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NUMERICAL METHODS FOR QUASILINEAR PARABOLIC DIFFERENTIAL FUNCTIONAL EQUATIONS WITH NEUMANN INITIAL BOUNDARY CONDITIONS

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ABSTRACT: The aim of this paper is to present a numerical approximation for quasilinear parabolic differential functional equations with initial boundary conditions of the Neumann type and coefficients satisfying Perron estimates. The convergence result is proved for a difference scheme with the property that the difference operators approximating mixed derivatives depend on local properties of the coefficients of the differential equation. A numerical example is given.

AMS (MOS) Subject Classification: 65M12, 35R10

1. INTRODUCTION

We will denote by $C(U,V)$ the class of all continuous functions $w : U \to V$ with U and V being any metric spaces. Let $M_{n\times n}$ be the set of $n \times n$ matrices with real elements. For $x = (x_1, \dots, x_n) \in R^n$ and $X \in M_{n\times n}, X = [X_{kj}]_{j,k=1,\dots,n}$ we put

$$\|x\| = |x_1| + \cdots + |x_n|, \quad \|X\| = \max\Big\{ \sum_{j=1}^{n} |x_{kj}| \ : \ 1 \le k \le n \Big\}.$$

Writing a vectorial inequality we mean that the same inequality holds for the corresponding components. Let $a > 0$, $R_+ = [0, +\infty)$, and $b = (b_1, \dots, b_n) \in R^n$ be given, where $b_k > 0$ for $1 \le k \le n$. Define the sets

$$E = [0, a] \times (-b, b),$$

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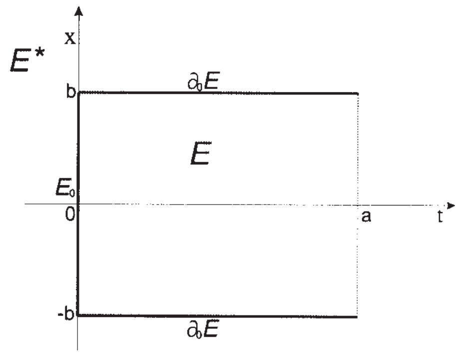


Figure 1. Sets E_h, $E_{0 \cdot h}$, $\partial_0 E_h$ and E_h^*.

and

$$E_0 = \{0\} \times [-b, b], \quad \partial_0 E = ([0, a] \times [-b, b]) \setminus E, \quad E^* = E_0 \cup E \cup \partial_0 E.$$

Assume that

$$\varrho = [\, \varrho_{kj} \,]_{j,\, k=1,\dots,n} \, : \, E \times C(E^*, R) \to M_{n \times n},$$

$$f \, : \, E \times C(E^*, R) \times R^n \to R$$

are given functions of the variables (t, x, w) and (t, x, w, p) respectively.

We consider the quasilinear differential functional equation with Neumann initial boundary conditions

$$\begin{cases} \partial_t z(t,x) = \displaystyle\sum_{k,j=1}^{n} \varrho_{kj}(t,x,z) \partial_{x_k x_j} z(t,x) + f(t,x,z,\partial_x z(t,x)) \, , \\ z(t,x) = \varphi_0(t,x) \quad \text{ for } \ (t,x) \in E_0 \, , \\ \partial_{x_j} z(t,x) = \varphi_j(t,x) \quad \text{ for } \ (t,x) \in \partial_0 E \ \text{ and } \ x_j = b_j \ \text{ or } \ x_j = -b_j \, , \end{cases} \tag{1}$$

where $\varphi_0 \, : \, E_0 \to R$ and $\varphi_j \, : \, \partial_0 E \to R$, $1 \le j \le n$ are given and $\partial_x z = (\partial_{x_1} z, \dots, \partial_{x_n} z)$.

The difference methods for nonlinear parabolic equations with Neumann boundary conditions were initiated in the papers by Malec [5], Malec [6], Malec [7] and Węglowski [9]. In Kamont and Kwapisz [2], authors introduce some general difference operators and investigate their stability. The results of Malec [5] Malec [6], Malec [7] and Węglowski [9] do not apply to quasilinear equations. The difference scheme applied in this paper has the property that the difference operators approximating mixed derivatives depend on local properties of the function ρ. We give sufficient conditions for convergence of difference method for problem (1). The convergence is proved by consistency and stability arguments. We are interested in the numerical approximation of a classical solution of the above problem.

