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STRONGLY SINGULAR INTEGRAL OPERATORS ON TRIEBEL-LIZORKIN SPACE

Yan Lin[1] and Shanzhen Lu[2]
[1,2]School of Mathematical Sciences
Beijing Normal University
Beijing, 100875, P.R. China
[1]yanlinwhat@sohu.com
[2]lusz@bnu.edu.cn

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ABSTRACT: In this paper, the boundedness of two classes of operators is discussed from the Lebesgue spaces to the Triebel-Lizorkin spaces. One is the commutator generated by the strongly singular convolution operator and Lipschitz function, the other one is the generalized Toeplitz operator generated by a class of strongly singular Calderón-Zygmund operators and Lipschitz function. Moreover, the corresponding result of the commutator generated by fractional integral operator and Lipschitz function can be deduced immediately.

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1. INTRODUCTION

The strongly singular integral operators have important background in multiple Fourier series. For a suitable function f, its Fourier transform is defined by

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \xi} dx.$$

Let $\theta(\xi)$ be a smooth radial cut-off function. $\theta(\xi) = 1$ if $|\xi| \geq 1$ and $\theta(\xi) = 0$ if $|\xi| \leq \frac{1}{2}$. The strongly singular integral operator is defined by

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$\mathcal{F}(T^{s,\alpha}f)(\xi)=\theta(\xi)\frac{e^{i|\xi|^s}}{|\xi|^\alpha}\hat{f}(\xi)$, where $0<s<1$, $0<\alpha\le\frac{ns}{2}$. Let $\lambda=\frac{ns/2-\alpha}{1-s}$, the convolution form of $T^{s,\alpha}$ can be roughly written as

$$T^{s,\alpha}f(x)=\text{p.v.}\int\frac{e^{i|x-y|^{-s'}}}{|x-y|^{n+\lambda}}\chi(|x-y|)f(y)dy.$$

Here $s'=\frac{s}{1-s}$ and χ denotes the characteristic function of the unit interval $(0,1)\subset\mathbb{R}$. Since s and α do not appear in the convolution form apparently, we would like to use $T^{s',\lambda}$ instead of $T^{s,\alpha}$ to denote this operator.

Fefferman [5], Hirschman [6] obtained that $T^{s',\lambda}$ was an (L^p,L^p) type operator when $|\frac{1}{p}-\frac{1}{2}|<\frac{1}{2}-\frac{\lambda}{ns'}$, namely $\frac{ns'}{ns'-\lambda}<p<\frac{ns'}{\lambda}$, and pointed out that the range of p is the best.

When $0<\beta<1$, the Lipschitz space $\dot{\Lambda}_\beta$ consists of functions satisfying

$$\|f\|_{\dot{\Lambda}_\beta}=\sup_{x,h\in\mathbb{R}^n,\,h\neq0}\frac{|f(x+h)-f(x)|}{|h|^\beta}<\infty.$$

Li [7] discussed the boundedness of the commutator

$$[b,T^{s',\lambda}](f)=T^{s',\lambda}(bf)-bT^{s',\lambda}(f)$$

generated by $T^{s',\lambda}$ and Lipschitz function b on $L^p(\mathbb{R}^n)$, $1<p<\infty$. In this paper, we are interested in the boundedness of this commutator from the Lebesgue space to the Triebel-Lizorkin space.

Another class of strongly singular non-convolution operators, which is called strongly singular Calderón-Zygmund operator, was introduced by Alvarez and Milman [1].

Definition 1.1. Let $T:\mathcal{S}\to\mathcal{S}'$ be a bounded linear operator. T is called a strongly singular Calderón-Zygmund operator if the following conditions are satisfied.

(1) T extends to a continuous operator from L^2 into itself.

(2) T is associated with a certain standard kernel. More precisely, there exists a function $K(x,y)$ continuous away the diagonal on $\mathbb{R}^{2n}$ such that

$$|K(x,y)-K(x,z)|+|K(y,x)-K(z,x)|\le C\frac{|y-z|^\delta}{|x-z|^{n+\frac{\delta}{\alpha}}},$$

if

$$2|y-z|^\alpha\le|x-z|\quad\text{for some}\quad 0<\delta\le1,0<\alpha<1,$$

$\langle Tf,g\rangle=\int K(x,y)f(y)g(x)dydx$, for $f,g\in\mathcal{S}$ with disjoint supports.

(3) For some $n(1-\alpha)/2\le\beta<n/2$, both operators T and T^* extend to continuous operators from L^q to L^2, where $1/q=1/2+\beta/n$.

The properties of the strongly singular Calderón-Zygmund operators are similar to those of Calderón-Zygmund operators, but the kernel is more singular near the diagonal than those of the standard case.

It is easy to see that the commutator generated by fractional integral operator and a locally integrable function b can be regarded as a special case of generalized Toeplitz operator $\Theta^b_{\alpha_0} = \sum_{j=1}^m (T_{j,1}M_b I_{\alpha_0} T_{j,2} + T_{j,3} I_{\alpha_0} M_b T_{j,4})$, where $T_{j,1}$ are Calderón-Zygmund operators or $\pm I$ (I is the identity operator), $T_{j,2}$, $T_{j,4}$ are the bounded linear operators on L^p, $T_{j,3} = \pm I$, $M_b f(x) = b(x)f(x)$, $I_{\alpha_0} f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha_0}} dy$ is the fractional integral operator. When $b \in BMO$ (see Qiu [10]) obtained the boundedness of $\Theta^b_{\alpha_0}$ on Homogeneous space from $L^p(X)$ to $L^q(X)$, $1/q = 1/p - \alpha_0$.

Now we consider the strongly singular Calderón-Zygmund operators instead of the standard Calderón-Zygmund operators in the definition of generalized Toeplitz operator $\Theta^b_{\alpha_0}$. We are interested in the boundedness of $\Theta^b_{\alpha_0} = \sum_{j=1}^m (T_{j,1}M_b I_{\alpha_0} T_{j,2} + T_{j,3} I_{\alpha_0} M_b T_{j,4})$ from the Lebesgue space to the Triebel-Lizorkin space, where $T_{j,1}$ are the strongly singular Calderón-Zygmund operators or $\pm I$, $T_{j,2}$, $T_{j,4}$ are the bounded linear operators on L^p, $T_{j,3} = \pm I$, b is a Lipschitz function.

Let us state our main results.

Theorem 1.1. *If* $0 < \lambda < \min\{1, \frac{ns/2}{1-s}\}$, $b \in \dot{\Lambda}_{\beta_1}$, $\lambda < \beta_1 < 1$, *then the commutator* $[b, T^{s',\lambda}]$ *is bounded from* L^p *to the homogeneous Triebel-Lizorkin space* $\dot{F}_p^{\beta_2,\infty}$, *where* $\frac{ns'}{ns'-\lambda} < p < \infty$, $0 < \beta_2 \leq (\beta_1 - \lambda)(1-s)$.

Theorem 1.2. *Suppose* $\Theta^1_{\alpha_0}(f) = 0$ *when* $f \in L^p(\mathbb{R}^n)$. *Let* α, β, δ *be as in Definition 1.1. If* $0 < \alpha_0 < \frac{n}{2}$, $b \in \dot{\Lambda}_{\beta_0}$, $0 < \beta_0 < \frac{\delta(n\alpha - n + 2\beta)}{2(\delta - \delta\alpha + \beta\alpha)}$, *then* $\Theta^b_{\alpha_0}$ *is bounded from* $L^p(\mathbb{R}^n)$ *to the homogeneous Triebel-Lizorkin space* $\dot{F}_q^{\beta_0,\infty}$, *where* $2 < p < \frac{n}{\alpha_0}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_0}{n}$.

It is obvious to see that the range of p and β_0 can be extended to $1 < p < \frac{n}{\alpha_0}$ and $0 < \beta_0 < 1$ if $T_{j,1} = \pm I$ from the proof of Theorem 1.2, which we will give in Section 3. Thus we can obtain the following corollary.

