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ASYMPTOTIC EXPRESSION AND A SUFFICIENT CONDITION ON THE OSCILLATING SOLUTIONS TO THE GENERAL SECOND PAINLEVÉ EQUATION

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ABSTRACT: In this paper, we study the general second Painlevé equation and find a sufficient condition for its solutions to be oscillating and the corresponding asymptotic expression as its independent variable x approaches negative infinity by using the uniform asymptotics and the monodromic data methods.

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1. INTRODUCTION

It is well-known now that the Painlevé equations are very important in mathematics theory as well as in applications. One of the important applications of the second Painlevé equation is its connection to the famous KDV equation. In about last twenty years, many mathematicians and physicists have spent dramatic efforts on studying the Painlevé equations, especially, the asymptotic behaviors of their solutions. Hastings and McLeod [2] developed a method, applied it to the second Painlevé equation

$$y'' = 2y^3 + xy + \alpha \qquad \text{(PII)}$$

with parameter $\alpha = 0$ and found a group of asymptotics as x approaches both $-\infty$ and ∞ and the corresponding connection formula. Joshi and Kruskal [4] studied the second Painlevé equation with $\alpha = 0$ and presented the

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asymptotic expressions of its solutions by finding the associated connection formula first. Its and Novokshnov [3] applied the WKB method to the second Painlevé equation with $\alpha = 0$ and found a group of asymptotics of its solutions. Bassom et al [1] developed the uniform asymptotics method, applied it to (PII) with $\alpha = 0$ and verified a connection formula of a group of its asymptotics. Qin and Shang [5] gave the asymptotic behaviour of the oscillating solutions of the Painlevé equations by using numerical methods. Qin and Shang [6] studied the general third Painlevé equations and found the asymptotic behaviour of its real solutions. In this paper, we study the real oscillating solutions of the general second Painlevé equation (PII) when its parameter α is real and prove the following theorem.

Theorem 1. *Let*

$$D_\alpha = \left\{ (x,y) | (2y^3 + xy + \alpha) < 0, x < -3\sqrt[3]{\frac{\alpha^2}{2}} \right\}.$$

If $(x_0, y_0) \in D_\alpha$ and $y_0 > 0$, the solutions $y(x)$ of (PII) satisfying $y(x_0) = y_0$ and $y'(x_0) = 0$ are oscillating as $x \to -\infty$ and have the following asymptotic expression:

$$y = c\,|x|^{-\frac{1}{4}} \cos\phi - \alpha x^{-1} - \frac{1}{16} c^3 \,|x|^{-\frac{7}{4}} \cos 3\phi + \alpha c^2 \,|x|^{-\frac{5}{2}} (-3 + \cos 2\phi) + O(|x|^{-\frac{13}{4}}), \quad (1.1)$$

$$y' = c\,|x|^{\frac{1}{4}} + \frac{c}{4} \,|x|^{-\frac{5}{4}} (3c^2 \sin\phi - \cos\phi - \frac{3}{4} c^2 \sin 3\phi) + x^{-2}(\alpha + 2\alpha c^2 \sin 2\phi) + O(|x|^{-\frac{11}{4}}), \quad (1.2)$$

where $\phi = \frac{2}{3}(-x)^{\frac{3}{2}} - \frac{3}{4} c^2 \ln(-x) + \phi_0 + O(|x|^{-\frac{3}{2}})$, c and ϕ_0 are real constants.

It is worth to point out that, in this theorem, we do not only derive an asymptotic expression, but also prove the existence, uniqueness and the differentiability of the oscillating solutions. A side product of this paper is the monodromic data for the solution which can be used to find the connection formula if we know the asymptotics of the solution as x approaches positive infinity. Our proof is divided into three steps. First, we use elementary analysis to establish some properties of the solutions of (PII). Then, we prove that $z = (-x)^{\frac{1}{4}}(y + \alpha x^{-1})$ and its derivative are bounded as $x \to -\infty$. Finally, we use the uniform asymptotics and isomonodromic deformation method to prove the uniqueness of the oscillating solutions and derive the desired asymptotic expressions.

2. SOME PROPERTIES OF THE SOLUTIONS OF (PII)

Noticing that equation (PII) does not change when we change y to $-y$ and α to $-\alpha$, we assume that $\alpha > 0$, $(x_0, y_0) \in D_\alpha$, $y_0 > 0$ and y is a solution of (PII) with $y(x_0) = y_0$ and $y'(x_0) = 0$ from this point on.

