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OSCILLATION CRITERIA FOR FIRST ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH ADVANCED ARGUMENTS

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ABSTRACT: In this paper some new oscillation criteria are obtained for the first order nonlinear differential equation

$$x'(t) = \sum_{i=1}^{n} p_i(t) f_i(x(\sigma_0(t)), \ldots, x(\sigma_m(t))), \qquad t \geq \alpha$$

with advanced arguments, where $n \geq 1$, $m \geq 1$ are integers, $p_i(t) \geq 0$, $\sigma_j(t) \geq t$ and $x_0 f_i(x_0, \ldots, x_m) > 0$ for $t \geq \alpha$, $x_0 x_j > 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m$.

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1. INTRODUCTION

Consider the first order nonlinear differential equation

$$x'(t) = \sum_{i=1}^{n} p_i(t) f_i(x(\sigma_0(t)), \ldots, x(\sigma_m(t))), \qquad t \geq \alpha \tag{1}$$

with several advanced arguments, where $n \geq 1$, $m \geq 1$ are integers; $p_i(t)$, $i = 1, \ldots, n$ are nonnegative continuous functions; $f_i(x_0, \ldots, x_m)$, $i = 1, \ldots, n$ are continuous functions and $x_0 f_i(x_0, \ldots, x_m) > 0$ for $x_0 x_j > 0$, $j = 0, \ldots, m$; $\sigma_j(t) \geq t$ for $t \geq \alpha$, $j = 0, \ldots, m$.

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The oscillatory behavior of the solutions of equation (1) have been studied by many authors (see the monographs Erbe et al [1], Györi and Ladas [2], Ladde et al [6] and the reference cited therein).

We note that various oscillation criteria for equation (1) can be obtained by using the same technique, which is applied in proving analogous oscillation criteria for the corresponding differential equation

$$x'(t) + \sum_{i=1}^{n} p_i(t) f_i(x(\tau_0(t)), \ldots, x(\tau_m(t))) = 0\,, \qquad t \geq \alpha \tag{2}$$

with several retarded arguments $\tau_i(t)$: $\tau_j(t) \leq t$, $t \geq \alpha$, $j = 0, \ldots, m$.

Important special cases of equations (1) and (2) are the equations:

$$x'(t) = p(t)x(t+\tau)\,, \tag{3}$$

$$x'(t) + p(t)x(t-\tau) = 0\,, \tag{4}$$

$$x'(t) = p(t)\prod_{j=1}^{m}\big[x(t+\tau_j)\big]^{\alpha_j}\,, \tag{5}$$

$$x'(t) + p(t)\prod_{j=1}^{m}\big[x(t-\tau_j)\big]^{\alpha_j} = 0\,, \tag{6}$$

$$x'(t) = \sum_{i=1}^{n} p_i(t)x(t+\delta_i)\,, \tag{7}$$

$$x'(t) + \sum_{i=1}^{n} p_i(t)x(t-\delta_i) = 0\,, \tag{8}$$

$$x'(t) = p(t)f(x(t+\tau_1), \ldots, x(t+\tau_m))\,, \tag{9}$$

$$x'(t) + p(t)f(x(t-\tau_1), \ldots, x(t-\tau_m)) = 0\,, \tag{10}$$

where $p(t) \geq 0$, $p_i(t) \geq 0$ for $t \geq \alpha$, $i = 1, \ldots, n$, $\tau > 0$, $0 < \tau_1 < \cdots < \tau_m$, $0 < \delta_1 < \cdots < \delta_n$ and $\alpha_j > 0$, $j = 1, \ldots, m$ are rational numbers with denominator of positive odd integers with $\sum_{j=1}^{m} \alpha_j = 1$.

