag{1.4}$$

where $\lambda > 0$ is a positive parameter. We prove that (1.4) has at least one positive solution for each λ in an explicit eigenvalue interval. Recently, several eigenvalue characterizations for different kinds of boundary value problems have appeared and we refer the reader to Chu and Jiang [3], Chu and Zhou [4], Erbe [7], Hao and Debnath [8].

To conclude the introduction we present some results for problem (1.1) which will be needed in Section 2 and Section 3. As in Jiang et al [11], suppose that x is a positive solution of problem (1.1). Then

$$x(t) = \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x(s))dsd\tau. \tag{1.5}$$

Here $G_1(t,s)$ is the Green's function for

$$-x'' = 0, \qquad x(0) = x(1) = 0$$

and $G_2(t,s)$ is the Green's function for

$$-x'' - \beta x = 0, \qquad x(0) = x(1) = 0.$$

Explicitly, we have

$$G_1(t,s) = \begin{cases} t(1-s) & \text{if} \quad 0 \le t \le s \le 1, \\ s(1-t) & \text{if} \quad 0 \le s \le t \le 1, \end{cases} \tag{1.6}$$

and one can show that

$$t(1-t)G_1(s,s) \le G_1(t,s) \le G_1(s,s) = s(1-s),\ \forall\ (t,s) \in [0,1] \times [0,1].$$

Set $\omega = \sqrt{\beta}$. Then $G_2(t,s)$ is explicitly given by

$$G_2(t,s) = \begin{cases} \dfrac{\sin \omega t \sin \omega(1-s)}{\omega \sin \omega} & \text{if} \quad 0 \le t \le s \le 1, \\ \dfrac{\sin \omega s \sin \omega(1-t)}{\omega \sin \omega} & \text{if} \quad 0 \le s \le t \le 1. \end{cases} \tag{1.7}$$

Clearly $G(t,s) > 0$ for all $(t,s) \in (0,1) \times (0,1)$.

2. REGULAR CASE

In this section, we establish two different existence results for (1.1) for the regular case. Throughout this section, we always assume that

($\mathtt{H}_1$) $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous.

As indicated in the introduction, the proof of our main results is based on a fixed point theorem for compact maps on conical shells. We recall the statement of this result below, after introducing the definition of a cone.

Definition. Let X be a Banach space and K be a closed, nonempty subset of X. K is a cone if:

(i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$.

(ii) $u, -u \in K$ implies $u = 0$.

We also recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If D is a subset X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 2.1. (see Krasnosel'skii [12]) *Let X be a Banach space and K a cone in X. Assume Ω^1, Ω^2 are open bounded subsets of X with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega^2_K$. Let*

$$T : \overline{\Omega^2}_K \to K$$

be a continuous and compact operator such that:

(i) $\|Tx\| \le \|x\|$ *for* $x \in \partial_K \Omega^1$, *and*

(ii) *there exists* $e \in K\backslash\{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial_K \Omega^2$ *and all* $\lambda > 0$.

Then T *has a fixed point in* $\overline{\Omega^2}_K \setminus \Omega^1_K$. *The same conclusion remains valid if* (i) *holds on* $\partial_K \Omega^2$ *and* (ii) *holds on* $\partial_K \Omega^1$.

In the applications below, we take $X = C[0,1]$ with the supremum norm $\|\cdot\|$ and define

$$K = \{x \in X : x(t) \ge 0 \text{ and } x(t) \ge t(1-t)\|x\| \text{ for all } t \in [0,1]\}, \tag{2.1}$$

and one may easily verify that K is a cone in X. Define an operator $T : X \to X$ by

$$Tx(t) = \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s,x(s)) ds d\tau \tag{2.2}$$

for $x \in X$ and $t \in [0,1]$.