Lemma 2. *If y is an oscillating solution of (PII), then $|y| \leq y_0$ for all $x < x_0$.*

Proof. We know that, when $x < -3\sqrt[3]{\frac{\alpha^2}{2}}$, the cubic equation $2y^3+xy+\alpha = 0$ has three real roots $y(x,k)$, $k = 1,2,3$, and $y(x,1) \sim \sqrt{-\frac{x}{2}}$, $y(x,2) \sim -\frac{\alpha}{x}$ and $y(x,3) \sim -\sqrt{-\frac{x}{2}}$. Clearly, $y(x,1) > y(x,2) > y(x,3)$, $2y^3 + xy + \alpha > 0$ for $y(x) > y(x,1)$, $2y^3+xy+\alpha < 0$ for $y(x,1) > y(x) > y(x,2)$, $2y^3+xy+\alpha > 0$ for $y(x,2) > y(x) > y(x,3)$, and $2y^3 + xy + \alpha < 0$ for $y(x) < y(x,3)$. By the continuity of $2y^3+xy+\alpha$ with respect to x and y, the half plane $x < -3\sqrt[3]{\frac{\alpha^2}{2}}$ is divided into four regions and all of them stretch to negative infinity. More precisely, the top closed region satisfies $y(x) \geq y(x,1)$ and does not belong to D_α, the second open region to the top satisfies $y(x,1) > y(x) > y(x,2)$ and belongs to D_α, the third closed region to the top satisfies $y(x,2) \geq y(x) \geq y(x,3)$ and does not belong to D_α, and the bottom open region satisfies $y(x,3) > y(x)$ and belongs to D_α. Now, let $y = y(x)$ be an oscillating solution of (PII), and $y_1 = y(x_1)$ and $y_2 = y(x_2)$ be two extreme value of y with $x_1 < x_2 < -3\sqrt[3]{\frac{\alpha^2}{2}}$ and $y_1 < y_2$. Then, $y''(x_1) \geq 0$ and $y''(x_2) \leq 0$. Thus, $(x_2, y_2) \in \overline{D}_\alpha$ and $(x_1, y_1) \notin D_\alpha$. Because $y_1 < y_2$, (x_2, y_2) cannot belong to the bottom region, and (x_1, y_1) cannot belong to the top region. Therefore, (x_2, y_2) must belong to the second to the top region, and (x_1, y_1) belongs to the third to the top region. Hence, $y(x_1, 3) < y_1 < y_2 \leq y(x_2, 1)$.

It suffices to prove that the first two extreme values of y to the left of x_0 satisfy the claim of the lemma. We will do this by dividing it into two cases.

1) Let $x_1 < x_0$ be the first point to the left of x_0 such that y has its minimum value at x_1. By the previous analysis, we have $y(x_1, 3) \leq y(x_1) \leq y(x_1, 2)$. If $|y(x_1)| \geq y_0$, Let $x_0 \geq x_2 > x_3 \geq x_4 \geq x_1$ such that $y(x_2) = y(x_2, 2)$, $y(x_3) = 0$, and $y(x_4) = -y(x_0)$. Then,

$$\begin{aligned} -y'^2(x_4) &= \int_{x_4}^{x_0} (4y^3 + 2xy + 2\alpha)dy \\ &= (\int_{x_4}^{x_2} + \int_{x_2}^{x_0})(4y^3 + 2xy + 2\alpha)dy \\ &> (\int_{x_4}^{x_3} + \int_{x_2}^{x_0})(4y^3 + 2xy + 2\alpha)dy. \end{aligned}$$

Noticing that

$$(\int_{x_4}^{x_3}+\int_{x_2}^{x_0})2xydy \geq x_3\int_{x_4}^{x_3}2ydy+x_2\int_{x_2}^{x_0}2ydy$$
$$= -x_3y^2(x_4)+x_2(y^2(x_0)-y^2(x_2)),$$

and $y(x_2)\sim \frac{\alpha}{x_2}$, we have

$$\begin{aligned}
&-y'^2(x_4)\\
&\quad > -y^4(x_2)-2\alpha y(x_2)+4\alpha y(x_0)-x_3y^2(x_0)+x_2y^2(x_0)-x_2y^2(x_2)\\
&\quad = -y^2(x_2)(y^2(x_2)+x_2)+2\alpha(2y(x_0)-y(x_2))+(x_2-x_3)y^2(x_0)>0.
\end{aligned}$$

This is clearly a contradiction.

2) Let $x_1 < x_0$ be the first point to the left of x_0 such that y has its maximum value at x_1. Then, $y'(x_1)=0$, $y''(x_1)\leq 0,$ and $y(x_1,2)\leq y(x_1)\leq y(x_1,1)$. Clearly, y has a minimum point x_2 in (x_1,x_0). By the previous case, $|y(x_2)|\leq y_0$. If $|y(x_1)|\geq y_0$, we pick x_3 with $x_1\leq x_3<x_2$ such that $y(x_3)=y_0$. Multiplying the equation (PII) by $2y'$ and integrating it from x_3 to x_0, we get

$$-y'^2(x_3)=\int_{x_3}^{x_0}(4y^3+2xy+2\alpha)dy=\int_{x_3}^{x_0}2xydy.$$

If $y(x_2)\geq 0$, we have, for any x_4 with $x_3<x_4<x_2$,

$$\begin{aligned}
&-y'^2(x_3)=(\int_{x_3}^{x_4}+\int_{x_4}^{x_2}+\int_{x_2}^{x_0})2xydy\\
&\quad \geq x_4(y^2(x_4)-y^2(x_3))+x_2(y^2(x_2)-y^2(x_4))+x_2(y^2(x_0)-y^2(x_2))\\
&\qquad\qquad > x_2(y^2(x_2)-y^2(x_3))+x_2(y^2(x_0)-y^2(x_2))=0,
\end{aligned}$$

and get a contradiction. If $y(x_2)<0$, we may pick $x_3<x_4<x_2<x_5<x_0$ such that $y(x_4)=y(x_5)=0$ and get

$$\begin{aligned}
-y'^2(x_3) &= (\int_{x_3}^{x_4}+\int_{x_4}^{x_2}+\int_{x_2}^{x_5}+\int_{x_5}^{x_0})2xydy\\
&\geq (x_5-x_4)(y_0^2-y^2(x_2))\\
&> 0.
\end{aligned}$$