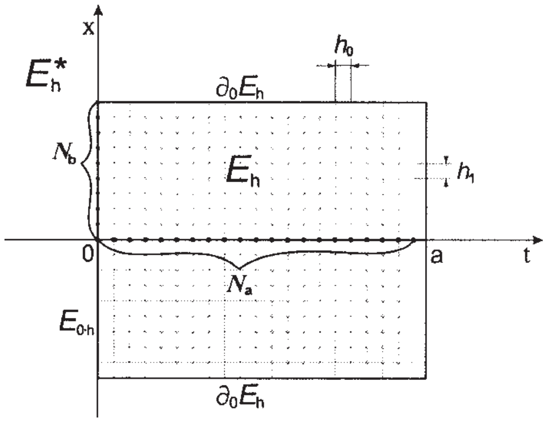


Figure 2. Sets E_h, $E_{0\cdot h}$, $\partial_0 E_h$ and E_h^*.

The norm of any $z \in C(E^*, R)$ is defined by

$$\|z\|_{E^*} = \max\{\,|z(t,x)| \;:\; (t,x) \in E^*\,\}.$$

We will need the norm

$$\|z\|_t = \max\{\,|z(\theta,x)| \;:\; 0 \le \theta \le t \text{ and } (\theta,x) \in E^*\,\}.$$

For $t \in [0,a]$ we write $H_t = [\,0,t\,] \times [-b,b]$. We assume that problem (1) is of Volterra type, that is, if $t \in [0,a]$ and $z,\, \bar{z} \in C(E^*, R)$ and $z(\theta) = \bar{z}(\theta)$ for $\theta \in H_t$ then $f(t,x,z,q) = f(t,x,\bar{z},q)$ and $\varrho_{kj}(t,x,z) = \varrho_{kj}(t,x,\bar{z})$ for $x \in [-b,b]$, $q \in R^n$ and $j,\, k = 1, \dots, n$.

2. DIFFERENCE FUNCTIONAL EQUATIONS

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of natural numbers and integers respectively. For $x, \bar{x} \in R^n$, $x = (x_1, \dots, x_n)$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n)$, we put $x*\bar{x} = (x_1\bar{x}_1, \dots, x_n\bar{x}_n)$. We define a mesh on the set E^* in the following way. Suppose that $h = (h_0, h') \in R_+^{1+n}$, where $h' = (h_1, \dots, h_n)$ stands for steps of the mesh. Denote by Δ the set of all $h = (h_0, h')$ such that there exists $N_b = (N_{b1}, \dots, N_{bn}) \in \mathbb{Z}^n$ with the properties that $N_b * h' = b$. For $h \in \Delta$ we write $\|h\| = h_0 + h_1 + \dots + h_n$ and $\|h'\| = h_1 + \dots + h_n$. It is required that $\Delta \ne \emptyset$ and that there exist a sequence $\{h^{(j)}\}$, $h^{(j)} \in \Delta$ such that $\lim_{j\to\infty} \|h^{(j)}\| = 0$.

Nodal points are defined by:

$$t^{(i)} = ih_0, \qquad x^{(m)} = m*h = (\,m_1h_1, \dots, m_nh_n) = \big(\,x_1^{(m_1)}, \dots, x_n^{(m_n)}\,\big),$$

where $(i,m) \in \mathbb{Z}^{1+n}$. Obviously there exists $N_a \in \mathbb{N}$ such that $N_a h_0 \le a < (N_a + 1)h_0$. Let

$$R_h^{1+n} = \big\{(t^{(i)}, x^{(m)}) \;:\; (i,m) \in \mathbb{Z}^{1+n}\,\big\}$$

and

$$E_h = \overline{E} \cap R_h^{1+n},$$

$$\partial_0 E_h = \partial_0 E \cap R_h^{1+n}, \qquad E_{0\cdot h} = E_0 \cap R_h^{1+n}, \qquad E_h^* = E_h \cup E_{0\cdot h} \cup \partial_0 E_h.$$

For $z \,:\, E_h^* \to R$ we write

$$z^{(i,m)} = z\,(t^{(i)}, x^{(m)}).$$

The norm of any $z \,:\, E_h^* \to R$ is defined by

$$\|z\|_h = \max\big\{\, |z^{(i,m)}| \;:\; (t^{(i)}, x^{(m)}) \in E_h^* \,\big\}.$$

For any $t^{(i)}$ we will need the norm

$$\|z\|_{h\cdot i} = \max\big\{\, |z^{(r,m)}| \;:\; 0 \le r \le i \text{ and } (t^{(r)}, x^{(m)}) \in E_h^* \,\big\}.$$