Ladas [5] and Koplatadze and Chanturija [4] obtained the well-known oscillation criterion for equations (3) and (4)

$$\lim_{t\to+\infty} \inf \int_{t-\tau}^{t} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}}\,. \tag{11}$$

Yu [10] extended the above result and proved that the nonlinear equation (5) (or resp. (6)) is oscillatory, if

$$\lim_{t\to+\infty} \inf \sum_{j=1}^{m} \alpha_j \int_{t}^{t+\tau_j} p(s)\mathrm{d}s > \frac{1}{\mathrm{e}}\,, \tag{12}$$

or

$$\lim_{t\to+\infty}\inf\sum_{j=1}^{m}\alpha_j\int_{t-\tau_j}^{t}p(s)\mathrm{d}s>\frac{1}{\mathrm{e}}\,,\qquad\text{respectively.}\tag{13}$$

Li [7] improved criterion (11) and proved that equation (4) is oscillatory if there exists $t_0\geq\alpha$ such that $\int_t^{t+\tau}p(s)\mathrm{d}s>0$ for $t\geq t_0$ and

$$\int_{t_0}^{\infty}p(t)\ln\left(\mathrm{e}\int_t^{t+\tau}p(s)\mathrm{d}s\right)\mathrm{d}t=+\infty\,.\tag{14}$$

It is also proved in Li [7], Theorem 2 that equation (8) is oscillatory, if

$$\lim_{t\to+\infty}\sup\int_t^{t+\delta_n}p_n(s)\mathrm{d}s>0\tag{15}$$

and there exists $t_0\geq 0$ such that

$$\sum_{i=1}^{n}\int_t^{t+\delta_i}p_i(s)\mathrm{d}s>0\,,\qquad t\geq t_0\tag{16}$$

and

$$\int_{t_0}^{\infty}\sum_{i=1}^{n}p_i(t)\ln\left(\mathrm{e}\sum_{i=1}^{n}\int_t^{t+\delta_i}p_i(s)\mathrm{d}s\right)\mathrm{d}t=+\infty\,.\tag{17}$$

Jiang [3] considered equations (9) and (10) in the case, when the function $f(x_1,\dots,x_m)$ is approximated in a neighbourhood of the origin by the function $g(x_1,\dots,x_m)=\prod_{j=1}^{m}x_j^{\alpha_j}$. It is proved in Jiang [3], Theorem 1 that equation (10) is oscillatory, if

$$\lim_{t\to+\infty}\inf\sum_{j=1}^{m}\alpha_j\int_t^{t+\tau_j}p(s)\mathrm{d}s>0\tag{18}$$

and there exists $t_0\geq\alpha$ such that

$$\int_{t_0}^{\infty}p(t)\ln\left(\mathrm{e}\sum_{j=1}^{m}\alpha_j\int_t^{t+\tau_j}p(s)\mathrm{d}s\right)\mathrm{d}t=+\infty\,.\tag{19}$$

An analogous oscillation criterion for equation (9) is formulated (without proof) in Jiang [3], Theorem 2. We will show below that this theorem is not true.

In Markova and Simeonov [9] oscillation criteria are obtained, which extend for equation (2) the results from Jiang [3], Theorem 1 and Li [7], Theorem 2.

In the present paper we will establish similar oscillation results for equation (1) in the case, when the functions $f_i(x_0,\dots,x_m)$ are approximated in a neighbourhood of infinity by the functions

$$g_i(x_0, \ldots, x_m) = \Big(\prod_{j=0}^{m} |x_j|^{\alpha_{ij}}\Big)\mathrm{sign}x_0,$$

where $\alpha_{ij} > 0$ are constants and $\sum_{j=0}^{m} \alpha_{ij} = 1$, $i = 1, \ldots, n$.

2. PRELIMINARY NOTES

Let $J = [\alpha, +\infty) \subseteq R_+ = [0, +\infty)$.

Introduce the following conditions:

H1. For each $j = 1, \ldots, m$ the function $\sigma_j \in C(J, R)$; there exists the inverse function $\mu_j(t)$; $\sigma_m(t) > t$, $t \in J$ and

$$t = \sigma_0(t) \le \sigma_1(t) \le \cdots \le \sigma_m(t)\,, \qquad t \in J\,. \tag{20}$$

H2. $p_i \in C(J, R_+)$, $i = 1, \ldots, n$.

H3. $f_i \in C(R^{m+1}, R)$ and $x_0 f_i(x_0, \ldots, x_m) > 0$ for $x_0 x_j > 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m$.