This gives another contradiction and finishes the proof of the lemma. □

Lemma 3. $|y(x)|<\sqrt{-\frac{2x}{3}}$ *for large* $-x$.

Proof. If y is oscillating, the claim is true by the previous lemma. Assume that $y(x)$ is not oscillating as $x \to -\infty$ and let $x_1 = \sup\{x|y(x)$ is monotone on $(-\infty, x_0]\}$. Then, $y'(x_1) = 0$ and $y(x_1, 3) < y(x_1) < y(x_1, 1)$ by the proof of the previous lemma. If $y(x)$ is not bounded, we find contradiction by dividing it into two cases:

Case 1. $y(x_1, 2) \leq y(x_1) < y(x_1, 1)$. In this case, $y''(x_1) = 2y^3(x_1) + x_1 y(x_1) + \alpha \leq 0$. Thus, $y(x)$ is monotonously increasing on$(-\infty, x_1]$, and then $y(x) \to -\infty$ as $x \to -\infty$. By the proof of case 1) in the previous lemma, we can show that this is impossible.

Case 2. $y(x_1, 3) \leq y(x_1) < y(x_1, 2)$. Since $y(x)$ is monotone on $(-\infty, x_1]$, $\lim\limits_{x\to-\infty}(2y^3 + xy + \alpha) = A$ exists. If $A < 0$, let $x_2 < x_1$ be a point such that $2y^3(x) + xy(x) + \alpha < \frac{A}{2}$ for all $x < x_2$. Then,

$$\begin{aligned} -y'^2(x) &= \left(\int_{y(x)}^{y(x_2)} + \int_{y(x_2)}^{y(x_1)}\right)(4y^3 + 2ty + 2\alpha)dy \\ &> -A(y(x) - y(x_2)) + \int_{y(x_2)}^{y(x_1)}(2y^3 + ty + \alpha)dy \\ &\to +\infty, \text{ as } x \to -\infty, \end{aligned}$$

and this is a contradiction. If $A = 0$, $y \sim -\frac{x}{2}$ as we want. If $A > 0$, there is a positive constant B such that $y''(x) > B$ for sufficiently large $-x$, and then $y(x) > \frac{B}{8}x^2$ for sufficiently large $-x$. We let $x_2 > x_1$ such that $y(x) > 0$ for all $x < x_2$. Then,

$$\begin{aligned} \frac{|\int_{y(x)}^{y(x_2)} 2ty(t)dy|}{y^4(x)} &= \frac{-2\int_{y(x_2)}^{y(x)} ty(t)dy}{y^4(x)} \\ &\leq \frac{-x(y^2(x) - y^2(x_2))}{y^4(x)} = \frac{-x}{y^2(x)} - \frac{y^2(x_2)}{y^4(x)} \\ &\leq -\frac{x}{\frac{B^2}{64}x^4} - \frac{y^2(x_2)}{y^4(x)} \to 0, \text{ as } x \to -\infty. \end{aligned}$$

Hence,

$$\begin{aligned} -y'^2(x) &= \left(\int_{y(x)}^{y(x_2)} + \int_{y(x_2)}^{y(x_1)}\right)(4y^3 + 2ty + 2\alpha)dy \\ &\sim -y^4(x), \text{ as } x \to -\infty, \end{aligned}$$

and therefore, $y'(x) \sim -y^2(x)$. Integrating this result, we get another contradiction and finish the proof of the lemma. □

Lemma 4. *If $y(x)$ is a solution of (PII) satisfying $|y(x)| < \sqrt{-\frac{2x}{3}}$ for sufficiently large $-x$, then both $u(x) = (-x)^{\frac{1}{4}}y(x)$ and $w(x) = (-x)^{-\frac{1}{4}}y'(x)$*

are bounded and

$$u^2+w^2=d^2+O((-x)^{-\frac{3}{4}})\,, \tag{2.1}$$

where d is a constant.