Corollary 1.1. *If* $b \in \dot{\Lambda}_{\beta_0}(\mathbb{R}^n)$, $0 < \beta_0 < 1$, $0 < \alpha_0 < n$, *then the commutator* $[b, I_{\alpha_0}]$ *is bounded from* $L^p(\mathbb{R}^n)$ *to* $\dot{F}_q^{\beta_0,\infty}$, *where* $1 < p < \frac{n}{\alpha_0}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_0}{n}$.

Remark 1.1. It should be pointed out that the result of Corollary 1.1 had been obtained by Lu and P. Zhang [8] (Theorem 1.6), but the method of the proof here is different.

2. MAIN LEMMAS

In this paper, denote Q a cube with sides parallel to the axes. For $c > 0$, cQ is the cube with the same center as Q, and the side length increased c times. Given $f \in L^p_{loc}(\mathbb{R}^n)$, define the maximal function by

$$f^*_{q,p}(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-qp/n}} \int_Q |f(y)|^p dy \right)^{1/p}, \quad 0 \le q < \infty, 1 \le p < \infty.$$

It is easy to see that $f^*_{0,1}(x)$ is the Hardy-Littlewood maximal function, $f^*_{0,p}(x) = [(|f|^p)^*_{0,1}]^{1/p}(x)$, and $f^*_{0,p_1}(x) \le f^*_{0,p_2}(x)$ for $p_1 \le p_2$ and $f \in L^{p_2}_{loc}(\mathbb{R}^n)$ by Hölder's inequality.

Lemma 2.1. (see Paluszyński [9]) *For $0 < \beta_0 < 1$, $1 < p < \infty$, we have*

$$\|h\|_{\dot{F}^{\beta_0,\infty}_p} \approx \left\| \sup_{Q \ni \cdot} \inf_{c \in \mathbb{C}} \frac{1}{|Q|^{1+\beta_0/n}} \int_Q |h(y) - c| dy \right\|_p.$$

Lemma 2.2. (see Chanillo [3]) *Let $\widetilde{K}^{\varepsilon}_{b,p}(x) = \frac{e^{i|x|^{-b}}}{|x|^{n(b+2)/p}} \chi_{(\varepsilon \le |x| \le 1)}(x)$, $x \in \mathbb{R}^n$, $0 < b < \infty$, $2 \le p \le \infty$. If $(2+b)/p < 1$, then $\|\widetilde{K}^{\varepsilon}_{b,p} * f\|_p \le C_p \|f\|_{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Lemma 2.3. (see Alvarez and Milman [1]) *If T is a strongly singular Calderón-Zygmund operator, then T can be defined to be a continuous operator from L^∞ to BMO.*

Alvarez and Milman [2] also proved that the strongly singular Calderón-Zygmund operator T is of weak (L^1, L^1) type. Thus we can get that T is bounded on L^p, $1 < p < \infty$, by the interpolation theory.

Lemma 2.4. (see Stein [11]) *The fractional integral operator I_{α_0} is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha_0}{n}$, $1 < p < \frac{n}{\alpha_0}$.*

Lemma 2.5. (see Chanillo [4]) *The maximal function $f^*_{\alpha,l}(x)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1 < l < p < n/\alpha$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < \alpha < n$.*

3. PROOF OF THEOREMS

Proof of Theorem 1.1. Since $\frac{ns'}{ns'-\lambda} < p < \infty$, there exists a q such that $\frac{ns'}{ns'-\lambda} < q < \min\{p, \frac{ns'}{\lambda}\}$. $T^{s',\lambda}$ is bounded on L^q; see Fefferman [5], Hirschman [6]. Choose a l' satisfying $\max\{2+s', q'\} < l' < \infty$, where q' is the conjugate exponent of q, then $\frac{2+s'}{l'} < 1$. Denote $\frac{1}{l} + \frac{1}{l'} = 1$, so $l < q$.

From Lemma 2.1, we know that

$$\|[b, T^{s',\lambda}](f)\|_{\dot{F}^{\beta_2,\infty}_p} \approx \left\| \sup_{Q \ni \cdot} \inf_{c \in \mathbb{C}} \frac{1}{|Q|^{1+\beta_2/n}} \int_Q |[b, T^{s',\lambda}](f)(y) - c| dy \right\|_p, \quad (3.1)$$

so we just need to estimate

$$\sup_{Q\ni x}\inf_{c\in\mathbb{C}}\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}](f)(y)-c|dy,$$

for any $x\in\mathbb{R}^n$.

Fix $x_0\in\mathbb{R}^n$. Let Q be a cube centered at x_0 whose side length is r. We only need to consider the situtzion when $r=2^k$, $k\in\mathbb{Z}$.

Case 1. $0<r\leq 1$. Let Q_1 be the cube centered at x_0 whose side length is r^{u_1}, where $u_1=\frac{\beta_2 l'+n}{(\beta_1-\lambda)l'+n(s'+1)}<1$.

Denote $f_1=f\chi_{2Q}$, $f_2=f\chi_{Q_1\backslash 2Q}$, $f_3=f-f_1-f_2$ for $r^{u_1}>2r$, $f_1=f\chi_{2Q}$, $f_2\equiv 0$, $f_3=f-f_1-f_2$ for $r^{u_1}\leq 2r$, and

$$c(Q)=\int_{\mathbb{R}^n}\frac{e^{i|x_0-z|^{-s'}}}{|x_0-z|^n}\chi(|x_0-z|)\frac{[b(x_0)-b(z)]}{|x_0-z|^{\lambda}}f_3(z)dz.$$

We have that

$$\begin{aligned}
&\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f(y)-c(Q)|dy\\
&\leq \frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f_1(y)|dy+\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f_2(y)|dy\\
&\quad +\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f_3(y)-c(Q)|dy\\
&:= I_1+I_2+I_3.
\end{aligned}$$

Let us estimate I_1 first. Since $T^{s',\lambda}$ is bounded on L^q and $0<r\leq 1$, let $b_Q=\frac{1}{|Q|}\int_Q b(y)dy$, then

$$\begin{aligned}
I_1 &\leq \frac{1}{|Q|^{1+\beta_2/n}}\int_Q |b(y)-b_Q||T^{s',\lambda}f_1(y)|dy\\
&\quad +\tfrac{1}{|Q|^{1+\beta_2/n}}\textstyle\int_Q |T^{s',\lambda}([b_Q-b]f_1)(y)|dy\\
&\leq C\frac{r^{\beta_1}}{|Q|^{1+\beta_2/n}}\|b\|_{\dot{\Lambda}_{\beta_1}}\int_Q |T^{s',\lambda}f_1(y)|dy\\
&\quad +\frac{1}{|Q|^{\beta_2/n}}\left(\frac{1}{|Q|}\int_Q |T^{s',\lambda}([b_Q-b]f_1)(y)|^q dy\right)^{\frac{1}{q}}\\
&\leq Cr^{\beta_1-\beta_2}\|b\|_{\dot{\Lambda}_{\beta_1}}\left(\frac{1}{|Q|}\int_Q |T^{s',\lambda}f_1(y)|^q dy\right)^{\frac{1}{q}}\\
&\quad +Cr^{-\beta_2}\left(\frac{1}{|Q|}\int_{2Q} |b_Q-b(y)|^q|f(y)|^q dy\right)^{\frac{1}{q}}\\
&\leq Cr^{\beta_1-\beta_2}\|b\|_{\dot{\Lambda}_{\beta_1}}\left(\frac{1}{|2Q|}\int_{2Q} |f(y)|^q dy\right)^{\frac{1}{q}}\\
&\leq C\|b\|_{\dot{\Lambda}_{\beta_1}}f^*_{0,q}(x_0).
\end{aligned}$$

Then we estimate I_2. Since $f_2 \equiv 0$ for $r^{u_1} \le 2r$, we only consider $r^{u_1} > 2r$ here.

