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ON THE STABILITY OF AN IMPULSIVE DIFFERENTIAL DIFFERENCE POPULATION MODEL

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ABSTRACT: Sufficient conditions are obtained for the various types of uniform stability of the zero solution of population system modelled by an impulsive differential-difference equation.

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1. INTRODUCTION

The delay differential equation

$$\dot{u}(t) = ru(t)[1 - \frac{u(t-\tau)}{K}], \quad t \geq 0 \tag{1}$$

called Hutchinson's equation, see Hutchinson [6], is a single species population growth model, where r, τ and K are positive constants.

This equation has been studied by many authors; see for example Cunningham [4], Gopalsamy [5], Kakutani and Markus [7], Kuang [8], May [9], Wright [12] and Zhang and Gopalsamy [13] and [14].

By making the change of variable $u(t) = K(1 + N(t))$, equation (1) is reduced to the form

$$\dot{N}(t) = -r[1 + N(t)]N(t-\tau), \quad t \geq 0. \tag{2}$$

Equation (2) has been a basis for the derivation of the Lorka-Volterra systems of equations Pianka [10], Pielou [11].

In this paper we consider the case where at certain moments biotic and anthropogeneous factors act on the population "momentarily" so that the population number varies by jamps Bainov et al [1], Bainov and Stamova

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[2], Bainov and Stamova [3]. Precisely we deal with the stability properties of the zero solution of equation of the form

$$\begin{cases} \dot{u}(t) = ru(t)[1 - \frac{u(t-\tau)}{K}], & t \neq t_k,\ t \geq 0, \\ \Delta u(t_k) = u(t_k + 0) - u(t_k) = KI_k(\frac{u(t_k)}{K}), & k = 1, 2, ..., \end{cases} \tag{3}$$

where $0 < t_1 < t_2 < ... < t_k < t_{k+1} < ...;\ \lim\limits_{k\to\infty} t_k = \infty$ and I_k are functions which characterize the magnitude of the impulse effect at the moments t_k.

If we let

$$u(t) = K[1 + x(t)]$$

in (3), we obtain

$$\begin{cases} \dot{x}(t) = -r[1 + x(t)]x(t - \tau), & t \neq t_k,\ t \geq 0, \\ \Delta x(t_k) = I_k(x(t_k) + 1), & k = 1, 2, ... \end{cases} \tag{4}$$

Due to the relation between (3) and (4), we are interested in only those solutions of (4) corresponding to initial conditions ϕ of the form

$$1 + \phi(s) \geq 0, \quad 1 + \phi(0) > 0, \quad s \in [-\tau, 0],$$

where ϕ is assumed to be piecewise continuous on its domain of definition.

2. PRELIMINARY NOTES AND DEFINITIONS

Let $R_+ = [0, \infty)$, $t_0 \in R_+$ and $D = \{\phi : [t_0 - \tau, t_0] \to [-1, \infty), \phi(t)$ is continuous everywhere except a finite number of points $\tilde{t}$ at which $\phi(t)$ is continuous from the left and $\phi(\tilde{t} + 0)$ exists $\}$. Let $\varphi_0 \in D$.

Denote by $x(t) = x(t; t_0, \varphi_0)$ the solution of (4) satisfying the initial condition

$$x(t; t_0, \varphi_0) = \varphi_0(t), \quad t \in [t_0 - \tau, t_0], \quad \varphi_0(t_0) > -1. \tag{5}$$

It is easy to see that the solution $x(t) = x(t; t_0, \varphi_0)$ which exists on $[t_0 - \tau, \infty)$ is a piecewise continuous function and $1 + x(t) > 0$ for all $t > t_0$.

Introduce the following notations:

$$G_k = \{(t, x) \in [t_0, \infty) \times [-1, \infty) : t_{k-1} < t < t_k\}, \quad k = 1, 2, ...;$$

$$G = \bigcup_{k=1}^{\infty} G_k;$$

$$\|\phi\| = \sup_{s \in [t_0 - \tau, t_0]} |\phi(s)| \text{ is the norm of the function } \phi \in D;$$

$$K = \{a \in C[R_+, R_+] :\ a(u)\ isstrictlyincreasingandsuchthata(0) = 0\ \}.$$