Let

$$E'_h = \big\{ (t^{(i)}, x^{(m)}) \in E_h \;:\; 0 \le i \le N_a - 1 \big\}$$

and denote by $\mathfrak{F}(E_h^*, R)$ the set of all functions $w \,:\, E_h^* \to R$. Suppose that

$$\varrho_h = [\varrho_{h\cdot kj}]_{j,\,k=1,\dots,n} \,:\, E'_h \times \mathfrak{F}(E_h^*, R) \to M_{n\times n},$$

$$f_h \,:\, E'_h \times \mathfrak{F}(E_h^*, R) \times R^n \to R,$$

$$\varphi_{0\cdot h} \,:\, E_{0\cdot h} \to R, \qquad \varphi_{j\cdot h} \,:\, \partial_0 E_h \to R \quad j = 1, \dots, n$$

are given functions.

We will approximate solutions of problem (1) by means of solutions of a difference equation with initial boundary condition of Neumann type. To do that, for every $(t^{(i)}, x^{(m)}) \in \partial_0 E_h$ we define:

$$\begin{aligned} \mathcal{A}^{(m)} = \big\{\, \alpha = (\alpha_1, \dots, \alpha_n) \,:\, & \alpha_j \in \{0,1\}, \quad \text{if } x_j^{(m_j)} = b_j, \\ & \alpha_j \in \{0,-1\}, \quad \text{if } x_j^{(m_j)} = -b_j, \\ & \alpha_j = 0, \quad \text{if } -b_j < x_j^{(m_j)} < b_j \\ & \text{and } \ \|\alpha\| = 1 \ \text{ or } \ \|\alpha\| = 2, \ \ 1 \le j \le n \,\big\}, \end{aligned}$$

where $\|\alpha\| = |\alpha_1| + \dots + |\alpha_n|$ and

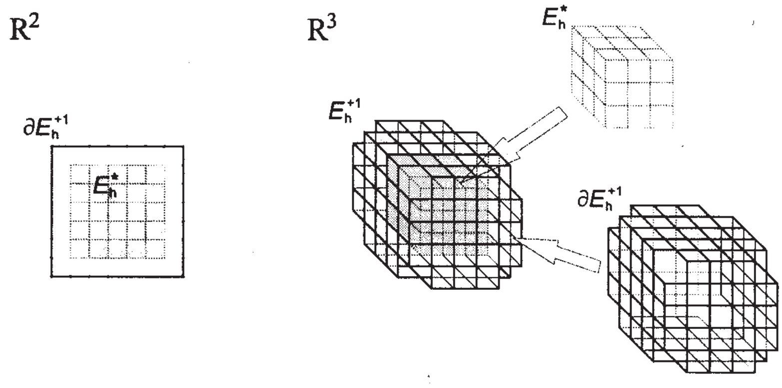
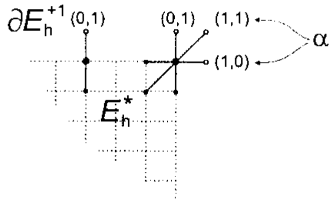


Figure 3. A cross section of E_h^{+1} for a given t^i in $R^n = R^2$ and R^3.

$$\partial E_h^{+1} = \big\{ (t^{(i)}, x^{(m+\alpha)}) : \quad 0 \le i \le N_a\,, \ \ (t^{(i)}, x^{(m)}) \in \partial_0 E_h \ \text{ and } \ \alpha \in \mathcal{A}^{(m)} \big\},$$

$$E_h^{+1} = \partial E_h^{+1} \cup E_h^{*}.$$



Now we consider the difference problem:

$$\delta_0\, z^{(i,m)} = \sum_{k,\,j=1}^{n} \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, z\,)\, \delta_{kj}^{(2)} z^{(i,m)} + f_h(\, t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}\,), \quad (2)$$

with Neumann boundary conditions

$$z^{(i,m)} = \varphi_{0\cdot h}^{(i,m)} \qquad \text{on } \ E_{0\cdot h}, \quad (3)$$

$$z_h^{(i,m+\alpha)} - z_h^{(i,m-\alpha)} = 2 \sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \qquad \text{on } \ \partial_0 E_h \ \text{ and } \ \alpha \in \mathcal{A}^{(m)}. \quad (4)$$

Let us notice that $(t^{(i)}, x^{(m+\alpha)}) \in \partial E_h^{+1}$ and $(t^{(i)}, x^{(m-\alpha)}) \in E_h$.