$$
\begin{aligned}
I_2 &\le \frac{1}{|Q|^{1+\beta_2/n}} \\
&\times \int_Q \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^n} \chi(|y-z|) \left[\frac{b(y)-b(z)}{|y-z|^\lambda} - \frac{b(x_0)-b(z)}{|x_0-z|^\lambda} \right] f_2(z)dz \right| dy \\
&\quad + \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^n} \chi(|y-z|) \frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda} f_2(z)dz \right| dy \\
&\le \frac{1}{|Q|^{1+\beta_2/n}} \\
&\times \int_Q \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^n} \chi(|y-z|) \left[\frac{b(y)-b(z)}{|y-z|^\lambda} - \frac{b(x_0)-b(z)}{|x_0-z|^\lambda} \right] f_2(z)dz \right| dy \\
&\quad + \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n(\frac{s'+2}{l'})}} \chi(|y-z|) \right. \\
&\times \left. \left[\frac{1}{|y-z|^{n(1-\frac{s'+2}{l'})}} - \frac{1}{|x_0-z|^{n(1-\frac{s'+2}{l'})}} \right] \frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda} f_2(z)dz \right| dy \\
&\quad + \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n(\frac{s'+2}{l'})}} \chi(|y-z|) \frac{1}{|x_0-z|^{n(1-\frac{s'+2}{l'})}} \right. \\
&\qquad \left. \times \frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda} f_2(z)dz \right| dy \\
&:= I_{21} + I_{22} + I_{23}.
\end{aligned}
$$

Using the fact that

$$
\begin{aligned}
&\frac{[b(y)-b(z)]}{|y-z|^\lambda} - \frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda} \\
&= \frac{[b(y)-b(z)-b(x_0)+b(z)]}{|y-z|^\lambda} \\
&\quad + \frac{[b(x_0)-b(z)]}{|y-z|^\lambda} - \frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda} \\
&= \frac{[b(y)-b(x_0)]}{|y-z|^\lambda} + [b(x_0)-b(z)] \left[\frac{1}{|y-z|^\lambda} - \frac{1}{|x_0-z|^\lambda} \right],
\end{aligned} \tag{3.2}
$$

we can estimate I_{21} as follows.

$$\begin{aligned}
I_{21} \quad &\leq \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \int_{(2Q)^c} \frac{|b(y)-b(x_0)|}{|y-z|^{n+\lambda}} |f(z)| dz dy \\
&\quad + \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \int_{(2Q)^c} \frac{|b(x_0)-b(z)|}{|y-z|^{n}} \left| \frac{1}{|y-z|^{\lambda}} - \frac{1}{|x_0-z|^{\lambda}} \right| |f(z)| dz dy \\
&\leq C r^{\beta_1-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \int_{(2Q)^c} \frac{|f(z)|}{|x_0-z|^{n+\lambda}} dz \\
&\quad + C r^{-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \int_{(2Q)^c} \frac{r}{|x_0-z|^{n+\lambda+1-\beta_1}} |f(z)| dz \\
&= C r^{\beta_1-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \frac{|f(z)|}{|x_0-z|^{n+\lambda}} dz \\
&\quad + C r^{1-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} \frac{|f(z)|}{|x_0-z|^{n+\lambda+1-\beta_1}} dz \\
&\leq C r^{\beta_1-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{\lambda}} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)| dz \\
&\quad + C r^{1-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \sum_{k=1}^{\infty} \frac{1}{(2^k r)^{\lambda+1-\beta_1}} \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)| dz \\
&\leq C \|b\|_{\dot{\Lambda}_{\beta_1}} r^{\beta_1-\beta_2-\lambda} f^*_{0,1}(x_0) \\
&\leq C \|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0).
\end{aligned}$$

We get the estimate of I_{22} by the similar method of I_{21}.

$$\begin{aligned}
I_{22} \quad &\leq \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \int_{(2Q)^c} \frac{1}{|y-z|^{n(\frac{s'+2}{l'})}} \frac{|b(x_0)-b(z)|}{|x_0-z|^{\lambda}} \\
&\qquad \times \left| \frac{1}{|y-z|^{n(1-\frac{s'+2}{l'})}} - \frac{1}{|x_0-z|^{n(1-\frac{s'+2}{l'})}} \right| |f(z)| dz dy \\
&\leq C r^{-\beta_2} \|b\|_{\dot{\Lambda}_{\beta_1}} \int_{(2Q)^c} \frac{r}{|x_0-z|^{n+\lambda+1-\beta_1}} |f(z)| dz \\
&\leq C \|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0).
\end{aligned}$$

Since $r^{u_1} > 2r$, there exists a $N(r) \in \mathbb{N}$ such that $2^N r < r^{u_1} \leq 2^{N+1} r$.

By Hölder's inequality and Lemma 2.2, we see

$$
\begin{aligned}
I_{23} &\le \frac{1}{|Q|^{1+\beta_2/n}}|Q|^{\frac{1}{l}}\left(\int_Q\left|\int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n(\frac{s'+2}{l'})}}\chi(|y-z|)\frac{1}{|x_0-z|^{n(1-\frac{s'+2}{l'})}}\right.\right.\\
&\qquad\left.\left.\times\frac{[b(x_0)-b(z)]}{|x_0-z|^{\lambda}}f_2(z)dz\right|^{l'}dy\right)^{\frac{1}{l'}}\\
&\le \frac{C}{|Q|^{1+\beta_2/n}}|Q|^{\frac{1}{l}}\left(\int_{Q_1\backslash 2Q}\frac{|b(x_0)-b(z)|^l}{|x_0-z|^{n(1-\frac{s'+2}{l'})l+\lambda l}}|f(z)|^l dz\right)^{\frac{1}{l}}\\
&\le \frac{C}{|Q|^{\frac{1}{l'}+\frac{\beta_2}{n}}}\|b\|_{\dot{\Lambda}_{\beta_1}}\left(\int_{Q_1\backslash 2Q}\frac{|x_0-z|^{l\beta_1}}{|x_0-z|^{n(1-\frac{s'+2}{l'})l+\lambda l}}|f(z)|^l dz\right)^{\frac{1}{l}}\\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{-\frac{n}{l'}-\beta_2}r^{u_1(\beta_1-\lambda)}\left(\int_{Q_1\backslash 2Q}\frac{|f(z)|^l}{|x_0-z|^{n(1-\frac{s'+2}{l'})l}}dz\right)^{\frac{1}{l}}\\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{-\frac{n}{l'}-\beta_2}r^{u_1(\beta_1-\lambda)}\left(\sum_{k=1}^{N}\int_{2^{k+1}Q\backslash 2^kQ}\frac{|f(z)|^l}{|x_0-z|^{n(1-\frac{s'+2}{l'})l}}dz\right)^{\frac{1}{l}}\\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{-\frac{n}{l'}-\beta_2}r^{u_1(\beta_1-\lambda)}\\
&\qquad\times\left(\sum_{k=1}^{N}\frac{1}{(2^kr)^{n(1-\frac{s'+2}{l'})l-n}}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(z)|^l dz\right)^{\frac{1}{l}}\\
&= C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{u_1(\beta_1-\lambda)-\frac{n}{l'}-\beta_2}\\
&\qquad\times\left(\sum_{k=1}^{N}(2^kr)^{n\frac{s'+1}{l'-1}}\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|f(z)|^l dz\right)^{\frac{1}{l}}\\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{u_1(\beta_1-\lambda+\frac{n}{l}\frac{s'+1}{l'-1})-\frac{n}{l'}-\beta_2}f^*_{0,l}(x_0)\\
&= C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{u_1(\beta_1-\lambda+\frac{n(s'+1)}{l'})-\frac{n}{l'}-\beta_2}f^*_{0,l}(x_0)\\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0).
\end{aligned}
$$

At last, we give the estimate of I_3.