Definition 1. The zero solution $x(t) \equiv 0$ of system (4) is said to be:

a) *uniformlystable* if

$$(\forall\varepsilon> 0)(\exists\delta = \delta(\varepsilon) > 0)(\forall t_0 \in R_+)(\forall\varphi_0 \in D: \ ||\varphi_0|| < \delta)$$
$$(\forall t > t_0): \ |x(t; t_0, \varphi_0)| < \varepsilon;$$

b) *uniformlyattractive* if

$$(\exists\lambda> 0)(\forall\varepsilon> 0)(\exists T = T(\varepsilon) > 0)(\forall t_0 \in R_+)$$
$$(\forall\varphi_0 \in D: \ ||\varphi_0|| < \lambda)(\forall t \geq t_0 + T):$$
$$|x(t; t_0, \varphi_0)| < \varepsilon;$$

c) *uniformlyasymptoticallystable* if it is uniformly stable and uniformly attractive;

d) *exponentiallyasymptoticallystable* if

$$(\exists c > 0)(\forall\alpha> 0)(\exists k = k(\alpha) > 0)(\forall t_0 \in R_+)$$
$$(\forall\varphi_0 \in D: \ ||\varphi_0|| < \alpha)(\forall t > t_0):$$
$$|x(t; t_0, \varphi_0)| \leq k(\alpha)||\varphi_0||exp[-c(t - t_0)].$$

In the further considerations we shall use the class V_0 of piecewise continuous functions $V: \ [t_0, \infty) \times [-1, \infty) \to R_+$ which are an analogue of Lyapunov's functions.

Definition 2. We shall say that the function $V: \ [t_0, \infty) \times [-1, \infty) \to R_+$ belongs to the class V_0, if:

1. The function V is continuous in G and locally Lipschitz continuous with respect to its second argument in each of the sets $G_k, k = 1, 2, ...$

2. $V(t, 0) = 0, \ t \geq t_0$.

3. For each $k = 1, 2, ...$ and $x \in [-1, \infty)$ there exist the finite limits

$$V(t_k - 0, x) = \lim_{\substack{(t,x)\to(t_k,x)\\ t<t_k}} V(t, x),$$
$$V(t_k + 0, x) = \lim_{\substack{(t,x)\to(t_k,x)\\ t>t_k}} V(t, x).$$

4. The equality $V(t_k - 0, x) = V(t_k, x)$ is valid.

We also introduce the following classes of functions:

$PC[[t_0, \infty), [-1, \infty)] = \{\sigma : [t_0, \infty) \to [-1, \infty): \ \sigma(t)$ is continuous everywhere except some points t_k at which $\sigma(t)$ is continuous from the left and $\sigma(t_k + 0)$ exists$\}$;

$$\Omega_0 = \Big\{x \in PC[[t_0, \infty), [-1, \infty)]: \ V(s, x(s)) \leq V(t, x(t)), \ t - \tau \leq s \leq t,$$
$$t \geq t_0, \ V \in V_0\Big\}.$$

Let $V \in V_0$. For $x \in PC[[t_0, \infty), [-1, \infty)]$ and $t \neq t_k$, $k = 1, 2, ...$ we define the function

$$D_{-}V(t, x(t))$$
$$= \lim_{h \to 0^-} \inf \frac{1}{h}[V(t + h, x(t) - hr[1 + x(t)]x(t - \tau)) - V(t, x(t))].$$

Introduce the following conditions:

H1. $t_0 < t_1 < t_2 < ....$

H2. $\lim\limits_{k \to \infty} t_k = \infty.$

H3. $I_k \in C[[-1, \infty), R],\ k = 1, 2, ...$

H4. $I_k(0) = 0,\ k = 1, 2, ...$

H5. The functions $(I + I_k) : [-1, \infty) \to [-1, \infty),\ k = 1, 2, ...$ where I is the identity in R.

In the proof of the main results we shall use the following lemma.

Lemma 1. (see Bainov et al [1]) *Let the following conditions hold*:

1. *Conditions* H1-H5 *are met.*
2. *The function* $V \in V_0$ *is such that*

$$D_{-}V(t, x(t)) \leq 0,\ t \neq t_k,\ k = 1, 2, ...,\ t \geq t_0, x \in \Omega_0;$$
$$V(t + 0, x(t) + I_k(1 + x(t))) \leq V(t, x(t)),\ t = t_k,\ k = 1, 2, ...$$

Then
$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0)),\ t \geq t_0. \tag{6}$$

3. MAIN RESULTS

Theorem 1. *Let the following conditions hold*:

1. *Conditions* H1-H5 *are met.*
2. *The functions* $V \in V_0$ *and* $a, b \in K$ *are such that*

$$a(|x|) \leq V(t, x) \leq b(|x|),\ (t, x) \in [t_0, \infty) \times [-1, \infty). \tag{7}$$

3. *The inequalities*

$$D_{-}V(t, x(t)) \leq 0, \quad t \neq t_k,\ k = 1, 2, ...,\ x \in \Omega_0\,,$$
$$V(t + 0, x(t) + I_k(1 + x(t))) \leq V(t, x(t)), \quad t = t_k,\ k = 1, 2, ...$$

are valid for $t \geq t_0$, *and* $V \in V_0$.

Then the zero solution of system (4) *is uniformly stable.*

Proof. Let $\varepsilon > 0$ be chosen. Choose $\delta = \delta(\varepsilon) > 0$ so that $b(\delta) < a(\varepsilon)$.