Let $e_j = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$ be the standard unit vector. The difference operators δ_0, $\delta = (\,\delta_1, \ldots, \delta_n\,)$, δ_α and $\delta^{(2)} = [\,\delta_{kj}\,]_{j,\,k=1,\ldots,n}$ are defined in the following way

$$\delta_0 z^{(i,m)} = \frac{1}{h_0}[\, z^{(i+1,m)} - z^{(i,m)}\,] \tag{5}$$

and

$$\delta_j z^{(i,m)} = \frac{1}{2h_j}[\, z^{(i,m+e_j)} - z^{(i,m-e_j)}\,], \qquad 1 \le j \le n, \tag{6}$$

$$\delta^{(2)}_{kk} z^{(i,m)} = \delta^+_k \delta^-_k z^{(i,m)}, \tag{7}$$

$$\delta^{(2)}_{kj} z^{(i,m)} = \frac{1}{2}\big[\, \delta^+_k \delta^+_j z^{(i,m)} + \delta^-_k \delta^-_j z^{(i,m)}\,\big] \quad \text{if} \quad \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, z) \ge 0, \tag{8}$$

$$\delta^{(2)}_{kj} z^{(i,m)} = \frac{1}{2}\big[\, \delta^+_k \delta^-_j z^{(i,m)} + \delta^-_k \delta^+_j z^{(i,m)}\,\big] \quad \text{if} \quad \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, z) < 0, \tag{9}$$

where

$$\delta^+_k z^{(i,m)} = \frac{1}{h_k}[\, z^{(i,m+e_k)} - z^{(i,m)}\,], \qquad \delta^-_k z^{(i,m)} = \frac{1}{h_k}[\, z^{(i,m)} - z^{(i,m-e_k)}\,].$$

There exists exactly one solution $u_h : E^* \to R$ of problem (2)-(4). Let the operator F_h be defined by

$$F_h[z]^{(i,m)} = \sum_{k,\,j=1}^{n} \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, z\,)\, \delta_{kj} z^{(i,m)} + f_h(\, t^{(i)}, x^{(m)}, z, \delta z^{(i,m)}\,). \tag{10}$$

Our purose is to examine the relation between the solution u_h of (2)-(4) and a function $v_h : E_h^{+1} \to R$ satisfying the condition

$$\big|\, \delta_0 v_h^{(i,m)} - F_h[v_h]^{(i,m)} \big| \le \gamma(h) \qquad \text{on } E'_h \tag{11}$$

and

$$\big|\, v_h^{(i,m)} - \varphi_{0\cdot h}^{(i,m)} \big| \le \gamma_0(h) \qquad \text{on } E_{0\cdot h} \tag{12}$$

$$\big|\, v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \big| \le C_\varphi \|h'\|^3 \qquad \text{on } \partial_0 E_h\,, \tag{13}$$

where

$$\gamma, \gamma_0 : \Delta \to R_+, \quad \lim_{h\to 0} \gamma_0(h) = 0,$$

$$\lim_{h\to 0} \gamma(h) = 0, \quad C_\varphi \in R_+ \quad \text{and} \quad \alpha \in \mathcal{A}^{(m)}.$$

The function v_h satisfying the above relations is considered to be an approximate solution of problem (2)-(4).

Assumption H $[\sigma]$**.** *Suppose that:*

1) $\sigma : [0,a] \times R_+ \to R_+$ *is continuous;*

2) $\sigma(t,0) = 0$ *for* $t \in [0,a]$*;*

3) σ *is nondecreasing with respect to both variables;*

4) for any $\tilde{c} > 1$ *the Cauchy problem*

$$y'(t) = \tilde{c}\,\sigma(t, y(t)), \qquad y(0) = 0$$

has the only solution $y(t) = 0$ *for* $t \in [0,a]$.