Proof. Let

$$\begin{aligned} z(t) &= (-x)^{\frac{1}{4}}y(x)+\alpha(-x)^{-\frac{3}{4}}\,,\\ t &= \frac{2}{3}(-x)^{\frac{3}{2}}. \end{aligned}$$

Then,

$$\frac{d^2z}{dt^2}+z=\frac{4}{3}t^{-1}z^3+4(\frac{2}{3})^{\frac{1}{2}}\alpha t^{-\frac{3}{2}}z^2+(\frac{8}{3}\alpha^2-\frac{5}{36})t^{-2}z \\ -\frac{8}{9}\sqrt{\frac{2}{3}}(2\alpha+\alpha^3)t^{-\frac{5}{2}}.$$

Multiplying it by $2z'$ and integrating it from t_0, we get

$$\begin{aligned} z'^2+z^2=&\frac{2}{3}t^{-1}z^4+\frac{8}{3}\sqrt{\frac{2}{3}}\alpha t^{-\frac{3}{2}}z^3+(\frac{8}{3}\alpha^2-\frac{5}{36})t^{-2}z^2\\ &-\frac{16}{9}\sqrt{\frac{2}{3}}(\alpha-\alpha^3)t^{-\frac{5}{2}}z+\frac{2}{3}\int_{t_0}^t t^{-2}z^4dt+4\sqrt{\frac{2}{3}}\alpha\int_{t_0}^t t^{-\frac{5}{2}}z^3dt\\ &+(\frac{16}{3}\alpha^2-\frac{5}{18})\int_{t_0}^t t^{-3}z^2dt-\frac{40}{9}\sqrt{\frac{2}{3}}(\alpha-\alpha^3)\int_{t_0}^t t^{-\frac{7}{2}}zdt+C_1. \end{aligned} \tag{2.2}$$

We select $-x_0$ large enough such that $-\frac{\alpha}{x_0}\le\frac{1}{10}\sqrt{-\frac{2x_0}{3}}$. Then, $t\ge t_0=\frac{2}{3}(-x_0)^{\frac{3}{2}}$,

$$\begin{aligned} |z|=(-x)^{\frac{1}{4}}|y+\alpha x^{-1}|&\le(-x)^{\frac{1}{4}}(\sqrt{-\frac{2x}{3}}-\frac{\alpha}{x})\\ &\le\frac{11}{10}\sqrt{\frac{2}{3}}(-x)^{\frac{3}{4}}=\frac{11}{10}t^{\frac{1}{2}}. \end{aligned} \tag{2.3}$$

Combining them, one obtains that there exists a constant C_2 such that

$$z'^2+\frac{29}{150}z^2\le\frac{11}{15}\int_{t_0}^t t^{-\frac{3}{2}}|z|^3dt+\frac{121}{25}\sqrt{\frac{2}{3}}\alpha\int_{t_0}^t t^{-\frac{3}{2}}\,|z|\,dt+C_2, \\ \text{for}\quad t>t_0. \tag{2.4}$$

Now, we let I be the right-hand side of this inequality. Then,

$$I'=\frac{11}{15}t^{-\frac{3}{2}}|z|^3+\frac{121}{25}\sqrt{\frac{2}{3}}\alpha^{-\frac{3}{2}}\,|z|\le C_3t^{-\frac{3}{2}}\sqrt{I}(I+C_4),\ \text{for } t>t_0,$$

where $C_3 = \frac{11}{15}$ and $C_4 = \frac{33}{5}\sqrt{\frac{2}{3}}\alpha$. Integrating this inequality, we get

$$\tan^{-1}\sqrt{\frac{I}{C_4}} \leq C_3\sqrt{C_4}t_0^{-\frac{1}{2}} + \tan^{-1}\sqrt{\frac{C_2}{C_4}}. \tag{2.5}$$

We now need a fine estimate of C_2 in terms of t_0. Using inequalities (2.3) and (2.4), we can find that

$$z'^2 + z^2 - \frac{2}{3}t^{-1}z^4 < \frac{2.1}{3}(\frac{11}{10})^4 t, \text{ for large value of } t. \tag{2.6}$$

Thus, we have $C_2 < \frac{2.1}{3}(\frac{11}{10})^4 t_0$ for large t_0. Therefore, we can select large t_0 such that

$$\begin{aligned} C_3\sqrt{C_4} + \tan^{-1}\sqrt{\frac{C_2}{C_4}} &= \frac{\pi}{2} + \frac{11}{15}\sqrt{C_4} - \sqrt{\frac{C_4}{C_2}} + O(C_2^{-\frac{3}{2}}) \\ &< \frac{\pi}{2} + \sqrt{C_4}(\frac{11}{15} - \sqrt{\frac{3}{2.1}}(\frac{100}{121}))t_0^{-\frac{1}{2}} + O(t_0^{-\frac{3}{2}}) \\ &< \frac{\pi}{2} - \frac{1}{4}\sqrt{C_4}t_0^{-\frac{1}{2}} + O(t_0^{-\frac{3}{2}}) < \frac{\pi}{2}, \text{ for large } t_0. \end{aligned} \tag{2.7}$$

Combining (2.4) and (2.5), we can conclude that I is bounded, and then both z'^2 and z^2 are bounded. It follows that (2.1) is also true. □