H4. There exist constants $M > 0$, $r > 0$ and $\varepsilon > 0$ such that

$$\left| f_i(x_0, \ldots, x_m) - \left(\prod_{j=0}^{m} |x_j|^{\alpha_{ij}}\right)\mathrm{sign}x_0 \right| \le M \prod_{j=0}^{m} |x_j|^{\alpha_{ij}} \big(\min_{0\le j\le m} |x_j|\big)^{-r}$$

for $|x_j| > \varepsilon$, $x_0 x_j > 0$, where $\alpha_{ij} \ge 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m$ and

$$\sum_{j=0}^{m} \alpha_{ij} = 1\,, \qquad i = 1, \ldots, n\,. \tag{21}$$

Definition 1. The solution $x(t)$ of equation (1) is said to be *proper* if it is defined in some interval $[T_x, +\infty)$ and $\sup\{|x(t)| : t \ge T\} > 0$ for each $T \ge T_x$.

Definition 2. The proper solution $x(t)$ of equation (1) is said to be:

2.1. *nonoscillatory*, if it is either eventually positive or eventually negative;

2.2. *oscillatory*, if it is neither eventually positive nor eventually negative.

Equation (1) is said to be *oscillatory* if all its proper solutions are oscillatory.

In order to prove our main results we will need some lemmas.

Lemma 1. *Assume that conditions H1–H3 hold and*

$$\int_{\alpha}^{\infty} \sum_{i=1}^{n} p_i(t)\mathrm{d}t = +\infty\,. \tag{22}$$

Then for every nonoscillatory solution $x(t)$ of equation (1) $\lim_{t\to+\infty} |x(t)| = +\infty$ monotonically.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) and suppose without loss of generality that $x(t)$ is eventually positive: $x(\sigma_j(t)) > 0$, $t \geq t_0 \geq \alpha$, $j = 0,\dots,m$. Then it follows from (1) and conditions H2 and H3 that $x(t)$ is a nondecreasing function for $t \geq T$. Hence the limit $\lim_{t\to+\infty} x(t) = \beta$ exists and $0 < \beta \leq +\infty$. If $\beta < +\infty$, then there exists $T \geq t_0$ such that $f_i(x(\sigma_0(t)),\dots,x(\sigma_m(t))) \geq \frac{1}{2} f_i(\beta,\dots,\beta) = c > 0$ for $t \geq T$, $i = 1,\dots,n$. From (1) we have

$$x'(t) \geq c \sum_{i=1}^{m} p_i(t)\,, \qquad t \geq T\,.$$

Integrating the above inequality from T to t we get

$$x(t) \geq x(T) + c \int_T^t \sum_{i=1}^{n} p_i(s)\mathrm{d}s$$

and keeping in mind (22) we conclude that $\lim_{t\to+\infty} x(t) = +\infty$. □

Lemma 2. *Assume that conditions H1–H4 and (22) hold.*

If equation (1) has a nonoscillatory solution, then there exists $c > 0$ such that for each $i = 1,\dots,n$, $j = 0,\dots,m$

$$\alpha_{ij} \int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s \leq c \qquad \textit{eventually.} \tag{23}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) and suppose without loss of generality that $x(t)$ is eventually positive. Since the function $x(t)$ is eventually nondecreasing and $\lim_{t\to+\infty} x(t) = +\infty$ (by Lemma 1), then it follows from condition (20) that there exists $t_1 \geq \alpha$ such that

$$\max(\varepsilon,(2M)^{\frac{1}{r}}) \leq x(\sigma_0(t)) \leq x(\sigma_1(t)) \leq \cdots \leq x(\sigma_m(t))\,, \quad t \geq t_1\,, \tag{24}$$

where $r > 0$, $M > 0$ and $\varepsilon > 0$ are given in condition H4. From (24) and condition H4 we conclude that

$$f_i(x(\sigma_0(t)),\dots,x(\sigma_m(t))) \geq \frac{1}{2} \prod_{j=0}^{m} \big[x(\sigma_j(t))\big]^{\alpha_{ij}} \tag{25}$$

for $i = 1,\dots,n$ and $t \geq t_1$. Then from (1) and (25) we have

$$x'(t) \geq \frac{1}{2} \sum_{i=1}^{n} p_i(t) \prod_{j=0}^{m} \big[x(\sigma_j(t))\big]^{\alpha_{ij}}\,, \qquad t \geq t_1\,. \tag{26}$$