Theorem 2.2. *Suppose that* (H_1) *holds. Then* T *is well defined and maps* X *into* K. *Moreover,* T *is continuous and completely continuous.*

Proof. Using a standard argument, one can show that T is a continuous and completely continuous operator since $f : [0,1] \times [0,\infty) \to [0,\infty)$ and $G_i : [0,1] \times [0,1] \to [0,\infty)$, $i = 1,2$, are continuous.

Next, we show that T maps X into K. Let $x \in X$, so we have for $t \in [0,1]$,

$$\begin{aligned} Tx(t) &= \int_0^1 \int_0^1 G_1(t,\tau) G_2(\tau,s) f(s,x(s)) ds d\tau \\ &\le \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) f(s,x(s)) ds d\tau. \end{aligned}$$

Therefore,

$$\|Tx\| \le \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) f(s,x(s)) ds d\tau.$$

On the other hand,

$$Tx(t) \ge t(1-t) \int_0^1 \int_0^1 G_1(\tau,\tau) G_2(\tau,s) f(s,x(s)) ds d\tau.$$

Thus, $Tx(t) \ge t(1-t)\|Tx\|$. Therefore $T(X) \subset K$. □

Now we state our first existence result for problem (1.1) in this section.

Theorem 2.3. *Assume that condition* (H$_1$) *holds. In addition, assume that there exist* $\alpha,\ \beta > 0$ *such that the following two hypotheses hold:*

(H$_2$) *There exist a continuous function* $\phi : [0, 1] \to (0, \infty)$ *and a continuous, nondecreasing function* $\omega_1 : [0, \infty) \to [0, \infty)$ *such that*

$$f(t, x) \le \phi(t)\omega_1(x), \qquad 0 \le t \le 1,\ 0 < x \le \alpha,$$

$$\omega_1(\alpha) \sup_{0\le t\le 1} \int_0^1 \int_0^1 G_1(t, \tau)G_2(\tau, s)\phi(s)dsd\tau \le \alpha.$$

(H$_3$) *There exist a constant* $a \in (0, \frac{1}{2})$, *a continuous function* $\psi : [a, 1 - a] \to (0, \infty)$, *and a continuous, nondecreasing function* $\omega_2 : [0, \infty) \to [0, \infty)$ *such that*

$$f(t, x) \ge \psi(t)\omega_2(x), \qquad a \le t \le 1 - a,\ c\beta \le x \le \beta,$$

$$x \le \omega_2(x) \inf_{a\le t\le 1-a} \int_0^1 \int_a^{1-a} G_1(t, \tau)G_2(\tau, s)\psi(s)dsd\tau, \quad c\beta \le x \le \beta;$$

here and henceforth $c = a(1 - a)$.

Then problem (1.1) has at least one positive solution $x \in K$ *with*

$$\min\{\alpha, \beta\} \le \|x\| \le \max\{\alpha, \beta\}.$$

Proof. We assume that $\alpha < \beta$. The case $\alpha > \beta$ is analogous.

Define the sets

$$\Omega^1 = \{x \in X : \|x\| < \alpha\} \text{ and } \Omega^2 = \{x \in X : \|x\| < \beta\}.$$

First we prove that

$$\|Tx\| \le \|x\|,\ \forall x \in \partial_K\Omega^1. \tag{2.3}$$

In fact, for any $x \in \partial_K\Omega^1$, we have for $t \in [0, 1]$,

$$\begin{aligned} Tx(t) &= \int_0^1 \int_0^1 G_1(t, \tau)G_2(\tau, s)f(s, x(s))dsd\tau \\ &\le \int_0^1 \int_0^1 G_1(t, \tau)G_2(\tau, s)\phi(s)\omega_1(x(s))dsd\tau \\ &\le \omega_1(\alpha) \sup_{0\le t\le 1} \int_0^1 \int_0^1 G_1(t, \tau)G_2(\tau, s)\phi(s)dsd\tau \\ &\le \alpha = \|x\|. \end{aligned}$$

Therefore, $\|Tx\| \le \|x\|$, i.e., (2.3) holds.