3. PROOF OF THE MAIN THEOREM

Now, we start to derive the asymptotics of y as $x \to -\infty$ by using the uniform asymptotics and monodromic data methods. By Lemma 4, we seek a solution of the form

$$\begin{aligned} y(x) &= (-x)^{-\frac{1}{4}}u(x), \\ y'(x) &= (-x)^{\frac{1}{4}}w(x), \end{aligned} \tag{3.1}$$

with both $u(x)$ and $w(x)$ bounded when $x \to -\infty$. It is well-known that the second Painlevé equation (PII) is related to the following linear systems:

$$\frac{d\Psi}{dx} = (-\lambda\sigma_3 + y\sigma_1)\Psi, \tag{3.2}$$

and

$$\frac{d\Psi}{d\lambda} = \{-i(4\lambda^2 + x + 2y^2)\sigma_3 + 4\lambda y\sigma_1 - 2y'\sigma_2 - \frac{\alpha}{\lambda}\sigma_1\}\Psi, \tag{3.3}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \text{ and } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Easy calculation can show that the compatibility condition of (3.2) and (3.3) reduces to (PII), and conversely, (3.2) and (3.3) are compatible if y is a solution of (PII). We first recall a result about the isomonodromic data of the Painlevé II by Its and Novokshnov [3].

Proposition 5. *Let the sectors Ω_k be defined by*

$$\Omega_k : \quad \frac{(k-2)\pi}{3} < \arg\lambda < \frac{k\pi}{3}$$

and Ψ_k, $k = 1, 2, ..., 7$ be canonical solutions of (3.3) satisfying

$$\Psi_k(\lambda) \sim e^{-\frac{4i}{3}\lambda^3\sigma_3 - ix\lambda\sigma_3}, \ \ as \quad |\lambda| \to \infty \ in \ \Omega_k.$$

Then, $\Psi_{k+1} = \Psi_k S_k$ for $k = 1, 2, ...7$ with

$$\begin{aligned} S_1 &= \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & \frac{p+q}{1-pq} \\ 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}, \\ S_4 &= \begin{pmatrix} 1 & -p \\ 0 & 1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} 1 & 0 \\ \frac{p+q}{pq-1} & 1 \end{pmatrix}, \quad S_6 = \begin{pmatrix} 1 & -q \\ 0 & 1 \end{pmatrix}, \end{aligned}$$

and p and q are independent of the variable x.

To apply uniform approximation, we need to turn (3.3) into a single second order equation. We first make the scaling

$$\xi = x^{\frac{3}{2}}, \quad \eta = x^{-\frac{1}{2}}\lambda,$$

so that (3.3) becomes

$$\frac{d\Psi}{d\eta} = \xi\{-i(4\eta^2 + 1 + \frac{2y^2}{x})\sigma_3 + (\frac{4\eta y}{\sqrt{x}} - \frac{\alpha}{\eta\xi})\sigma_1 - \frac{2y'}{x}\sigma_2\}\Psi,$$

which, with

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$

is equivalent to

$$\begin{aligned} \frac{d\psi_1}{d\eta} &= \xi\{-i(4\eta^2 + 1 + \frac{2y^2}{x})\psi_1 + (\frac{4\eta y}{\sqrt{x}} - \frac{\alpha}{\eta\xi} + \frac{2iy'}{x})\psi_2\}, \\ \frac{d\psi_2}{d\eta} &= \xi\{i(4\eta^2 + 1 + \frac{2y^2}{x})\psi_2 + (\frac{4\eta y}{\sqrt{x}} - \frac{\alpha}{\eta\xi} - \frac{2iy'}{x})\psi_1\}. \end{aligned}$$

Eliminating ψ_1, we obtain

$$\begin{aligned} \frac{d^2\psi_2}{d\eta^2} &= \xi\{-\xi(4\eta^2 + 1 + \frac{2y^2}{x})^2\psi_2 + 8i\eta\psi_2 + \xi[(\frac{4\eta y}{\sqrt{x}} - \frac{\alpha}{\eta\xi})^2 + \frac{4y'^2}{x^2}]\psi_2 \\ &+ \frac{1}{\xi^2}(1 + \frac{\alpha}{4\eta^2 xy})(\eta - \frac{iy'}{2y\sqrt{x}} - \frac{\alpha}{4x\eta y})^{-1}[\frac{d\psi_2}{d\eta} - i\xi(4\eta^2 + 1 + \frac{2y^2}{x})\psi_2]\}. \end{aligned}$$

We let

$$\phi = (\eta - \frac{iy'}{2y\sqrt{x}} - \frac{\alpha}{4x\eta y})^{-\frac{1}{2}}\psi_2.$$

Then

$$\begin{aligned}\frac{d^2\phi}{d\eta^2} = \xi^2\{-(4\eta^2+1)^2 + \frac{8i\eta}{\xi} + (\frac{4y'^2}{x^2} - \frac{4y^2}{x} - \frac{4y^4}{x^2} - \frac{8\alpha y}{x^2} + \frac{\alpha^2}{x^3\eta^2})\\ - \frac{i}{\xi}(1 + \frac{\alpha}{4xy\eta^2})(4\eta^2 + 1 + \frac{2y^2}{x})(\eta - \frac{iy'}{2y\sqrt{x}} - \frac{\alpha}{4x\eta y})^{-1}\\ + \frac{1}{\xi^2}\frac{\alpha}{4xy\eta^3}((\eta - \frac{iy'}{2y\sqrt{x}} - \frac{\alpha}{4x\eta y})^{-1}\\ + \frac{3}{4\xi^2}(1 + \frac{\alpha}{4xy\eta^2})^2(\eta - \frac{iy'}{2y\sqrt{x}} - \frac{\alpha}{4x\eta y})^{-2}\}\phi\\ = -\xi^2 F(\eta,\xi)\phi. \quad (3.4)\end{aligned}$$