In particular, for fixed $i = 1, \dots, n$

$$x'(t) \geq \frac{1}{2} p_i(t) \prod_{j=0}^{m} \big[x(\sigma_j(t))\big]^{\alpha_{ij}}\,, \qquad t \geq t_1\,. \tag{27}$$

According to condition (21) there exists $k = k(i) \in \{0, \dots, m\}$ such that $k = \max\{j : \alpha_{ij} > 0\}$. Set $\beta_i = \alpha_{ik} > 0$. Then from (24) we have

$$0 < x(t) = x(\sigma_0(t)) \leq \dots \leq x(\sigma_k(t))\,, \qquad t \geq t_1$$

and it follows from (27) and (21) that

$$x'(t) \geq \frac{1}{2} p_i(t) \big[x(t)\big]^{1-\beta_i} \big[x(\sigma_k(t))\big]^{\beta_i}\,, \qquad t \geq t_1\,.$$

Set $y(t) = \big[x(t)\big]^{\beta_i}$. Then $y(t)$ satisfies the inequality

$$y'(t) \geq \frac{1}{2} \beta_i p_i(t) y(\sigma_k(t))\,, \qquad t \geq t_1$$

and proceeding as in [1], [8] it is easy to show that

$$\frac{1}{2}\beta_i \int_{\mu_k(t)}^{t} p_i(s)\mathrm{d}s \leq 1 \qquad \text{eventually.} \tag{28}$$

If $j > k$, then $\alpha_{ij} = 0$ and (23) is fulfilled.

If $j \leq k$, then $\sigma_j(t) \leq \sigma_k(t)$, $\mu_j(t) \geq \mu_k(t)$ and from (28) it follows that

$$\frac{1}{2}\beta_i \int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s \leq 1 \qquad \text{eventually,}$$

or

$$\alpha_{ij} \int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s \leq \frac{2\alpha_{ij}}{\beta_i} \leq 2 \max\left(\frac{1}{\beta_1}, \dots \frac{1}{\beta_n}\right) = c$$

eventually. □

Lemma 3. *Assume that conditions H1–H4 hold and there exists $i \in \{1, \dots, n\}$ such that $\alpha_{im} > 0$ and*

$$\lim_{t\to+\infty} \inf \int_{\mu_m(t)}^{t} p_i(s)\mathrm{d}s > 0\,. \tag{29}$$

If $x(t)$ is nonoscillatory solution of equation (1), then $\frac{x(\sigma_m(t))}{x(t)}$, which is well defined for large t, is bounded.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) and suppose without loss of generality that $x(t)$ is eventually positive. Since (29) implies (22) we conclude as in the proof of Lemma 2 that there exists $t_1 \geq \alpha$ such that (24) and (26) are fulfilled. In particular, we have

$$x'(t) \geq \frac{1}{2} p_i(t) \big[x(t)\big]^{1-\alpha_{im}} \big[x(\sigma_m(t))\big]^{\alpha_{im}}\,, \qquad t \geq t_1\,.$$

Then $y(t) = \big[x(t)\big]^{\alpha_{im}}$ satisfies the inequality

$$y'(t) \geq \frac{\alpha_{im}}{2} p_i(t) y(\sigma_m(t)) \,, \qquad t \geq t_1 \,. \tag{30}$$

From (29) it follows that there exist $\beta > 0$ and $T \geq t_1$ such that

$$\frac{\alpha_{im}}{2} \int_{\mu_m(t)}^{t} p_i(s) \mathrm{d}s \geq 2\beta \,, \qquad t \geq T \,.$$

Then for each $t \geq T$ there exists an unique $\xi(t) \in (\mu_m(t), t)$ such that

$$\frac{\alpha_{im}}{2} \int_{\mu_m(t)}^{\xi(t)} p_i(s) \mathrm{d}s = \beta \qquad \text{and} \qquad \frac{\alpha_{im}}{2} \int_{\xi(t)}^{t} p_i(s) \mathrm{d}s \geq \beta \,. \tag{31}$$