$$
\begin{aligned}
I_3 &\le \frac{1}{|Q|^{1+\beta_2/n}}\int_Q\left|\int\frac{e^{i|y-z|^{-s'}}}{|y-z|^n}\chi(|y-z|)\left[\frac{b(y)-b(z)}{|y-z|^{\lambda}}-\frac{b(x_0)-b(z)}{|x_0-z|^{\lambda}}-\right]\right.\\
&\qquad\left.\times f_3(z)dz\right|dy\\
&\quad+\frac{1}{|Q|^{1+\beta_2/n}}\int_Q\left|\int\left[\frac{e^{i|y-z|^{-s'}}}{|y-z|^n}\chi(|y-z|)-\frac{e^{i|x_0-z|^{-s'}}}{|x_0-z|^n}\chi(|x_0-z|)\right]\right.\\
&\qquad\left.\times\frac{[b(x_0)-b(z)]}{|x_0-z|^{\lambda}}f_3(z)dz\right|dy\\
&:= I_{31}+I_{32}.
\end{aligned}
$$

From the definition of f_3, $\mathrm{supp} f_3 \subset (Q_1 \cup 2Q)^c \subset (2Q)^c$, we can use the similar method of I_{21} to get the estimate of I_{31}.

$$I_{31} \le C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0).$$

It is easy to see that $u_1 \le \frac{1-\beta_2}{1+s'+\lambda-\beta_1}$ from the hypothesis of the theorem. Using the inequality

$$\left|\frac{e^{i|y-z|^{-s'}}}{|y-z|^n} - \frac{e^{i|x_0-z|^{-s'}}}{|x_0-z|^n}\right| \le C\frac{r}{|x_0-z|^{n+s'+1}},$$

and noticing the fact that $\mathrm{supp} f_3 \subset (Q_1 \cup 2Q)^c \subset (Q_1)^c$, $0 < r \le 1$, we have

$$\begin{aligned}
I_{32} &\le Cr^{-\beta_2}\int_{Q_1^c} \frac{r}{|x_0-z|^{n+s'+1}}\frac{|b(x_0)-b(z)|}{|x_0-z|^{\lambda}}|f(z)|dz \\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{1-\beta_2}\int_{Q_1^c} \frac{|f(z)|}{|x_0-z|^{n+s'+1+\lambda-\beta_1}}dz \\
&= C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{1-\beta_2}\sum_{k=0}^{\infty}\int_{2^{k+1}Q_1\setminus 2^kQ_1} \frac{|f(z)|}{|x_0-z|^{n+s'+1+\lambda-\beta_1}}dz \\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{1-\beta_2}\sum_{k=0}^{\infty}(2^k r^{u_1})^{-(s'+1+\lambda-\beta_1)}\frac{1}{|2^{k+1}Q_1|}\int_{2^{k+1}Q_1}|f(z)|dz \\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{1-\beta_2-u_1(s'+1+\lambda-\beta_1)} f^*_{0,1}(x_0) \\
&\le C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0).
\end{aligned}$$

Thus,

$$\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b, T^{s',\lambda}]f(y) - c(Q)|dy \le C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0). \qquad (3.3)$$

Case 2. $r > 1$. Let Q_2 be the cube centered at x_0 whose side length is r^{u_2}, where $u_2 = [\frac{\beta_1-\beta_2}{\lambda}] + 1$, then $u_2 > \frac{\beta_1-\beta_2}{\lambda} > 1$. Here for $s \in \mathbb{R}$, $[s]$ denotes the largest integer no more than s.

Since $r > 1$, there exists a $K_0(r) \in \mathbb{N}$, such that $r = 2^{K_0}$. We decompose Q_2 into $2^{nu_2K_0}$ unit cube A_j. Number these cubes like $Q = \bigcup_{j\le 2^{nK_0}} A_j$, $Q_2 = \bigcup_{j\le 2^{nu_2K_0}} A_j$.

Set $j \sim k$ to mean that A_j and A_k are adjacent. For a fixed j, there are only finite k such that $j \sim k$, and the finite number just depends on the space dimension n.

Denote $f_4 = f\chi_{Q_2}$, $f_5 = f - f_4$, and

$$c'(Q) = \int_{\mathbb{R}^n} \frac{e^{i|x_0-z|^{-s'}}}{|x_0-z|^n}\chi(|x_0-z|)\frac{[b(x_0)-b(z)]}{|x_0-z|^{\lambda}}f_5(z)dz.$$

We have that

$$\begin{aligned}&\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f(y)-c'(Q)|dy\\&\le \frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f_4(y)|dy\\&+\frac{1}{|Q|^{1+\beta_2/n}}\int_Q |[b,T^{s',\lambda}]f_5(y)-c'(Q)|dy\\&:= II_1+II_2.\end{aligned}$$

Let us estimate II_2 first.

$$\begin{aligned}II_2 \;&\le \frac{1}{|Q|^{1+\beta_2/n}}\int_Q\left|\int \frac{e^{i|y-z|^{-s'}}}{|y-z|^n}\chi(|y-z|)\right.\\&\quad\times\left.\left(\tfrac{[b(y)-b(z)]}{|y-z|^\lambda}-\tfrac{[b(x_0)-b(z)]}{|x_0-z|^\lambda}\right)f_5(z)dz\right|dy\\&\quad+\frac{1}{|Q|^{1+\beta_2/n}}\int_Q\left|\int\left[\frac{e^{i|y-z|^{-s'}}}{|y-z|^n}\chi(|y-z|)-\frac{e^{i|x_0-z|^{-s'}}}{|x_0-z|^n}\chi(|x_0-z|)\right]\right.\\&\qquad\qquad\left.\times\frac{[b(x_0)-b(z)]}{|x_0-z|^\lambda}f_5(z)dz\right|dy\\&:= II_{21}+II_{22}.\end{aligned}$$

By using (3.2) we can get the estimate of II_{21}.

$$\begin{aligned}II_{21}\;&\le \frac{1}{|Q|^{1+\beta_2/n}}\int_Q\int_{Q_2^c}\frac{|b(y)-b(x_0)|}{|y-z|^{n+\lambda}}|f(z)|dzdy\\&\quad+\frac{1}{|Q|^{1+\beta_2/n}}\int_Q\int_{Q_2^c}\frac{|b(x_0)-b(z)|}{|y-z|^n}\left|\frac{1}{|y-z|^\lambda}-\frac{1}{|x_0-z|^\lambda}\right||f(z)|dzdy\\&\le Cr^{\beta_1-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\int_{Q_2^c}\frac{|f(z)|}{|x_0-z|^{n+\lambda}}dz\\&\quad+Cr^{-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\int_{Q_2^c}\frac{r}{|x_0-z|^{n+\lambda+1-\beta_1}}|f(z)|dz\\&= Cr^{\beta_1-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\sum_{k=0}^\infty\int_{2^{k+1}Q_2\setminus 2^kQ_2}\frac{|f(z)|}{|x_0-z|^{n+\lambda}}dz\\&\quad+Cr^{1-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\sum_{k=0}^\infty\int_{2^{k+1}Q_2\setminus 2^kQ_2}\frac{|f(z)|}{|x_0-z|^{n+\lambda+1-\beta_1}}dz\\&\le Cr^{\beta_1-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\sum_{k=0}^\infty(2^kr^{u_2})^{-\lambda}\frac{1}{|2^{k+1}Q_2|}\int_{2^{k+1}Q_2}|f(z)|dz\\&\quad+Cr^{1-\beta_2}\|b\|_{\dot\Lambda_{\beta_1}}\sum_{k=0}^\infty(2^kr^{u_2})^{-(\lambda+1-\beta_1)}\frac{1}{|2^{k+1}Q_2|}\int_{2^{k+1}Q_2}|f(z)|dz\\&\le C\|b\|_{\dot\Lambda_{\beta_1}}\left(r^{\beta_1-\beta_2-\lambda u_2}+r^{1-\beta_2-u_2(\lambda+1-\beta_1)}\right)f^*_{0,1}(x_0)\\&\le C\|b\|_{\dot\Lambda_{\beta_1}}\left(1+r^{\frac{(\beta_1-\lambda-\beta_2)(\beta_1-1)}{\lambda}}\right)f^*_{0,1}(x_0)\end{aligned}$$

$$\leq C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(f)(x_0).$$

Like the method of I_{32} we can estimate II_{22} as follows.