Equation (3.4) has four turning points

$$\begin{aligned}\eta_{1,2} &= \frac{i}{2} \pm \frac{1}{2}\xi^{-\frac{1}{2}}\sqrt{1 - iu^2 - iw^2}(1 + o(1)),\\ \eta_{3,4} &= -\frac{i}{2} \pm \frac{1}{2}\xi^{-\frac{1}{2}}\sqrt{-iu^2 - iw^2 - 1}(1 + o(1)),\end{aligned}$$

which merge to $\frac{i}{2}$ and $-\frac{i}{2}$ as $\xi \to \infty$. By (3.1), the dominant term on the right-hand side of (3.4) is $-\xi^2(4\eta^2+1)^2\phi$ and thus the Stokes' directions are determined by

$$\arg\eta = -\frac{1}{3}\arg\xi \pm \frac{1}{3}k\pi, \qquad k = 0, 1, 2, ...$$

Now, we define a constant β by

$$\frac{1}{2}\pi i\beta^2 = \int_{-\beta}^{\beta}(\tau^2 - \beta^2)^{\frac{1}{2}}d\tau = \int_{\eta_1}^{\eta_2} F^{\frac{1}{2}}(\eta,\xi)d\eta, \quad (3.5)$$

and a new variable ζ by

$$\int_{\beta}^{\zeta}(\tau^2 - \beta^2)^{\frac{1}{2}}d\tau = \int_{\eta_1}^{\eta} F^{\frac{1}{2}}(s,\xi)ds. \quad (3.6)$$

A. P. Bassom, P. A. Clarkson, C.K. Law, and J. B. McLeod have proved the following theorem, see Bassom et al [1].

Theorem 6. *Given any solution ϕ of equation (3.4), there exist constants c_1 and c_2 such that, uniformly for η on the Stokes' curve, as $\xi \to \infty$,*

$$\begin{aligned}
&\left(\frac{\zeta^2-\beta^2}{F(\eta,\xi)}\right)^{-\frac{1}{4}} \phi(\eta,\xi) \\
&\quad = \{(c_1+o(1))D_v(e^{\frac{\pi i}{4}}\sqrt{2\xi}\zeta) + (c_2+o(1))D_{-\nu-1}(e^{-\frac{\pi i}{4}}\sqrt{2\xi}\zeta)\}\,, \quad (3.7)
\end{aligned}$$

where $\nu = -\frac{1}{2}+\frac{1}{2}i\xi\beta^2$ and $D_\nu(z)$, $D_{-\nu-1}(z)$ are the solutions of the parabolic cylinder equation.

To use this theorem for finding the monodromic data, we need to find the asymptotic expression of ζ first. Carrying out the integration in the left-hand side of the definition of ζ in (3.6), we have

$$\frac{1}{2}\zeta^2 - \frac{\beta^2}{2}\log(2\zeta) - \frac{\beta^2}{4} + \frac{\beta^2}{2}\log\beta + O(\beta^4\zeta^{-2}) = \int_{\eta_1}^{\eta} F^{\frac{1}{2}}(s,\xi)ds. \quad (3.8)$$

We split the right-hand side of (3.6) into two integrals:

$$\int_{\eta_1}^{\eta} F^{\frac{1}{2}}(s,\xi)ds = (\int_{\eta_1}^{\eta^*} + \int_{\eta^*}^{\eta})F^{\frac{1}{2}}(s,\xi)ds = I_1 + I_2,$$

where $\eta^* = \frac{i}{2} + T\xi^{-\frac{1}{2}}$ and T is a large parameter to be specified later. By using the substitution

$$s - \frac{i}{2} = \tau\xi^{-\frac{1}{2}},$$

I_1 can be evaluated as following.

$$\begin{aligned}
I_1 = \frac{4i}{\xi}\int_{\frac{1}{2}\sqrt{1-iu^2-iw^2}}^{T} &(\sqrt{\tau^2 - \frac{1}{4}(1-iu^2-iw^2)} + o(1))d\tau \\
&= \frac{2iT^2}{\xi} - \frac{i+u^2+w^2}{4\xi} - \frac{i+u^2+w^2}{2\xi}\log(2T) \\
&\quad + \frac{i+u^2+w^2}{4\xi}\log\frac{1}{4}(1-iu^2-iw^2) + o(\xi^{-1}). \quad (3.9)
\end{aligned}$$

We can also evaluate I_2 as following.