Integrating (30) in the intervals $[\mu_m(t), \xi(t)]$ and $[\xi(t), t]$ we obtain

$$y(\xi(t)) - y(\mu_m(t)) \geq \frac{\alpha_{im}}{2} \int_{\mu_m(t)}^{\xi(t)} p_i(s) y(\sigma_m(s)) \mathrm{d}s \,, \qquad t \geq T \tag{32}$$

and

$$y(t) - y(\xi(t)) \geq \frac{\alpha_{im}}{2} \int_{\xi(t)}^{t} p_i(s) y(\sigma_m(s)) \mathrm{d}s \,, \qquad t \geq T \,. \tag{33}$$

Since $y(t)$ is nondecreasing we conclude from (31)–(33) that

$$y(\xi(t)) \geq \beta y(\sigma_m(\mu_m(t))) = \beta y(t) \,, \qquad t \geq T \tag{34}$$

and

$$y(t) \geq \beta y(\sigma_m(\xi(t))) \,, \qquad t \geq T \,. \tag{35}$$

From (34) and (35) it follows that

$$\frac{y(\sigma_m(\xi(t)))}{y(\xi(t))} \leq \left(\frac{1}{\beta}\right)^2 \,, \qquad t \geq T \,,$$

which means that

$$\frac{x(\sigma_m(t))}{x(t)} \leq \left(\frac{1}{\beta}\right)^{\frac{2}{\alpha_{im}}} < +\infty \,, \qquad t \geq T \,.$$

□

Lemma 4. *Assume that conditions H1–H4 and (22) hold. If $x(t)$ is a nonoscillatory solution of equation (1), then there exist constants $A > 0$ and $T > \alpha$ such that*

$$|x(t)| \geq A \exp\left(\frac{1}{2} \int_{T}^{t} \sum_{i=1}^{n} p_i(s) \mathrm{d}s\right) \,, \qquad t \geq T \,. \tag{36}$$

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1) and suppose without loss of generality that $x(t)$ is eventually positive. Then it follows as in the proof of Lemma 2 that there exists $T \geq \alpha$ such that (24) and (26) hold for $t \geq T$. Then

$$x'(t) \geq \frac{1}{2}\sum_{i=1}^{n} p_i(t)x(t)\,, \qquad t \geq T\,,$$

whence it follows (36) with $A = x(T)$.

3. MAIN RESULTS

Theorem 1. *Assume that conditions H1–H4 hold and there exist $\beta > 0$, $t_0 \geq \alpha$ and $i \in \{1,\dots,n\}$ such that $\alpha_{im} > 0$,*

$$\int_{\mu_m(t)}^{t} p_i(s)\mathrm{d}s \geq \beta\,, \qquad t \geq t_0\,, \tag{37}$$

and

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i(t) \ln\left(\mathrm{e}\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s\right)\mathrm{d}t = +\infty\,. \tag{38}$$

Then equation (1) is oscillatory.

Proof. First we note that (37) implies (22) and

$$S(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij}\int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s > 0\,, \qquad t \geq t_0\,.$$

Assume that equation (1) has a nonoscillatory solution $x(t)$ and suppose without loss of generality that $x(t)$ is eventually positive. Then $x(t)$ is eventually nondecreasing, $\lim_{t\to+\infty} x(t) = +\infty$ and there exists $t_1 \geq t_0$ such that (24) holds. From (1), (24) and condition H4 we obtain

$$x'(t) \geq \sum_{i=1}^{n} p_i(t)\prod_{j=0}^{m}\big[x(\sigma_j(t))\big]^{\alpha_{ij}} - M\sum_{i=1}^{n} p_i(t)x(\sigma_m(t))\big[x(t)\big]^{-r} \tag{39}$$

for $t \geq t_1$.