$$\begin{aligned} II_{22} &\leq C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{1-\beta_2-u_2(1+s'+\lambda-\beta_1)} f^*_{0,1}(x_0) \\ &\leq C\|b\|_{\dot{\Lambda}_{\beta_1}} r^{-\frac{(1-\beta_1)(\beta_1-\lambda-\beta_2)+(\beta_1-\beta_2)s'}{\lambda}} f^*_{0,1}(x_0) \\ &\leq C\|b\|_{\dot{\Lambda}_{\beta_1}} f^*_{0,q}(x_0). \end{aligned}$$

At last, we give the estimate of II_1. Let x_j be the center of A_j.

$$\begin{aligned} II_1 &= \frac{1}{|Q|^{1+\beta_2/n}} \int_Q \left| \int_{Q_2} \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|)[b(y)-b(z)]f(z)dz \right| dy \\ &\leq \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{nK_0}} \int_{A_j} |b(y)-b(x_j)| \\ &\quad \times \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \sum_{k=1}^{2^{nu_2K_0}} f\chi_{A_k}(z)dz \right| dy \\ &\quad + \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{nK_0}} \int_{A_j} \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \right. \\ &\quad + \left. \sum_{k=1}^{2^{nu_2K_0}} [b(x_j)-b(z)] f\chi_{A_k}(z)dz \right| dy \\ &\leq \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{nK_0}} \int_{A_j} |b(y)-b(x_j)| \\ &\quad \times \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \sum_{k\sim j} f\chi_{A_k}(z)dz \right| dy \\ &\quad + \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{nK_0}} \int_{A_j} \\ &\quad \times \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \sum_{k\sim j} [b(x_j)-b(z)] f\chi_{A_k}(z)dz \right| dy. \end{aligned}$$

When $y \in A_j$, $|y-x_j| \leq \frac{\sqrt{n}}{2}$. It is true that $|z-x_j| < C_n$ when $z \in A_k$ and $k \sim j$. By using Hölder's inequality and the boundedness of $T^{s',\lambda}$ on L^q, we can obtain

$$
\begin{aligned}
II_1 &\le C\|b\|_{\dot\Lambda_{\beta_1}} \frac{1}{|Q|^{1+\beta_2/n}} \\
&\times \textstyle\sum_{j=1}^{2^{nK_0}} |A_j|^{\frac{1}{q'}} \left(\int_{A_j} \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \sum_{k\sim j} f\chi_{A_k}(z)dz \right|^q dy \right)^{\frac{1}{q}} \\
&\quad + \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{nK_0}} |A_j|^{\frac{1}{q'}} \left(\int_{A_j} \left| \int \frac{e^{i|y-z|^{-s'}}}{|y-z|^{n+\lambda}} \chi(|y-z|) \right.\right. \\
&\quad \left.\left. \times \sum_{k\sim j} [b(x_j)-b(z)] f\chi_{A_k}(z)dz \right|^q dy \right)^{\frac{1}{q}} \\
&\le C\|b\|_{\dot\Lambda_{\beta_1}} \frac{1}{|Q|^{1+\beta_2/n}} \sum_{j=1}^{2^{n(K_0+1)}} |A_j|^{\frac{1}{q'}} \left(\int_{A_j} |f(z)|^q dz \right)^{\frac{1}{q}} \\
&\le C\|b\|_{\dot\Lambda_{\beta_1}} \frac{1}{|Q|^{1+\beta_2/n}} \left(\sum_{j=1}^{2^{n(K_0+1)}} |A_j| \right)^{\frac{1}{q'}} \left(\sum_{j=1}^{2^{n(K_0+1)}} \int_{A_j} |f(z)|^q dz \right)^{\frac{1}{q}} \\
&= C\|b\|_{\dot\Lambda_{\beta_1}} \frac{1}{|Q|^{1+\beta_2/n}} |2Q|^{\frac{1}{q'}} \left(\int_{2Q} |f(z)|^q dz \right)^{\frac{1}{q}} \\
&= C\|b\|_{\dot\Lambda_{\beta_1}} r^{-\beta_2} \left(\frac{1}{|2Q|} \int_{2Q} |f(z)|^q dz \right)^{\frac{1}{q}} \\
&\le C\|b\|_{\dot\Lambda_{\beta_1}} f^*_{0,q}(x_0).
\end{aligned}
$$

Thus,

$$
\frac{1}{|Q|^{1+\beta_2/n}} \int_Q |[b, T^{s',\lambda}] f(y) - c'(Q)| dy \le C\|b\|_{\dot\Lambda_{\beta_1}} f^*_{0,q}(x_0). \tag{3.4}
$$

In conclusion, by (3.3) and (3.4),

$$
\sup_{Q\ni x_0} \inf_{c\in\mathbb{C}} \frac{1}{|Q|^{1+\beta_2/n}} \int_Q |[b, T^{s',\lambda}](f)(y) - c| dy \le C\|b\|_{\dot\Lambda_{\beta_1}} f^*_{0,q}(x_0),\ x_0 \in \mathbb{R}^n.
$$

Thus by (3.1) and the boundedness of the Hardy-Littlewood maximal function on Lebesgue spaces,

$$
\|[b, T^{s',\lambda}](f)\|_{\dot F_p^{\beta_2,\infty}} \le C\|b\|_{\dot\Lambda_{\beta_1}} \|f^*_{0,q}\|_p \le C\|b\|_{\dot\Lambda_{\beta_1}} \|f\|_p.
$$

This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. We just need to estimate

$$
\sup_{Q\ni x} \inf_{c\in\mathbb{C}} \frac{1}{|Q|^{1+\beta_0/n}} \int_Q |\Theta^b_{\alpha_0}(f)(y) - c| dy,
$$

for any $x \in \mathbb{R}^n$.

Denote B a ball in $\mathbb{R}^n$ with radius $r = r(B)$. For $c > 0$, cB is the ball with the same center as B, and the radius increased c times. It is easy to see that we can discuss the ball instead of the cube in the above formula.

$$\begin{aligned}
&\sup_{B\ni x}\inf_{c\in\mathbb{C}}\frac{1}{|B|^{1+\beta_0/n}}\int_B|\Theta^b_{\alpha_0}(f)(y)-c|dy\\
&=\max\Big\{\sup_{B\ni x,\,0<r(B)\le 1}\inf_{c\in\mathbb{C}}\frac{1}{|B|^{1+\beta_0/n}}\int_B|\Theta^b_{\alpha_0}(f)(y)-c|dy,\\
&\qquad\sup_{B\ni x,\,r(B)>1}\inf_{c\in\mathbb{C}}\frac{1}{|B|^{1+\beta_0/n}}\int_B|\Theta^b_{\alpha_0}(f)(y)-c|dy\Big\}. \qquad (3.5)
\end{aligned}$$

For fixed $x_0 \in \mathbb{R}^n$, let B be a ball centered at x_0. Denote $\frac{1}{q_0} = \frac{1}{2} + \frac{\beta}{n}$, then the conjugate exponent q_0' of q_0 satisfies $\frac{1}{q_0'} = \frac{1}{2} - \frac{\beta}{n}$. Since

$$0 < \beta_0 < \frac{\delta(n\alpha - n + 2\beta)}{2(\delta - \delta\alpha + \beta\alpha)} = \frac{n\delta[1/2 - 1/(q_0'\alpha)]}{\delta/\alpha - \delta + n/2 - n/q_0'},$$

$$\frac{n}{2}(\delta - \beta_0 - \frac{2\delta}{q_0'\alpha} + \frac{2\beta_0}{q_0'}) > \beta_0\delta(\frac{1}{\alpha} - 1).$$