$$
\begin{aligned}
I_2 &= \int_{\eta^*}^{\eta} \sqrt{(4\eta^2+1)^2 - \frac{8i\eta}{\xi} - \frac{4iw^2+4iu^2}{\xi} + \frac{i(4\eta^2+1)}{\xi(\eta+\frac{w}{2u})} + O(\xi^{-\frac{3}{2}})}d\eta \\
&= \int_{\eta^*}^{\eta} (4\eta^2+1 - \frac{4i\eta}{\xi(4\eta^2+1)} - \frac{2iu^2+2iw^2}{\xi(4\eta^2+1)} + \frac{i}{2\xi(\eta+\frac{w}{2u})})d\eta + O(\xi^{-\frac{3}{2}}) \\
&= (\frac{4}{3}\eta^3+\eta-\frac{i}{2\xi}\log(4\eta^2+1)-\frac{u^2+w^2}{2\xi}\log\frac{2\eta-i}{2\eta+i}+\frac{i}{2\xi}\log(\eta+\frac{w}{2u}))|_{\eta^*}^{\eta}+O(\xi^{-\frac{3}{2}}) \\
&= \frac{4}{3}\eta^3 + \eta - \frac{i}{2\xi}\log(4\eta) - \frac{i}{3} - 2iT^2\xi^{-1} + \frac{i}{2\xi}\log(4iT\xi^{-\frac{1}{2}}) \\
&+ \frac{u^2+w^2}{2\xi}\log(-iT\xi^{-\frac{1}{2}}) - \frac{i}{2\xi}\log(\frac{i}{2}+\frac{w}{2u}) + O(\eta^{-1}\xi^{-1}) + O(\xi^{-\frac{3}{2}}). \quad (3.10)
\end{aligned}
$$

Using definition (3.5) and let $T = -\frac{1}{2}\sqrt{1-iu^2-iw^2}$ in I_1, we get the following expression for β :

$$
\beta^2 = \frac{i+u^2+w^2}{\xi} + o(\xi^{-1}). \quad (3.11)
$$

Combining (3.8), (3.9) and (3.10), we have the asymptotic representation for ζ

$$
\begin{aligned}
\frac{1}{2}\zeta^2 - \frac{i+u^2+w^2}{2\xi}\log\zeta &= \frac{4}{3}\eta^3 + \eta - \frac{i}{2\xi}\log\eta - \frac{i}{3} - \frac{i}{2\xi}\log(\frac{i}{2}+\frac{w}{2u}) \\
&+ \frac{a}{\xi} + O(\eta^{-1}\xi^{-1}) + O(\xi^{-\frac{3}{2}}) \quad (3.12)
\end{aligned}
$$

where $a = \frac{\pi i(i-3u^2-3w^2)}{4} + \frac{3i-u^2-w^2}{2}\log 2$. Now, we look for a solution of (3.3) that satisfies the conditions:

$$
\Psi_k(\lambda) \sim e^{-\frac{4i}{3}\lambda^3\sigma_3 - ix\lambda\sigma_3}, \text{ as } \lambda \to \infty. \quad (3.13)
$$

By the well-known asymptotic expressions of the Parabolic functions:

$$
D_\nu(z) \sim \begin{cases}
z^\nu e^{-\frac{1}{4}z^2}, \quad \text{if } |\arg z| < \frac{3}{4}\pi, \\
z^\nu e^{-\frac{1}{4}z^2} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\pi\nu} z^{-\nu-1} e^{\frac{1}{4}z^2}, \quad \text{on } \arg z = \frac{3}{4}\pi, \\
e^{-2i\pi\nu} z^\nu e^{-\frac{1}{4}z^2} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{\frac{1}{4}z^2}, \quad \text{on } \arg z = \frac{5}{4}\pi, \\
e^{-2i\pi\nu} z^\nu e^{-\frac{1}{4}z^2}, \quad \text{if } \frac{5}{4}\pi < \arg z < \frac{11}{4}\pi,
\end{cases}
$$

we have, as $\lambda \to \infty$ along $\arg\lambda = 0$,

$$D_v(e^{\frac{\pi i}{4}}\sqrt{2\xi}\zeta) \sim (e^{\frac{\pi i}{4}}\sqrt{2\xi}\zeta)^{\nu} e^{-\frac{1}{2}i\xi\zeta^2}$$
$$\sim (\frac{i}{2}+\frac{w}{2u})^{-\frac{1}{2}}(2\xi)^{-\frac{1}{2}+\frac{iu^2+iw^2}{4}}\zeta^{-\frac{1}{2}}\eta^{-\frac{1}{2}}e^{-\frac{4i}{3}\xi\eta^3-i\xi\eta-\frac{\xi}{3}-\frac{\pi i}{4}-\frac{\pi(u^2+w^2)}{8}-ia},$$

$$D_{-\nu-1}(e^{-\frac{\pi i}{4}}\sqrt{2\xi}\zeta) \sim (e^{-\frac{\pi i}{4}}\sqrt{2\xi}\zeta)^{-\nu-1} e^{\frac{1}{2}i\xi\zeta^2}$$
$$\sim (\frac{i}{2}+\frac{w}{2u})^{\frac{1}{2}}(2\xi)^{-\frac{iu^2+iw^2}{4}}\zeta^{-\frac{1}{2}}\eta^{\frac{1}{2}}e^{\frac{4i}{3}\xi\eta^3+i\xi\eta+\frac{\xi}{3}-\frac{\pi(u^2+w^2)}{8}+ia}.$$