Let $\varphi_0 \in D: ||\varphi_0|| < \delta$ and $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (4), (5).

Since the conditions of Lemma 1 are met, then from (6) and (7) we get the inequalities

$$a(|x(t; t_0, \varphi_0)|) \leq V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0))$$
$$\leq b(|\varphi_0(t_0)|) \leq b(||\varphi_0||) < b(\delta) < a(\varepsilon),\ t \geq t_0$$

from which it follows that $|x(t; t_0, \varphi_0)| < \varepsilon$ for $t \geq t_0$. This proves uniform stability of the zero solution of system (4). □

Theorem 2. *Let the following conditions hold*:

1. *Conditions* H1-H5 *are met.*
2. *The functions* $V \in V_0$ *and* $a, b \in K$ *are such that*

$$a(|x|) \leq V(t, x) \leq b(|x|), \quad (t, x) \in [t_0, \infty) \times [-1, \infty). \tag{8}$$

3. *The inequalities*

$$D_{-}V(t, x(t)) \leq -c(|x(t)|), \quad t \neq t_k,\ k = 1, 2, ...,\ x \in \Omega_0, \tag{9}$$

$$V(t + 0, x(t) + I_k(1 + x(t))) \leq V(t, x(t)), \quad t = t_k,\ k = 1, 2, ... \tag{10}$$

are valid for $t \geq t_0$ *and* $V \in V_0$, $c \in K$.

Then the zero solution of system (4) *is uniformly asymptotically stable.*

Proof. Let $\varepsilon > 0$ be chosen.

Choose $\lambda > 0$ so that $b(\lambda) < a(\varepsilon)$ and let $T = T(\varepsilon) > 0$ is such that $T > \frac{b(\lambda)}{c(\lambda)}$.

Let $\varphi_0 \in D: ||\varphi_0|| < \lambda$ and $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (4), (5).

If we suppose that for each $t \in [t_0, t_0 + T]$ the inequality $|x(t; t_0, \varphi_0)| \geq \lambda$ is valid, then from (9) and(10) we deduce the inequalities

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0))$$
$$- \int_{t_0}^{t} c(|x(s; t_0, \varphi_0)|)\, dx \leq b(\lambda) - c(\lambda)T < 0,$$

which contradict (8). Hence there exists $t^* \in [t_0, t_0 + T]$ such that

$$|x(t^*; t_0, \varphi_0)| < \lambda.$$

Then from (8), (9) and (10) we obtain that for $t \geq t^*$ (hence for $t \geq t_0 + T$ too) the following inequalitues are valid

$$a(|x(t; t_0, \varphi_0)|) \leq V(t, x(t; t_0, \varphi_0))$$

$$\leq V(t^*, x(t^*; t_0, \varphi_0)) \leq b(|x(t^*; t_0, \varphi_0)|)$$
$$< b(\lambda) < a(\varepsilon),$$

whence we deduce that the zero solution of (4) is uniformly attractive and since Theorem 1 implies that it is uniformly stable, then the zero solution of (4) is uniformly asymptotically stable. □

Theorem 3. *Let the following conditions hold*:

1. *Conditions* H1-H5 *are met.*
2. *The function* $V \in V_0$ *is such that*
$$|x(t)| \leq V(t, x(t)) \leq k(\alpha)|x(t)|,\ t \in [t_0, \infty), \tag{11}$$

$$x \in PC[[t_0, \infty), [-1, \infty)],\ k = k(\alpha) = \text{const.} > 0,\ \alpha > 0.$$

3. *The inequalities*
$$D_- V(t, x(t)) \leq -cV(t, x(t)),\ t \neq t_k,\ k = 1, 2, ...,\ x \in \Omega_0, \tag{12}$$
$$V(t + 0, x(t) + I_k(1 + x(t))) \leq V(t, x(t)),\ t = t_k,\ k = 1, 2, ... \tag{13}$$

are valid for $t \geq t_0$ *and* $V \in V_0$, $c =$const.> 0.

Then the zero solution of system (4) is exponentially asymptotically stable.

Proof. Let $\alpha > 0$ and $\varphi_0 \in D$ is such that $||\varphi_0|| < \alpha$. Let $x(t) = x(t; t_0, \varphi_0)$ be the solution of problem (4), (5).

From the inequalities (12) and (13) we get the estimate

$$V(t, x(t; t_0, \varphi_0)) \leq V(t_0 + 0, \varphi_0(t_0))[-c(t - t_0)], \quad t > t_0.$$

From the above estimate and (11) we deduce the inequalities

$$|x(t; t_0, \varphi_0)| \leq k(\alpha)|\varphi_0(t_0)| \exp[-c(t - t_0)]$$
$$\leq k(\alpha)||\varphi_0|| \exp[-c(t - t_0)], \quad t > t_0.$$

This shows that the zero solution of system (4) is exponentially asymptotocally stable.

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