We can find $2 < l < p$ such that

$$\frac{n}{l}(\delta - \beta_0 - \frac{2\delta}{q_0'\alpha} + \frac{2\beta_0}{q_0'}) > \beta_0\delta(\frac{1}{\alpha} - 1).$$

So

$$\frac{(2n)/(lq_0') + \beta_0}{n/l + \beta_0} < \frac{\delta - \beta_0}{\delta/\alpha - \beta_0}.$$

If T is a strongly singular Calderón-Zygmund operator, then from (3) of Definition 1.1, T is bounded from L^2 into $L^{q_0'}$, by interpolating between $(L^2, L^{q_0'})$ and (L^∞, BMO), we can get that T is bounded from L^l to L^s, where $\frac{1}{s} = \frac{2}{lq_0'}$, $0 < l/s \le \alpha$. So

$$\frac{n/s + \beta_0}{n/l + \beta_0} < \frac{\delta - \beta_0}{\delta/\alpha - \beta_0}.$$

Denote $\Theta^b_{\alpha_0} = H^b_{\alpha_0} + W^b_{\alpha_0}$, where

$$H^b_{\alpha_0} = \sum_{j=1}^m T_{j,1}M_bI_{\alpha_0}T_{j,2},$$

$$W^b_{\alpha_0} = \sum_{j=1}^m T_{j,3}I_{\alpha_0}M_bT_{j,4}.$$

Since $\Theta^1_{\alpha_0}(f) = 0$, let $b_B = \frac{1}{|B|}\int_B b(y)dy$, then $\Theta^b_{\alpha_0}(f) = \Theta^{b-b_B}_{\alpha_0}(f) = H^{b-b_B}_{\alpha_0}(f) + W^{b-b_B}_{\alpha_0}(f)$. We consider $W^{b-b_B}_{\alpha_0}(f)$ first.

$$\begin{aligned}
&W^{b-b_B}_{\alpha_0}(f)\\
&= \sum_{j=1}^m T_{j,3}I_{\alpha_0}M_{(b-b_B)\chi_{2B}}T_{j,4}(f) + \sum_{j=1}^m T_{j,3}I_{\alpha_0}M_{(b-b_B)\chi_{(2B)^c}}T_{j,4}(f)\\
&:= f_{11} + f_{12}.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{1}{|B|^{1+\beta_0/n}}\int_B |W^{b-b_B}_{\alpha_0}(f)(y) - f_{12}(x_0)|dy\\
&\le \frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{11}(y)|dy + \frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{12}(y) - f_{12}(x_0)|dy\\
&:= I_1 + I_2.
\end{aligned}$$

Denote $\frac{1}{t} = \frac{1}{l} - \frac{\alpha_0}{n}$, then by Lemma 2.4 we can get that I_{α_0} is bounded from L^l to L^t. Let $\|I_{\alpha_0}\|_{(l,t)}$ be the operator norm. We have

$$\begin{aligned}
I_1 &\le \sum_{j=1}^m \frac{1}{|B|^{1+\beta_0/n}}\int_B |T_{j,3}I_{\alpha_0}M_{(b-b_B)\chi_{2B}}T_{j,4}(f)(y)|dy\\
&\le \sum_{j=1}^m |B|^{-\beta_0/n}\left(\frac{1}{|B|}\int_B |I_{\alpha_0}M_{(b-b_B)\chi_{2B}}T_{j,4}(f)(y)|^t dy\right)^{1/t}\\
&\le \sum_{j=1}^m |B|^{-\beta_0/n}|B|^{-1/t}\|I_{\alpha_0}\|_{(l,t)}\left(\int_{2B} |b(y)-b_B|^l|T_{j,4}(f)(y)|^l dy\right)^{1/l}\\
&\le C\sum_{j=1}^m |B|^{-\beta_0/n}|B|^{-1/t}\|I_{\alpha_0}\|_{(l,t)}\|b\|_{\dot{\Lambda}_{\beta_0}} r^{\beta_0}\left(\int_{2B} |T_{j,4}(f)(y)|^l dy\right)^{1/l}\\
&\le C\sum_{j=1}^m \|I_{\alpha_0}\|_{(l,t)}\|b\|_{\dot{\Lambda}_{\beta_0}}|B|^{-\frac{1}{t}+\frac{1}{l}-\frac{\alpha_0}{n}}\\
&\qquad \times\left(\frac{1}{|2B|^{1-\alpha_0 l/n}}\int_{2B} |T_{j,4}(f)(y)|^l dy\right)^{1/l}\\
&\le C\sum_{j=1}^m \|I_{\alpha_0}\|_{(l,t)}\|b\|_{\dot{\Lambda}_{\beta_0}}[T_{j,4}(f)]^*_{\alpha_0,l}(x_0),
\end{aligned}$$

$$
\begin{aligned}
I_2 &\le \sum_{j=1}^{m} \frac{1}{|B|^{1+\beta_0/n}} \\
&\quad \times \int_B \int_{(2B)^c} |b(z)-b_B||T_{j,4}(f)(z)| \left| \frac{1}{|y-z|^{n-\alpha_0}} - \frac{1}{|x_0-z|^{n-\alpha_0}} \right| dzdy \\
&\le C \sum_{j=1}^{m} |B|^{-\beta_0/n} \int_{(2B)^c} |b(z)-b_B||T_{j,4}(f)(z)| \frac{r}{|x_0-z|^{n-\alpha_0+1}} dz \\
&= C \sum_{j=1}^{m} |B|^{-\beta_0/n} \sum_{k=1}^{\infty} \int_{2^{k+1}B\setminus 2^k B} |b(z)-b_B||T_{j,4}(f)(z)| \frac{r}{|x_0-z|^{n-\alpha_0+1}} dz \\
&\le C \sum_{j=1}^{m} |B|^{-\beta_0/n} \|b\|_{\dot{\Lambda}_{\beta_0}} \sum_{k=1}^{\infty} \frac{(2^{k+1}r)^{\beta_0} r}{(2^k r)^{n-\alpha_0+1}} \int_{2^{k+1}B} |T_{j,4}(f)(z)| dz \\
&\le C \sum_{j=1}^{m} \|b\|_{\dot{\Lambda}_{\beta_0}} \sum_{k=1}^{\infty} \frac{(2^{k+1})^{\beta_0} r}{(2^k r)^{-\alpha_0+1}} \left(\frac{1}{|2^{k+1}B|} \int_{2^{k+1}B} |T_{j,4}(f)(z)|^l dz \right)^{1/l} \\
&\le C \sum_{j=1}^{m} \|b\|_{\dot{\Lambda}_{\beta_0}} \sum_{k=1}^{\infty} 2^{-(1-\beta_0)k} \left(\frac{1}{|2^{k+1}B|^{1-\alpha_0 l/n}} \int_{2^{k+1}B} |T_{j,4}(f)(z)|^l dz \right)^{1/l} \\
&\le C \sum_{j=1}^{m} \|b\|_{\dot{\Lambda}_{\beta_0}} [T_{j,4}(f)]^*_{\alpha_0,l}(x_0).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\frac{1}{|B|^{1+\beta_0/n}} \int_B |W_{\alpha_0}^{b-b_B}(f)(y) - f_{12}(x_0)| dy \\
&\le C \sum_{j=1}^{m} \|I_{\alpha_0}\|_{(l,t)} \|b\|_{\dot{\Lambda}_{\beta_0}} [T_{j,4}(f)]^*_{\alpha_0,l}(x_0) + C \sum_{j=1}^{m} \|b\|_{\dot{\Lambda}_{\beta_0}} [T_{j,4}(f)]^*_{\alpha_0,l}(x_0).
\end{aligned} \tag{3.6}
$$

Then we consider $H_{\alpha_0}^{b-b_B}(f)$.