Let $e(t) \equiv 1$. Then $e \in K \setminus \{0\}$. Next we prove that

$$x \neq Tx + \lambda e, \ \forall\, x \in \partial_K \Omega^2 \text{ and } \lambda > 0. \tag{2.4}$$

If not, there would exist $x_0 \in \partial_K \Omega^2$ and $\lambda_0 > 0$ such that $x_0 = Tx_0 + \lambda_0 e$.

Since $x_0 \in \partial_K \Omega^2$, we have $x_0(t) \geq c\|x_0\| = c\beta$, $a \leq t \leq 1-a$. Let $\mu = \min\limits_{a \leq t \leq 1-a} x_0(t) \geq c\beta$. Then for $a \leq t \leq 1-a$, we have

$$\begin{aligned}
x_0(t) = Tx_0(t) + \lambda_0 &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x_0(s))dsd\tau + \lambda_0 \\
&\geq \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)f(s,x_0(s))dsd\tau + \lambda_0 \\
&\geq \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)\omega_2(x_0(s))dsd\tau + \lambda_0 \\
&\geq \omega_2(\mu) \inf_{a \leq t \leq 1-a} \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau + \lambda_0 \geq \mu + \lambda_0.
\end{aligned}$$

This implies $\mu \geq \mu + \lambda_0$, a contradiction. Therefore, (2.4) holds. □

It follows from Theorem 2.1, (2.3) and (2.4) that T has a fixed point $\overline{\Omega^2_K} \setminus \Omega^1_K$. Clearly, this fixed point is a positive solution of problem (1.1) satisfying $\alpha \leq \|x\| \leq \beta$.

Before stating another existence result, we need to select adequate open bounded sets. Following Infante and Webb [9], Lan [13], define

$$\Omega^r = \{x \in X : \ q(x) < cr\}, \quad B^r = \{x \in X : \|x\| < r\},$$

where $q(x) = \min\{x(t) : a \leq t \leq 1-a\}$.

Recall from Infante and Webb [9], Lan [13] the following result.

Lemma 2.4. *Ω^r and B^r defined above have the following properties:*

(a) *Ω^r_K and B^r_K are open relative to K.*
(b) *$B^{cr}_K \subset \Omega^r_K \subset B^r_K$.*
(c) *$x \in \partial_K \Omega^r$ if and only if $q(x) = cr$.*
(d) *If $x \in \partial_K \Omega^r$, then $cr \leq x(t) \leq r$, $a \leq t \leq 1-a$.*

It is clear that the sets Ω^r are unbounded sets for each $r > 0$, so we can not use Theorem 2.1 with Ω^r. However we will be able to apply Theorem 2.1 taking into account that, for each $\delta > r$, the following relations hold:

$$\Omega^r_K = (\Omega^r \cap B^\delta)_K \quad \text{and} \quad \overline{\Omega^r_K} = (\overline{\Omega^r \cap B^\delta})_K.$$

The first equality follows immediately from Lemma 2.4 (b). For the second let $x \in \overline{\Omega^r}_K$, then from Lemma 2.4 (c) we have that

$$c\|x\| \leq \min_{a \leq t \leq 1-a} x(t) \leq cr < c\delta,$$

so $x \in (\overline{\Omega^r} \cap B^\delta) \cap K$. Now, since Ω^r and B^δ are open sets we have $\overline{\Omega^r} \cap B^\delta \subset \overline{\Omega^r \cap B^\delta}$. Thus $x \in (\overline{\Omega^r \cap B^\delta})_K$, and therefore $\overline{\Omega^r}_K \subseteq (\overline{\Omega^r \cap B^\delta})_K$. The reverse inclusion is trivial.