Set $\lambda(t) = \frac{x'(t)}{x(t)}$ for $t \geq t_1$. Then $\lambda(t) \geq 0$, $t \geq t_1$ and

$$\frac{x(\sigma_j(t))}{x(t)} = \exp\left(\int_{t}^{\sigma_j(t)} \lambda(s)\mathrm{d}s\right), \qquad t \geq t_1\,. \tag{40}$$

From (39) and (40) we have

$$\lambda(t) \geq \sum_{i=1}^{n} p_i(t)\exp\left(\sum_{j=1}^{m}\alpha_{ij}\int_{t}^{\sigma_j(t)}\lambda(s)\mathrm{d}s\right) - \varphi(t)$$

for $t \geq T$, where

$$\varphi(t) = M \sum_{i=1}^{n} p_i(t) \left[\frac{x(\sigma_m(t))}{x(t)} \right] \big[x(t)\big]^{-r}, \qquad t \geq T\,. \tag{41}$$

Then

$$S(t)\lambda(t) \geq \sum_{i=1}^{n} p_i(t) S(t) \exp\left(\sum_{j=1}^{m} \alpha_{ij} \int_t^{\sigma_j(t)} \lambda(s)\mathrm{d}s \right) - S(t)\varphi(t)$$

and taking into account the inequality $\gamma \mathrm{e}^x \geq x + \ln \mathrm{e}\gamma$ for $\gamma > 0$ we obtain

$$S(t)\lambda(t) \geq \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} p_i(t) \int_t^{\sigma_j(t)} \lambda(s)\mathrm{d}s + \sum_{i=1}^{n} p_i(t) \ln(\mathrm{e}S(t)) - S(t)\varphi(t)\,, \qquad t \geq T\,. \tag{42}$$

Integrating (42) from T to $N > \sigma_m(T)$ we get

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_T^N \lambda(s) \int_{\mu_j(s)}^{s} p_i(t)\mathrm{d}t\mathrm{d}s \geq \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_T^N p_i(t) \int_t^{\sigma_j(t)} \lambda(s)\mathrm{d}s\mathrm{d}t + \int_T^N \sum_{i=1}^{n} p_i(t) \ln\big(\mathrm{e}S(t)\big)\mathrm{d}t - \int_T^N S(t)\varphi(t)\mathrm{d}t\,. \tag{43}$$

Interchanging the order of integration we find

$$\int_T^N p_i(t) \int_t^{\sigma_j(t)} \lambda(s)\mathrm{d}s\mathrm{d}t \geq \int_{\sigma_j(T)}^N \lambda(s) \int_{\mu_j(s)}^{s} p_i(t)\mathrm{d}t\mathrm{d}s\,.$$

Then from (43) it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_T^{\sigma_j(T)} \lambda(s) \int_{\mu_j(s)}^{s} p_i(t)\mathrm{d}t\mathrm{d}s \geq \int_T^N \sum_{i=1}^{n} p_i(t) \ln\big(\mathrm{e}S(t)\big)\mathrm{d}t - \int_T^N S(t)\varphi(t)\mathrm{d}t\,. \tag{44}$$

Taking into account Lemma 2 we obtain the estimate

$$S(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s \leq \sum_{i=1}^{n}\sum_{j=1}^{m} c = cmn = C_0\,, \qquad t \geq T\,. \tag{45}$$

By Lemma 3 there exists $c_m > 1$ such that

$$\frac{x(\sigma_m(t))}{x(t)} \leq c_m\,, \qquad t \geq T\,.$$

Then from (41), (45) and Lemma 4 we obtain the estimate

$$S(t)\varphi(t) \le C_0 M \sum_{i=1}^{n} p_i(t)\Big[\frac{x(\sigma_m(t))}{x(t)}\Big]\big[x(t)\big]^{-r}$$
$$\le C_0 M c_m A^{-r} \sum_{i=1}^{n} p_i(t)\exp\Big(-\frac{r}{2}\int_T^t \sum_{i=1}^{n} p_i(s)\mathrm{d}s\Big)$$

for $t \ge T$. This implies

$$\int_T^\infty S(t)\varphi(t)\mathrm{d}t \le 2C_0 M c_m A^{-r}\int_0^\infty \mathrm{e}^{-ru}\mathrm{d}u = c_1 < +\infty\,. \tag{46}$$

Moreover,

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{ij}\int_T^{\sigma_j(T)}\lambda(s)\int_{\mu_j(s)}^{s} p_i(t)\mathrm{d}t\mathrm{d}s \le c_2 = c_2(T)\,. \tag{47}$$