Case 1. $0 < r \le 1$. Let $\widetilde{B}$ be the ball centered at x_0 whose radius is r^σ, $\sigma = \frac{\delta-\beta_0}{\delta/\alpha-\beta_0} < \alpha$.

$$
\begin{aligned}
&H_{\alpha_0}^{b-b_B}(f) \\
&\quad = \sum_{j=1}^{m} T_{j,1} M_{(b-b_B)\chi_{2\widetilde{B}}} I_{\alpha_0} T_{j,2}(f) + \sum_{j=1}^{m} T_{j,1} M_{(b-b_B)\chi_{(2\widetilde{B})^c}} I_{\alpha_0} T_{j,2}(f) \\
&\quad := f_{21} + f_{22}.
\end{aligned}
$$

So

$$\begin{aligned}&\frac{1}{|B|^{1+\beta_0/n}}\int_B |H_{\alpha_0}^{b-b_B}(f)(y)-f_{22}(x_0)|dy\\&\le \frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{21}(y)|dy+\frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{22}(y)-f_{22}(x_0)|dy\\&:= II_1+II_2.\end{aligned}$$

If $T_{j,1}$ is a strongly singular Calderón-Zygmund operator, then T is bounded from L^l into L^s. Denote $\|T_{j,1}\|_{(l,s)}$ the operator norm. Noticing that $\sigma > \frac{n/s+\beta_0}{n/l+\beta_0}$, we have

$$\begin{aligned}&\frac{1}{|B|^{1+\beta_0/n}}\int_B |T_{j,1}M_{(b-b_B)\chi_{2\widetilde{B}}}I_{\alpha_0}T_{j,2}(f)(y)|dy\\&\le |B|^{-\beta_0/n}\left(\frac{1}{|B|}\int_B |T_{j,1}M_{(b-b_B)\chi_{2\widetilde{B}}}I_{\alpha_0}T_{j,2}(f)(y)|^s dy\right)^{1/s}\\&\le \|T_{j,1}\|_{(l,s)}|B|^{-\beta_0/n}|B|^{-1/s}\left(\int_{2\widetilde{B}}|b(y)-b_B|^l|I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\&\le C\|T_{j,1}\|_{(l,s)}\|b\|_{\dot{\Lambda}_{\beta_0}}r^{-\beta_0-\frac{n}{s}}r^{\sigma(\frac{n}{l}+\beta_0)}\left(\frac{1}{|2\widetilde{B}|}\int_{2\widetilde{B}}|I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\&\le C\|T_{j,1}\|_{(l,s)}\|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0).\end{aligned}$$

If $T_{j,1}=\pm I$, then

$$\begin{aligned}&\frac{1}{|B|^{1+\beta_0/n}}\int_B |T_{j,1}M_{(b-b_B)\chi_{2\widetilde{B}}}I_{\alpha_0}T_{j,2}(f)(y)|dy\\&=\frac{1}{|B|^{1+\beta_0/n}}\int_B |b(y)-b_B||I_{\alpha_0}T_{j,2}(f)(y)|dy\\&\le C\|b\|_{\dot{\Lambda}_{\beta_0}}|B|^{-\beta_0/n}r^{\beta_0}\left(\frac{1}{|B|}\int_B |I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\&\le C\|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0).\end{aligned}$$

So

$$\begin{aligned}II_1&\le \sum_{j=1}^m \frac{1}{|B|^{1+\beta_0/n}}\int_B |T_{j,1}M_{(b-b_B)\chi_{2\widetilde{B}}}I_{\alpha_0}T_{j,2}(f)(y)|dy\\&\le C\sum_{j=1}^m \|T_{j,1}\|\|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0),\end{aligned}\tag{3.7}$$

where $\|T_{j,1}\|$ is a certain operator norm of $T_{j,1}$.

Since $\sigma<\alpha$ and $0<r\le 1$, $2|y-x_0|^\alpha\le |x_0-z|$ for $y\in B$ and $z\in(2\widetilde{B})^c$, then by (2) of Definition 1.1,

$$\begin{aligned}
II_2 &\le \sum_{j=1}^{m} \frac{1}{|B|^{1+\beta_0/n}} \\
&\quad \times \int_B \int_{(2\widetilde{B})^c} |K(y,z)-K(x_0,z)||b(z)-b_B||I_{\alpha_0}T_{j,2}(f)(z)|dzdy \\
&\le C\sum_{j=1}^{m} \frac{\|b\|_{\dot\Lambda_{\beta_0}}}{|B|^{1+\beta_0/n}} \int_B \int_{(2\widetilde{B})^c} \frac{|y-x_0|^{\delta}}{|x_0-z|^{n+\delta/\alpha}}|x_0-z|^{\beta_0}|I_{\alpha_0}T_{j,2}(f)(z)|dzdy \\
&\le C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} r^{-\beta_0+\delta} \sum_{k=1}^{\infty} \int_{2^{k+1}\widetilde{B}\backslash 2^k\widetilde{B}} \frac{|x_0-z|^{\beta_0}}{|x_0-z|^{n+\delta/\alpha}}|I_{\alpha_0}T_{j,2}(f)(z)|dz \\
&\le C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} r^{-\beta_0+\delta} \sum_{k=1}^{\infty} \frac{(2^{k+1}r^{\sigma})^{\beta_0}}{(2^k r^{\sigma})^{\delta/\alpha}} \frac{1}{|2^{k+1}\widetilde{B}|}\int_{2^{k+1}\widetilde{B}} |I_{\alpha_0}T_{j,2}(f)(z)|dz \\
&\le C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} r^{\sigma(\beta_0-\delta/\alpha)+\delta-\beta_0} \sum_{k=1}^{\infty} 2^{-k(\delta/\alpha-\beta_0)}[I_{\alpha_0}T_{j,2}(f)]^{*}_{0,1}(x_0) \\
&= C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} [I_{\alpha_0}T_{j,2}(f)]^{*}_{0,1}(x_0).
\end{aligned} \tag{3.8}$$

The above estimate is obviously right when $T_{j,1} = \pm I$.
Thus by (3.6), (3.7) and (3.8),

$$\begin{aligned}
&\sup_{B\ni x_0,\, 0<r(B)\le 1} \inf_{c\in\mathbb{C}} \frac{1}{|B|^{1+\beta_0/n}} \int_B |\Theta^b_{\alpha_0}(f)(y)-c|dy \\
&\le \sup_{B\ni x_0,\, 0<r(B)\le 1} \frac{1}{|B|^{1+\beta_0/n}} \int_B |\Theta^b_{\alpha_0}(f)(y)-[f_{12}(x_0)+f_{22}(x_0)]|dy \\
&\le \sup_{B\ni x_0,\, 0<r(B)\le 1} \left\{ \frac{1}{|B|^{1+\beta_0/n}} \int_B |H^{b-b_B}_{\alpha_0}(f)(y)-f_{22}(x_0)|dy \right. \\
&\quad \left. + \frac{1}{|B|^{1+\beta_0/n}} \int_B |W^{b-b_B}_{\alpha_0}(f)(y)-f_{12}(x_0)|dy \right\} \\
&\le C\sum_{j=1}^{m} \|T_{j,1}\|\|b\|_{\dot\Lambda_{\beta_0}} [I_{\alpha_0}T_{j,2}(f)]^{*}_{0,l}(x_0) + C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} [I_{\alpha_0}T_{j,2}(f)]^{*}_{0,1}(x_0) \\
&\quad + C\sum_{j=1}^{m} \|I_{\alpha_0}\|_{(l,t)} \|b\|_{\dot\Lambda_{\beta_0}} [T_{j,4}(f)]^{*}_{\alpha_0,l}(x_0) + C\sum_{j=1}^{m} \|b\|_{\dot\Lambda_{\beta_0}} [T_{j,4}(f)]^{*}_{\alpha_0,l}(x_0).
\end{aligned} \tag{3.9}$$

Case 2. $r > 1$.