Theorem 2.5. *Assume that condition* (H_1) *holds. In addition, we assume that the following two hypotheses hold:*

(H_4) *there exist a constant $\alpha > 0$ and a continuous function $\psi : [a, 1-a] \to (0, \infty)$ such that*

$$f(t, x) \geq c\alpha\psi(t), \qquad a \leq t \leq 1-a,\ c\alpha \leq x \leq \alpha,$$

$$\inf_{a \leq t \leq 1-a} \int_0^1 \int_a^{1-a} G_1(t, \tau)G_2(\tau, s)\psi(s)dsd\tau \geq 1.$$

(H_5) *there exist a constant $\beta > 0$ and a continuous function $\phi : [0, 1] \to (0, \infty)$ such that*

$$f(t, x) \leq \beta\phi(t), \qquad 0 \leq t \leq 1,\ 0 < x \leq \beta,$$

$$\sup_{0 \leq t \leq 1} \int_0^1 \int_0^1 G_1(t, \tau)G_2(\tau, s)\phi(s)dsd\tau \leq 1.$$

Then, the following results hold:

(a) *if $\beta < c\alpha$, then problem (1.1) has at least one positive solution x satisfying*

$$\beta \leq \|x\| \leq \alpha \text{ and } c\beta \leq \min_{a \leq t \leq 1-a} x(t) \leq c\alpha;$$

(b) *if $\alpha < \beta$, then problem (1.1) has at least one positive solution x satisfying*

$$c\alpha \leq \|x\| \leq \beta \text{ and } c\alpha \leq \min_{a \leq t \leq 1-a} x(t).$$

Proof. We claim that:

(i) $\|Tx\| \leq \|x\|$ for $x \in \partial_K B^\beta$, and

(ii) there exists $e \in K\backslash\{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial_K \Omega^\alpha$ and all $\lambda > 0$.

We start with (i). In fact, for any $x \in \partial_K B^\beta$ and $t \in [0,1]$,

$$
\begin{aligned}
Tx(t) &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x(s))dsd\tau \\
&\le \beta \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)dsd\tau \\
&\le \beta \sup_{0\le t\le 1} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)dsd\tau \\
&\le \beta = \|x\|.
\end{aligned}
$$

This implies that (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, then $e \in K \setminus \{0\}$. Next, suppose that there exists $x \in \partial_K \Omega^\alpha$ and $\lambda > 0$ such that $x = Tx + \lambda e$. Since $x \in \partial_K \Omega^\alpha$, then from Lemma 2.4 (d) we have $\alpha \ge x_0(t) \ge c\|x_0\| = c\alpha,\ a \le t \le 1-a$. Then for $a \le t \le 1-a$, we have

$$
\begin{aligned}
x_0(t) = Tx_0(t) + \lambda_0 &= \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x_0(s))dsd\tau + \lambda_0 \\
&\ge \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)f(s,x_0(s))dsd\tau + \lambda_0 \\
&\ge c\alpha \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau + \lambda_0 \\
&\ge c\alpha \inf_{a\le t\le 1-a} \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau + \lambda_0 \ge c\alpha + \lambda_0.
\end{aligned}
$$

Hence $\min_{a\le t\le 1-a} x(t) \ge c\alpha + \lambda_0 > c\alpha$, contradicting the statement of Lemma 2.4 (c). This contradiction proves part (ii) of our claim.

Now suppose that $\beta < c\alpha$. Then one has from Lemma 2.4 that $\overline{B^\beta}_K \subset B_K^{c\alpha} \subset \Omega_K^\alpha$ and therefore it follows from Theorem 2.1 that T has at least one fixed point $x \in \overline{\Omega_K^\alpha} \setminus B_K^\beta$. Hence $c\beta \le \min_{a\le t\le b} x(t) \le c\alpha$ and $\|x\| \ge \beta$ hold. On the other hand, $c\|x\| \le \min_{a\le t\le b} x(t) \le c\alpha$ and therefore $\|x\| \le \alpha$.