Assumption H$[\varrho_h, f_h]$**.** *The functions* $\varrho_h : E'_h \times \mathfrak{F}(E^*_h, R) \to M_{n\times n}$ *and* $f_h : E'_h \times \mathfrak{F}(E^*_h, R) \times R^n \to R$ *satisfy the following conditions:*

1) there is $\sigma : [0,a] \times R_+ \to R_+$ *satisfying Assumption* $H[\sigma]$ *and such that*

$$\| \varrho_h(t^{(i)}, x^{(m)}, w) - \varrho_h(t^{(i)}, x^{(m)}, \bar{w})\| \le \sigma(t^{(i)}, \|w - \bar{w}\|_{h\cdot i}),$$

$$| f_h(t^{(i)}, x^{(m)}, w, p) - f_h(t^{(i)}, x^{(m)}, \bar{w}, p)| \le \sigma(t^{(i)}, \|w - \bar{w}\|_{h\cdot i});$$

2) $\partial_p f_h = (\partial_{p_1} f_h, \ldots, \partial_{p_n} f_h)$ *exists on* $E'_h \times \mathfrak{F}(E^*_h, R) \times R^n$ *and* $\partial_p f_h(t,x,w,\cdot) \in C(R^n, R^n)$.

Theorem 1. *Suppose that Assumption* $H[\varrho_h, f_h]$ *holds and:*

1) $h \in \Delta$ *and*

$$1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2} \varrho_{h\cdot kk}(Q) + h_0 \sum_{\substack{k,j=1\\ j\ne k}}^{n} \frac{1}{h_k h_j} |\varrho_{h\cdot kj}(Q)| \ge 0$$

$$\frac{1}{h_k} \varrho_{h\cdot kk}(Q) - \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{1}{h_j} |\varrho_{h\cdot kj}(Q)| - \frac{1}{2}\left|\partial_{p_k} f_h(P)\right| \ge 0, \quad 1 \le k \le n,$$

where $Q = (t,x,w) \in E'_h \times \mathfrak{F}(E^*_h, R)$ *and* $P = (t,x,w,p) \in E'_h \times \mathfrak{F}(E^*_h, R) \times R^n$*;*

2) $u_h : E_h^{+1} \to R$ *is the solution of problem* (2)-(4)*;*

3) $v_h : E_h^{+1} \to R$ *satisfies relations* (11)-(13)*;*

4) there exists $c_0 \in R_+$ *such that*

$$| \delta^{(2)}_{kj} v_h^{(i,m)} | \le c_0 \qquad \text{on } E_h \quad \text{for } 1 \le k, j \le n;$$

5) there exists $\widetilde{C} \in R_+$ *such that* $\|h'\|^2 \le \widetilde{C} h_0$.

Under these assumptions there is $\eta : \Delta \to R_+$ *such that*

$$\left| u_h^{(i,m)} - v_h^{(i,m)} \right| \le \eta(h) \quad \text{on } E_h \quad \text{and} \quad \lim_{h \to 0} \eta(h) = 0. \tag{14}$$

Proof. Let $\Gamma : E'_h \to R$, $\Gamma_{0 \cdot h} : E_{0 \cdot h} \to R$ and $\Gamma_{\partial \cdot h} : \partial_0 E_h \to R$ be defined by

$$\begin{aligned} \delta_0 v_h^{(i,m)} &= F_h[v_h]^{(i,m)} + \Gamma_h^{(i,m)} \quad \text{on } E'_h, \\ v_h^{(i,m)} &= \varphi_{0 \cdot h}^{(i,m)} + \Gamma_{0 \cdot h}^{(i,m)} \quad \text{on } E_{0 \cdot h}, \\ v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} &= 2 \sum_{j=1}^{n} \alpha_j h_j \varphi_{j \cdot h}^{(i,m)} + \Gamma_{\partial \cdot h}^{(i,m)} \quad \text{on } \partial_0 E_h \quad \text{and} \quad \alpha \in \mathcal{A}^{(m)}. \end{aligned}$$

Then

$$\begin{aligned} & |\, \Gamma_h^{(i,m)} \,| \le \gamma(h) && \text{on } E'_h \text{ with } \lim_{h \to 0} \gamma(h) = 0, \\ & |\, \Gamma_{0 \cdot h}^{(i,m)} \,| \le \gamma_0(h) && \text{on } E_{0 \cdot h} \text{ with } \lim_{h \to 0} \gamma_0(h) = 0, \\ & |\, \Gamma_{\partial \cdot h}^{(i,m)} \,| \le C_\varphi \|h'\|^3 && \text{on } \partial_0 E_h. \end{aligned}$$