Then from (44), (46) and (47) it follows the inequality

$$\int_T^\infty \sum_{i=1}^{n} p_i(t)\ln\big(\mathrm{e}S(t)\big)\mathrm{d}t \le c_1 + c_2\,,$$

which contradicts (38). □

Example 1. Consider the equation

$$x'(t) = \frac{a}{t}x(\lambda t)\arctan|x(t)|\,, \qquad t \ge 1\,, \tag{48}$$

where $\lambda > 1$ and $a > 0$ are constants such that

$$\mathrm{e}\pi a \ln\lambda > 2\,. \tag{49}$$

Equation (49) is of the form (1) with $n = 1$, $m = 1$, $\sigma_0(t) = t$, $\sigma_1(t) = \lambda t$, $\mu_1(t) = \frac{t}{\lambda}$, $p_1(t) = \frac{\pi a}{2t}$, $f_1(x_0, x_1) = \frac{2}{\pi}x_1\arctan|x_0|$, $\alpha_{10} = 0$, $\alpha_{11} = 1$. Moreover,

$$S(t) = \alpha_{11}\int_{\mu_1(t)}^{t} p_1(s)\mathrm{d}s = \int_{\frac{t}{\lambda}}^{t}\frac{\pi a}{2s}\mathrm{d}s = \frac{\pi a\ln\lambda}{2} > 0\,, \qquad t \ge 1$$

and we have from (49) that

$$\int_1^\infty p_1(t)\ln(\mathrm{e}S(t))\mathrm{d}t = \int_1^\infty \frac{\pi a}{2t}\ln\Big(\frac{\mathrm{e}\pi a\ln\lambda}{2}\Big)\mathrm{d}t = +\infty\,.$$

Hence conditions (37) and (38) are fulfilled and by Theorem 1 equation (48) is oscillatory. □

Consider equation (9) which satisfies the following conditions

H. $0 < \tau_1 < \tau_2 < \cdots < \tau_m$, $p \in C(J, R_+)$, $f \in C(R^m, R)$,

$$x_1 f(x_1, \ldots, x_m) > 0 \ \text{ for } \ x_1 x_j > 0, \quad j = 1, \ldots, m$$

and there exist constants $M > 0$, $r > 0$ and $\varepsilon > 0$ such that

$$\Big|f(x_1, \dots, x_m) - \prod_{j=1}^{m} x_j^{\alpha_j}\Big| \le M \prod_{j=1}^{m} |x_j|^{\alpha_j} \big(\max_{1\le j\le m} |x_j|\big)^r \tag{50}$$

for $|x_j| < \varepsilon$, $j = 1, \dots, m$, where $\alpha_j > 0$ are rational numbers with denominator of positive odd integers and $\sum_{j=1}^{m} \alpha_j = 1$.

In Jiang [3] Theorem 2 is formulated without proof, which asserts that equation (9) is oscillatory, if conditions H hold,

$$\lim_{t\to+\infty} \inf \sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} p(s)\mathrm{d}s > 0 \tag{51}$$

and there exists $t_0 \ge \alpha$ such that

$$\int_{t_0}^{\infty} p(t) \ln\left(\mathrm{e} \sum_{j=1}^{m} \alpha_j \int_{t-\tau_j}^{t} p(s)\mathrm{d}s\right)\mathrm{d}t = +\infty\,. \tag{52}$$

Equation (9) is of the type (1), when $n = 1$, $\sigma_j(t) = t + \tau_j$, $\mu_j(t) = t - \tau_j$, $p_1(t) = p(t)$, $f_1(x_0, \dots, x_m) = f(x_1, \dots, x_m)$. From conditions H, (51) and (52) it follows that equation (9) satisfies conditions H1–H3, (37) and (38) but condition H4 differs from condition (50).

We note that the approximate equality

$$f(x_1, \dots, x_m) \approx \left(\prod_{j=1}^{m} |x_j|^{\alpha_j}\right)\mathrm{sign} x_0 \tag{53}$$

is applied in Theorem 1 when $|x_j|$ are large enough and $x_0 x_j > 0$, $j = 1, \dots, m$. The validity of (53) in Theorem 1 is ensured by condition H4.

Since condition (50) does not ensure (53) in a neighbourhood of infinity we can suppose that Theorem 2 from Jiang [3] is not true and the following example confirms this.