Finally, if $\alpha < \beta$ one has $\overline{\Omega^\alpha}_K \subset B_K^\beta$, and then Theorem 2.1 guarantees the existence of at least one fixed point $x \in \overline{B^\beta}_K \setminus \Omega_K^\alpha$ of T. Hence we obtain the inequalities

$$c\alpha \le \|x\| \le \beta \text{ and } c\alpha \le \min_{a\le t\le 1-a} x(t). \qquad \square$$

As an application, now we consider the existence of positive solutions for the fourth-order boundary value problem (1.3). We assume the following conditions hold:

($\mathtt{h}_1$) $g : [0, \infty) \to [0, \infty)$ is continuous.

($\mathtt{h}_2$) $h : [0, 1] \to [0, \infty)$ is continuous and there exists $a \in (0, \frac{1}{2})$ such that $h(t) > 0$ for $t \in [a, 1-a]$.

Theorem 2.6. *Suppose that conditions* ($\mathtt{h}_1$)-($\mathtt{h}_2$) *hold. Then problem (1.3) has at least one positive solution x with $x(t) \not\equiv 0$ for $t \in (0, 1)$ if one of the following conditions holds:*

($\mathtt{h}_3$) $0 \le g^0 < M$ *and* $m < g_\infty \le \infty$;
($\mathtt{h}_4$) $0 \le g^\infty < M$ *and* $m < g_0 \le \infty$.
Here

$$g_0 = \liminf_{x \to 0} \frac{g(x)}{x}, \quad g_\infty = \liminf_{x \to \infty} \frac{g(x)}{x},$$

$$g^0 = \limsup_{x \to 0} \frac{g(x)}{x}, \quad g^\infty = \limsup_{x \to \infty} \frac{g(x)}{x}$$

and

$$M = \left(\max_{0 \le t \le 1} \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) h(s) ds d\tau \right)^{-1},$$

$$m = \left(\min_{a \le t \le 1-a} \int_0^1 \int_a^{1-a} G_1(t, \tau) G_2(\tau, s) h(s) ds d\tau \right)^{-1}.$$

Proof. To see this, we will apply Theorem 2.5 with $f(t, x) = h(t)g(x)$. We assume that ($\mathtt{h}_3$) holds. The case when ($\mathtt{h}_4$) holds is completely analogous.

By the first part of ($\mathtt{h}_3$), there exists $\beta > 0$ such that $g(x) \le M\beta$ for $0 \le x \le \beta$. Choosing $\phi(t) = Mh(t)$, then

$$f(t, x) = h(t)g(x) \le M\beta h(t) = \beta\phi(t) \quad \text{if } 0 \le t \le 1 \text{ and } 0 \le x \le \beta$$

and

$$\begin{aligned} &\int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) \phi(s) ds d\tau \\ &\qquad = M \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) h(s) ds d\tau \\ &\qquad\qquad \le M \max_{0 \le t \le 1} \int_0^1 \int_0^1 G_1(t, \tau) G_2(\tau, s) h(s) ds d\tau \le 1 \end{aligned}$$

Thus hypothesis ($\mathtt{H}_5$) holds.

From the second part of ($\mathtt{h}_3$), there exists $\alpha > 0$ such that $c\alpha > \beta$ and $g(x) \geq mx$ for $x \geq c\alpha$. Choose $\psi(t) = mh(t)$, then for $a \leq t \leq 1-a$, we have

$$f(t,x) = h(t)g(x) \geq mc\alpha h(t) = c\alpha\psi(t), \ x \geq c\alpha,$$

so in particular for $c\alpha \leq x \leq \alpha$) and

$$\begin{aligned}\int_0^1 \int_a^{1-a} & G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau \\ &= m\int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)h(s)dsd\tau \\ &\geq m \min_{a\leq t\leq 1-a} \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)h(s)dsd\tau \geq 1\,.\end{aligned}$$

This implies that hypothesis ($\mathtt{H}_4$) holds. The result now follows from Theorem 2.1. □

Now we consider the nonlinear eigenvalue problem (1.4). By applying Theorem 2.6, we easily obtain the following result.