By Proposition 5, we need, as $|\lambda| \to \infty$ along $\arg\lambda = 0$,

$$\Psi_1^{(22)} \sim e^{\frac{4i}{3}\lambda^3+ix\lambda},$$
$$\Psi_1^{(21)} \sim \frac{1}{2}e^{-\frac{\pi i}{4}}\xi^{-\frac{1}{2}}\eta^{-1}e^{-\frac{4i}{3}\lambda^3-ix\lambda}.$$

Hence, by (3.7), we can take

$$\Psi_1^{(22)} = (\frac{i}{2}+\frac{w}{2u})^{-\frac{1}{2}}(2\xi)^{\frac{iu^2+iw^2}{4}}e^{-\frac{\xi}{3}+\frac{\pi(u^2+w^2)}{8}-ia}\eta^{-\frac{1}{2}}\zeta^{\frac{1}{2}}D_{-\nu-1}(e^{-\frac{\pi i}{4}}\sqrt{2\xi}\zeta),$$

$$\Psi_1^{(21)} = \frac{1}{2}e^{-\frac{\pi i}{4}}\xi^{-\frac{1}{2}}(\frac{i}{2}+\frac{w}{2u})^{\frac{1}{2}}(2\xi)^{\frac{1}{2}-\frac{iu^2+iw^2}{4}}e^{\frac{\xi}{3}+\frac{\pi i}{4}+\frac{\pi(u^2+w^2)}{8}+ia} \times \eta^{-\frac{1}{2}}\zeta^{\frac{1}{2}}D_v(e^{\frac{\pi i}{4}}\sqrt{2\xi}\zeta).$$

By Proposition 5 again, we have

$$\Psi_2^{(21)} = \Psi_1^{(21)} + p\Psi_1^{(22)},$$
$$\Psi_2^{(21)} \sim \frac{u}{2}e^{-\frac{\pi i}{4}}\xi^{-\frac{1}{2}}\eta^{-1}e^{-\frac{4i}{3}\lambda^3-ix\lambda}, \text{ as } \lambda \to \infty \text{ along } \arg\lambda = \frac{\pi}{3}.$$

Since, as $\lambda \to \infty$ along $\arg\lambda = \frac{\pi}{2}$ which is between 0 and $\frac{2\pi}{3}$,

$$\Psi_1^{(21)} + p\Psi_1^{(22)} \sim \frac{u}{2}e^{-\frac{\pi i}{4}}\xi^{-\frac{1}{2}}\eta^{-1}e^{-\frac{4i}{3}\lambda^3-ix\lambda}$$
$$+ (p - \frac{\sqrt{\pi}(\frac{iu}{2}+\frac{w}{2})}{\Gamma(1-\frac{iu^2+iw^2}{2})}(2\xi)^{-\frac{iu^2+iw^2}{2}}e^{\frac{2\xi}{3}-\frac{\pi(u^2+w^2)}{4}+2ia})e^{\frac{4i}{3}\xi\eta^3+i\xi\eta}.$$

Thus,

$$p = \frac{\sqrt{\pi}(\frac{iu}{2}+\frac{w}{2})}{\Gamma(1-\frac{iu^2+iw^2}{2})}(2\xi)^{-\frac{iu^2+iw^2}{2}}e^{\frac{2\xi}{3}-\frac{\pi(u^2+w^2)}{4}+2ia}.$$

Applying the result from Lemma 4, we get

$$p = \frac{\sqrt{\pi}(\frac{iu}{2} + \frac{w}{2})}{\Gamma(1 - \frac{i}{2}d^2 + O((-x)^{-\frac{3}{4}})}(2\xi)^{-\frac{i}{2}d^2 + O((-x)^{-\frac{3}{4}})} \times e^{\frac{2\xi}{3} - \frac{\pi d^2}{4} + 2ia + O((-x)^{-\frac{3}{4}})}.$$

Solving the equation for u and w, we find the asymptotics:

$$u = IM(\frac{2p\Gamma(1 - \frac{iu^2+iw^2}{2})}{\sqrt{\pi}}(2\xi)^{\frac{iu^2+iw^2}{2}} e^{-\frac{2\xi}{3} + \frac{\pi(u^2+w^2)}{4} - 2ia}) = (d^2 + O((-x)^{-\frac{3}{4}}))\sin(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{d^2}{2}\log(-x) + \phi_0), \quad (3.14)$$

$$w = Re(\frac{2p\Gamma(1 - \frac{iu^2+iw^2}{2})}{\sqrt{\pi}}(2\xi)^{\frac{iu^2+iw^2}{2}} e^{-\frac{2\xi}{3} + \frac{\pi(u^2+w^2)}{4} - 2ia}) = (d^2 + O((-x)^{-\frac{3}{4}}))\cos(\frac{2}{3}(-x)^{\frac{3}{2}} + \frac{d^2}{2}\log(-x) + \phi_0).$$

Combining (3.14) with the transformations in Lemma 4, we get the major terms of the expected asymptotics. Finding the lower order terms of the asymptotics in the theorem is a straight forward task.

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