$$\begin{aligned}
&H^{b-b_B}_{\alpha_0}(f) \\
&= \sum_{j=1}^{m} T_{j,1} M_{(b-b_B)\chi_{2B}} I_{\alpha_0} T_{j,2}(f) + \sum_{j=1}^{m} T_{j,1} M_{(b-b_B)\chi_{(2B)^c}} I_{\alpha_0} T_{j,2}(f) \\
&:= f_{31} + f_{32}.
\end{aligned}$$

So

$$\begin{aligned}
&\frac{1}{|B|^{1+\beta_0/n}}\int_B |H_{\alpha_0}^{b-b_B}(f)(y)-f_{32}(x_0)|dy\\
&\le \frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{31}(y)|dy+\frac{1}{|B|^{1+\beta_0/n}}\int_B |f_{32}(y)-f_{32}(x_0)|dy\\
&:= II_3+II_4.
\end{aligned}$$

Let us estimate II_3. $T_{j,1}$ is bounded on L^l. Denote $\|T_{j,1}\|_{(l,l)}$ the operator norm.

$$\begin{aligned}
II_3 &\le \sum_{j=1}^m \frac{1}{|B|^{1+\beta_0/n}}\int_B |T_{j,1}M_{(b-b_B)\chi_{2B}}I_{\alpha_0}T_{j,2}(f)(y)|dy\\
&\le \sum_{j=1}^m |B|^{-\beta_0/n}\left(\frac{1}{|B|}\int_B |T_{j,1}M_{(b-b_B)\chi_{2B}}I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\
&\le \sum_{j=1}^m \|T_{j,1}\|_{(l,l)}|B|^{-\beta_0/n}\left(\frac{1}{|B|}\int_{2B} |b(y)-b_B|^l|I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\
&\le C\sum_{j=1}^m \|T_{j,1}\|_{(l,l)}\|b\|_{\dot\Lambda_{\beta_0}}|B|^{-\beta_0/n}r^{\beta_0}\left(\frac{1}{|2B|}\int_{2B} |I_{\alpha_0}T_{j,2}(f)(y)|^l dy\right)^{1/l}\\
&\le C\sum_{j=1}^m \|T_{j,1}\|_{(l,l)}\|b\|_{\dot\Lambda_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0).
\end{aligned} \tag{3.10}$$

Noticing that $r>1$ and $0<\alpha<1$, by (2) of Definition 1.1,

$$\begin{aligned}
II_4 &\le \sum_{j=1}^m \frac{1}{|B|^{1+\beta_0/n}}\\
&\quad\times\int_B\int_{(2B)^c} |K(y,z)-K(x_0,z)||b(z)-b_B||I_{\alpha_0}T_{j,2}(f)(z)|dzdy\\
&\le C\sum_{j=1}^m \frac{\|b\|_{\dot\Lambda_{\beta_0}}}{|B|^{1+\beta_0/n}}\int_B\int_{(2B)^c}\frac{|y-x_0|^\delta}{|x_0-z|^{n+\delta/\alpha}}|x_0-z|^{\beta_0}|I_{\alpha_0}T_{j,2}(f)(z)|dzdy\\
&\le C\sum_{j=1}^m \|b\|_{\dot\Lambda_{\beta_0}}r^{-\beta_0+\delta}\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\frac{|x_0-z|^{\beta_0}}{|x_0-z|^{n+\delta/\alpha}}|I_{\alpha_0}T_{j,2}(f)(z)|dz\\
&\le C\sum_{j=1}^m \|b\|_{\dot\Lambda_{\beta_0}}r^{-\beta_0+\delta}\sum_{k=1}^\infty\frac{(2^{k+1}r)^{\beta_0}}{(2^kr)^{\delta/\alpha}}\frac{1}{|2^{k+1}B|}\int_{2^{k+1}B}|I_{\alpha_0}T_{j,2}(f)(z)|dz\\
&\le C\sum_{j=1}^m \|b\|_{\dot\Lambda_{\beta_0}}r^{\delta-\delta/\alpha}\sum_{k=1}^\infty 2^{-k(\delta/\alpha-\beta_0)}[I_{\alpha_0}T_{j,2}(f)]^*_{0,1}(x_0)\\
&\le C\sum_{j=1}^m \|b\|_{\dot\Lambda_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,1}(x_0).
\end{aligned} \tag{3.11}$$

The above estimate is obviously right when $T_{j,1}=\pm I$.

Thus by (3.6), (3.10) and (3.11),

$$
\begin{aligned}
&\sup_{B\ni x_0,\, r(B)>1} \inf_{c\in\mathbb{C}} \frac{1}{|B|^{1+\beta_0/n}} \int_B |\Theta^b_{\alpha_0}(f)(y)-c|dy \\
&\leq \sup_{B\ni x_0,\, r(B)>1} \frac{1}{|B|^{1+\beta_0/n}} \int_B |\Theta^b_{\alpha_0}(f)(y)-[f_{12}(x_0)+f_{32}(x_0)]|dy \\
&\leq \sup_{B\ni x_0,\, r(B)>1} \left\{ \frac{1}{|B|^{1+\beta_0/n}} \int_B |H^{b-b_B}_{\alpha_0}(f)(y)-f_{32}(x_0)|dy \right.\\
&\quad \left. + \frac{1}{|B|^{1+\beta_0/n}} \int_B |W^{b-b_B}_{\alpha_0}(f)(y)-f_{12}(x_0)|dy \right\} \\
&\leq C\sum_{j=1}^m \|T_{j,1}\|_{(l,l)}\|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0) \\
&\quad +C\sum_{j=1}^m \|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,1}(x_0) \\
&\quad +C\sum_{j=1}^m \|I_{\alpha_0}\|_{(l,t)}\|b\|_{\dot{\Lambda}_{\beta_0}}[T_{j,4}(f)]^*_{\alpha_0,l}(x_0) \\
&\quad +C\sum_{j=1}^m \|b\|_{\dot{\Lambda}_{\beta_0}}[T_{j,4}(f)]^*_{\alpha_0,l}(x_0).
\end{aligned} \tag{3.12}
$$

In conclusion, from (3.5), (3.9) and (3.12), we can get that

$$
\begin{aligned}
&\sup_{B\ni x_0} \inf_{c\in\mathbb{C}} \frac{1}{|B|^{1+\beta_0/n}} \int_B |\Theta^b_{\alpha_0}(f)(y)-c|dy \\
&\leq C\sum_{j=1}^m \|T_{j,1}\|\|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,l}(x_0) + C\sum_{j=1}^m \|b\|_{\dot{\Lambda}_{\beta_0}}[I_{\alpha_0}T_{j,2}(f)]^*_{0,1}(x_0) \\
&\quad +C\sum_{j=1}^m \|I_{\alpha_0}\|_{(l,t)}\|b\|_{\dot{\Lambda}_{\beta_0}}[T_{j,4}(f)]^*_{\alpha_0,l}(x_0) + C\sum_{j=1}^m \|b\|_{\dot{\Lambda}_{\beta_0}}[T_{j,4}(f)]^*_{\alpha_0,l}(x_0),
\end{aligned}
$$

where $\|T_{j,1}\|$ is a certain operator norm of $T_{j,1}$.

By using Lemma 2.1, Lemma 2.4 and Lemma 2.5, we obtain

$$
\begin{aligned}
&\|\Theta^b_{\alpha_0}(f)\|_{\dot{F}^{\beta_0,\infty}_q} \\
&\leq C\|I_{\alpha_0}\|\left[\left(\sum_{j=1}^m \|T_{j,1}\|\right)\left(\sum_{j=1}^m \|T_{j,2}\|_{(p,p)}\right) + \sum_{j=1}^m \|T_{j,4}\|_{(p,p)}\right]\|b\|_{\dot{\Lambda}_{\beta_0}}\|f\|_p,
\end{aligned}
$$

where $\|I_{\alpha_0}\|$ is a certain operator norm of I_{α_0}, $\|T_{j,2}\|_{(p,p)}$ and $\|T_{j,4}\|_{(p,p)}$ are the operator norms on L^p. This completes the proof of the theorem. □

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