Theorem 2.7. *Suppose that conditions* ($\mathtt{h}_1$)-($\mathtt{h}_2$) *hold. Then problem (1.4) has at least one positive solution for every* $\lambda \in (m/g_\infty, M/g^0)$ *if* $m/g_\infty < M/g^0$*. The same result remains valid for every* $\lambda \in (m/g_0,\, M/g^\infty)$ *if* $m/g_0 < M/g^\infty$*. Here we write* $m/g_\alpha = 0$ *if* $g_\alpha = \infty$ *and* $M/g^\alpha = \infty$ *if* $g^\alpha = 0$*, where* $\alpha = 0, \infty$*.*

Remark. The conclusions of Theorem 2.6 and Theorem 2.7 remain valid if $h(t)$ is allowed to have singularities at $t = 0, 1$. Our results improve those in Hao and Debnath [8].

3. SINGULAR CASE

In this section, we establish one existence result for (1.1) in the singular case. Now (1.1) has a singularity at $x = 0$ if

$$\lim_{x\to 0^+} f(t,x) = +\infty \qquad \text{uniformly in } t \in [0,1].$$

To state the results in this section, we introduce the following hypotheses:

($\mathtt{G}_1$) $f : [0,1] \times (0,\infty) \to (0,\infty)$ is continuous.

($\mathtt{G}_2$) There exist a continuous function $\phi(t) : [0,1] \to (0,\infty)$ and non-negative continuous functions $g(x)$ and $h(x)$ defined on $(0,\infty)$ such that

$$f(t,x) \leq \phi(t)(g(x) + h(x)) \qquad \text{for all } (t,x) \in [0,1] \times (0,\infty),$$

where $g(x)$ is non-increasing and $h(x)/g(x)$ is non-decreasing in $x \in (0,\infty)$.

($\mathtt{G}_3$) $\Lambda_0 = \sup_{0\le t\le 1} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)g(s(1-s))dsd\tau < \infty.$

($\mathtt{G}_4$) There exists a constant $K_0 > 0$ such that $g(xy) \le K_0 g(x)g(y)$ for all $x \ge 0$, $y \ge 0$.

($\mathtt{G}_5$) There exists a constant $a \in (0, \frac{1}{2})$ and a continuous function $\psi(t) : [0,1] \to (0,\infty)$, non-negative continuous functions $g_1(x)$ and $h_1(x)$ defined on $(0,\infty)$ such that

$$f(t,x) \ge \psi(t)(g_1(x) + h_1(x)) \qquad \text{for all } (t,x) \in [a, 1-a] \times (0,\infty),$$

where $g_1(x) > 0$ is non-increasing and $h_1(x)/g_1(x)$ is non-decreasing in $x \in (0,\infty)$.

($\mathtt{G}_6$) There exists a constant $r > 0$ such that

$$K_0\Lambda_0 \le \frac{r}{g(r)+h(r)}.$$

($\mathtt{G}_7$) There exists a constant $R > r$ such that

$$\inf_{a\le t\le 1-a} \int_0^1 \int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau \ge \frac{R}{g_1(R)\left(1 + \frac{h_1(cR)}{g_1(cR)}\right)},$$

where $c = a(1-a)$.

Theorem 3.1. *Assume that conditions* ($\mathtt{G}_1$) $-$ ($\mathtt{G}_4$) *hold, then the operator* $T : \overline{K}_{r,R} \to K$ *is continuous and compact, here* $\overline{K}_{r,R} = \{x \in K : r \le \|x\| \le R\}$.

Proof. First we show that T is well defined. To see this note that if $x \in \overline{K}_{r,R}$, then $r \le \|x\| \le R$ and $x(t) \ge t(1-t)\|x\| \ge t(1-t)r$. Therefore,

$$f(t,x(t)) \le \phi(t)(g(x(t)) + h(x(t))) \le K_0\phi(t)(g(r)+h(r))g(t(1-t)).$$

This inequality together with condition ($\mathtt{G}_3$) guarantees that $T : \overline{K}_{r,R} \to C[0,1]$.