Example 2. Consider the equation

$$x'(t) = \frac{x(t+1) + \sin x(t+1)}{\mathrm{e}(1 + \mathrm{e}^{-t}\sin(\mathrm{e}^t))}\,, \qquad t \ge 0\,. \tag{54}$$

Equation (54) is of the form (9), when $m = 1$, $\tau_1 = 1$, $\alpha_1 = 1$, $p(t) = \frac{2}{\mathrm{e}(1+\mathrm{e}^{-t}\sin(\mathrm{e}^t))}$, $f_1(x_1) = \frac{x_1+\sin x_1}{2}$. It is easy to verify that conditions H are fulfilled. Since $\mathrm{e}^{-t} \le \frac{1}{2}$ for $t \ge \ln 2 = t_0$, then

$$p(t) \ge \frac{2}{\mathrm{e}\big(1 + \frac{1}{2}\big)} = \frac{4}{3\mathrm{e}}\,, \qquad t \ge t_0\,,$$

$$S(t) = \alpha_1 \int_{t-\tau_1}^{t} p(s)\mathrm{d}s \ge \int_{t-1}^{t} \frac{4}{3\mathrm{e}}\mathrm{d}s = \frac{4}{3\mathrm{e}}\,, \qquad t \ge t_0$$

and

$$p(t)\ln(\mathrm{e}S(t)) \geq \frac{4}{3\mathrm{e}}\ln\left(\frac{4}{3}\right) > 0\,, \qquad t \geq t_0\,,$$

which implies that conditions (51) and (52) also hold. Then according Jiang [3], Theorem 2 equation (54) is oscillatory. But this equation has a nonoscillatory solution $x = \mathrm{e}^{t-1}$, $t \geq 0$. Hence Theorem 2 from Jiang [3] is not true. □

Consider the equation

$$x'(t) = \sum_{i=1}^{n} p_i(t)\left(\prod_{j=0}^{m} |x(\sigma_j(t))|^{\alpha_{ij}}\right)\mathrm{sign}x(t)\,, \qquad t \geq \alpha\,, \tag{55}$$

where the constants $\alpha_{ij} \geq 0$, $i = 1, \ldots, n$, $j = 0, \ldots, m$ satisfy (21).

Theorem 2. *Assume that conditions H1 and H2 hold and there exists* $t_0 \geq \alpha$ *such that*

$$S(t) = \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_{\mu_j(t)}^{t} p(s)\mathrm{d}s > 0\,, \qquad t \geq t_0 \tag{56}$$

and condition (38) is fulfilled.

Then equation (55) is oscillatory.

Proof. Proceeding as in the proof of Theorem 1 we obtain the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{ij} \int_{T}^{\sigma_j(T)} \lambda(s) \int_{\mu_j(s)}^{s} p_i(t)\mathrm{d}t\mathrm{d}s \geq \int_{T}^{N} \sum_{i=1}^{n} p_i(t)\ln(\mathrm{e}S(t))\mathrm{d}t\,, \qquad N \geq \sigma_m(T)\,.$$

This implies the relation

$$\int_{T}^{\infty} \sum_{i=1}^{n} p_i(t)\ln(\mathrm{e}S(t))\mathrm{d}t < +\infty\,,$$

which contradicts (38). □

Corollary 1. *Assume that conditions H1 and H2 hold and there exists* $t_0 \geq \alpha$ *such that*

$$\sum_{i=1}^{n} \int_{\mu_j(t)}^{t} p_i(s)\mathrm{d}s > 0\,, \qquad t \geq t_0 \tag{57}$$

and

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i(t) \ln \left(\mathrm{e} \sum_{i=1}^{n} \int_{\mu_i(t)}^{t} p_i(s) \mathrm{d}s \right) \mathrm{d}t = +\infty \,. \tag{58}$$

Then the equation

$$x'(t) = \sum_{i=1}^{n} p_i(t) x(\sigma_i(t)) \,, \qquad t \geq \alpha \tag{59}$$

is oscillatory.

Corollary 2. *Assume that condition H2 holds and there exists* $t_0 \geq \alpha$ *such that conditions (57) and (58) are fulfilled with* $\mu_i(t) = t - \delta_i$*,* $i = 1, \dots, n$*, where* $0 < \delta_1 < \cdots < \delta_n$*.*

Then equation (7) is oscillatory.

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