Next we show that T maps $\overline{K}_{r,R}$ into K. If $x \in \overline{K}_{r,R}$, then for $t \in [0,1]$ we have

$$Tx(t) \le \int_0^1 \int_0^1 G_1(\tau,\tau)G_2(\tau,s)f(s,x(s))dsd\tau$$

and

$$Tx(t) \ge t(1-t) \int_0^1 \int_0^1 G(\tau,\tau)G_2(\tau,s)f(s,x(s))dsd\tau.$$

Therefore, $Tx(t) \ge t(1-t)\|Tx\|$, i.e., $Tx \in K$.

Now we show $T : \overline{K}_{r,R} \to K$ is continuous. Let $x_n, \ x \in \overline{K}_{r,R}$ with $\|x_n - x\| \to 0$ as $n \to \infty$. Of course $r \le \|x_n\| \le R, \ r \le \|x\| \le R, \ x_n(t) \ge t(1-t)r, \ x(t) \ge t(1-t)r$ and

$$x_n(t) \in [t(1-t)r, R] \quad n \in \{1, 2, \dots\} \qquad \text{and} \quad x(t) \in [t(1-t)r, R] \quad \text{for } t \in [0,1].$$

Notice also that

$$\rho_n(s) = |f(s, x_n(s)) - f(s, x(s))| \to 0 \quad n \to \infty \text{ for } s \in (0,1)$$

and

$$\rho_n(s) \le 2\phi(s)\{1 + \frac{h(R)}{g(R)}\}g(s(1-s)r) \text{ for } s \in (0,1).$$

Now these together with the Lebesgue dominated convergence theorem guarantee that

$$\|Tx_n - Tx\| \le \sup_{0 \le t \le 1} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)\rho_n(s)dsd\tau \to 0 \quad \text{as } n \to \infty.$$

Hence $T : \overline{K}_{r,R} \to K$ is continuous. Finally we prove that $T : \overline{K}_{r,R} \to K$ is completely continuous. In fact, for $x \in \overline{K}_{r,R}$,

$$\|Tx\| \le \left(1 + \frac{h(R)}{g(R)}\right) \sup_{0 \le \le 1} \int_0^1 \int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)g(s(1-s)r)\, dsd\tau$$

and for $t, t' \in [0,1]$, we have

$$\|Tx(t) - Tx(t')\| \le \left(1 + \frac{h(R)}{g(R)}\right) \int_0^1 \int_0^1 |G_1(t,\tau) - G_1(t',\tau)|G_2(\tau,s)\phi(s)g(s(1-s)r)dsd\tau.$$

Now the Arzela-Ascoli Theorem guarantees that $T : \overline{K}_{r,R} \to K$ is compact. □

We need the following well known fixed point theorem of compression and expansion of cones.

Theorem 3.2. (see Krasnosel'skii [12]) *Let X be a Banach space and K $(\subset X)$ be a cone. Assume that Ω^1, Ω^2 are open subsets of X with $0 \in \Omega^1$, $\overline{\Omega^1} \subset \Omega^2$, and let*

$$T : (\overline{\Omega^2} \backslash \Omega^1)_K \to K$$

be a continuous and completely continuous operator such that either

(i) $\|Tu\| \ge \|u\|, \ u \in \partial_K\Omega^1$ *and* $\|Tu\| \le \|u\|, \ u \in \partial_K\Omega^2$; *or*
(ii) $\|Tu\| \le \|u\|, \ u \in \partial_K\Omega^1$ *and* $\|Tu\| \ge \|u\|, \ u \in \partial_K\Omega^2$.

Then T has a fixed point in $(\overline{\Omega^2}\backslash\Omega^1)_K$.

In the application below, we also take $X = C[0,1]$ with the supremum norm $\|\cdot\|$ and define a cone K as given by (2.1).

Theorem 3.3. *Assume that conditions $(\mathtt{G}_1)$-$(\mathtt{G}_7)$ hold. Then problem (1.1) has at least one positive solution x with $r \le \|x\| \le R$.*

Proof. Define the open sets $\Omega^1 = \{x \in X : \|x\| < r\}$ and $\Omega^2 = \{x \in X : \|x\| < R\}$.

First we prove that

$$\|Tx\| \le \|x\|, \ \forall x \in \partial_K\Omega^1. \tag{3.1}$$

In fact, for any $x \in \partial_K\Omega^1$, we have for $t \in [0,1]$,

$$\begin{aligned} Tx(t) &= \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x(s))dsd\tau \\ &\le \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)(g(x(s))+h(x(s)))dsd\tau \\ &\le K_0(h(r)+g(r))\sup_{0\le t\le 1}\int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)\phi(s)g(s(1-s))dsd\tau \\ &\le r = \|x\|. \end{aligned}$$

Therefore, $\|Tx\| \le \|x\|$, i.e., (3.1) holds.

Next we prove that

$$\|Tx\| \ge \|x\|, \quad \forall x \in \partial_K\Omega^2. \tag{3.2}$$

In fact, for $a \le t \le 1-a$, we have

$$\begin{aligned} Tx(t) &= \int_0^1\int_0^1 G_1(t,\tau)G_2(\tau,s)f(s,x(s))dsd\tau \\ &\ge \int_0^1\int_a^{1-a} G_1(t,\tau)G_2(\tau,s)f(s,x(s))dsd\tau \\ &\ge \int_0^1\int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)\{g_1(x(s))+h_1(x(s))\}dsd\tau \\ &\ge g_1(R)\left(1+\frac{h_1(cR)}{g_1(cR)}\right)\inf_{a\le t\le 1-a}\int_0^1\int_a^{1-a} G_1(t,\tau)G_2(\tau,s)\psi(s)dsd\tau \\ &\ge R = \|x\|. \end{aligned}$$

This implies (3.2) holds. □

It follows from Theorem 3.2, (3.1) and (3.2) that T has a fixed point $(\overline{\Omega^2}\backslash\Omega^1)_K$. Clearly, this fixed point is a positive solution of (1.1) satisfying $r \le \|x\| \le R$.

Remark. In the literature, one usually discusses first an appropriate family of regular problems, in order to establish the existence of a positive solution to a singular problem. The idea is to establish the existence of a positive solution x_n to the regular problem, and then a positive solution to the singular problem is obtained by letting $n \to +\infty$ in a subset of N. A key step is a truncation technique and the application of the Arzela-Ascoli Theorem. In particular one considers the "modified" problems

$$\begin{cases} x^{(4)}(t) + \beta x''(t) = f^*(t, x), \quad 0 < t < 1\,, \\ x(0) = x(1) = \frac{1}{n},\ x''(0) = x''(1) = 0, \end{cases}$$

where

$$f^*(t,x) = \begin{cases} f(t,x) & \text{if} \quad x \ge \frac{1}{n}, \\ f(t, \frac{1}{n}) & \text{if} \quad x \le \frac{1}{n}. \end{cases}$$

Let $n \to +\infty$ to obtain the existence of a positive solution of (1.1).

However, in this paper, we deal with the singular problem (1.1) by using Theorem 3.2 directly. The main reason is that we only need to show that $T : (\overline{\Omega^2}\backslash\Omega^1)_K \to K$ is continuous and completely continuous in Theorem 3.2. This point is important and has some advantages for singular problems. In fact, Agarwal and O'Regan have used this technique to deal with singular conjugate, focal and $(N,\ P)$ problems successfully in Agarwal and O'Regan [2]. The papers Jiang et al [10] and Torres [17] study singular periodic problems using the same technique.

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