The function $\varepsilon_h = u_h - v_h$ satisfies the difference functional equation

$$\begin{aligned} \delta_0 \varepsilon_h^{(i,m)} = & \sum_{k,j=1}^{n} \varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, u_h)\, \delta_{kj}^{(2)} \varepsilon_h^{(i,m)} \\ & + f_h(t^{(i)}, x^{(m)}, v_h, \delta u_h{}^{(i,m)}) - f_h(t^{(i)}, x^{(m)}, v_h, \delta v_h{}^{(i,m)}) + \Lambda_h^{(i,m)}, \end{aligned} \tag{15}$$

where

$$\begin{aligned} \Lambda_h^{(i,m)} = & \sum_{k,j=1}^{n} \Big[\varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, u_h) - \varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, v_h) \Big] \delta_{kj}^{(2)} v_h^{(i,m)} \\ & + f_h(t^{(i)}, x^{(m)}, u_h, \delta u_h{}^{(i,m)}) - f_h(t^{(i)}, x^{(m)}, v_h, \delta u_h{}^{(i,m)}) - \Gamma_h^{(i,m)} \end{aligned} \tag{16}$$

on E'_h and

$$\varepsilon_h^{(i,m+\alpha)} = \varepsilon_h^{(i,m-\alpha)} + \Gamma_{\partial \cdot h}^{(i,m)} \tag{17}$$

on $\partial_0 E'_h$. Let us deal with ε_h on E'_h first. Write

$$\begin{aligned} I_+^{(i,m)} &= \{ (k,j) : 1 \le k, j \le n, \quad k \ne j, \quad \varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, u_h) \ge 0 \}, \\ I_-^{(i,m)} &= \{ (k,j) : 1 \le k, j \le n, \quad k \ne j, \quad \varrho_{h \cdot kj}(t^{(i)}, x^{(m)}, u_h) < 0 \}. \end{aligned}$$

From (5) and the Mean Value Theorem, we can rewrite (15) as

$$\begin{aligned}\varepsilon_h^{(i+1,m)} = \varepsilon_h^{(i,m)} &+ h_0 \sum_{k=1}^{n} \varrho_{h\cdot kk}(\, t^{(i)}, x^{(m)}, u_h\,) \delta_{kk}^{(2)} \varepsilon_h^{(i,m)} \\ &+ h_0 \sum_{(k,\,j)\,\in I_+^{(i,m)}} \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,) \delta_{kj}^{(2)} \varepsilon_h^{(i,m)} \\ &+ h_0 \sum_{(k,\,j)\,\in I_-^{(i,m)}} \varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,) \delta_{kj}^{(2)} \varepsilon_h^{(i,m)} \\ &+ h_0 \sum_{k=1}^{n} \partial_{p_k} f_h(P) \frac{\varepsilon_h^{(i,m+e_k)} - \varepsilon_h^{(i,m-e_k)}}{2h_k} + h_0 \Lambda_h^{(i,m)}.\end{aligned}$$

Finally by (7)-(9) and after regrouping, function ε_h satisfies on E'_h the recursive equation

$$\begin{aligned}\varepsilon_h^{(i+1,m)} &= \varepsilon_h^{(i,m)} \boldsymbol{A}^{(i,m)} + h_0 \sum_{k=1}^{n} \varepsilon_h^{(i,m+e_k)} \boldsymbol{B}_k^{(i,m)} + h_0 \sum_{k=1}^{n} \varepsilon_h^{(i,m-e_k)} \boldsymbol{C}_k^{(i,m)} \\ &+ h_0 \sum_{(k,\,j)\,\in I_+^{(i,m)}} \left[\varepsilon_h^{(i,m+e_k+e_j)} + \varepsilon_h^{(i,m-e_k-e_j)} \right] \boldsymbol{D}_{k,j}^{(i,m)} \\ &+ h_0 \sum_{(k,\,j)\,\in I_-^{(i,m)}} \left[\varepsilon_h^{(i,m+e_k-e_j)} + \varepsilon_h^{(i,m-e_k+e_j)} \right] \boldsymbol{D}_{k,j}^{(i,m)} + h_0 \Lambda_h^{(i,m)},\end{aligned} \tag{18}$$

where

$$\begin{aligned}\boldsymbol{A}^{(i,m)} &= 1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2} \varrho_{h\cdot kk}(\, t^{(i)}, x^{(m)}, u_h\,) \\ &\quad + h_0 \sum_{\substack{k,j=1\\ j\neq k}}^{n} \frac{1}{h_k h_j} |\varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,)|\,, \\ \boldsymbol{B}_k^{(i,m)} &= \frac{1}{h_k^2} \varrho_{h\cdot kk}(\, t^{(i)}, x^{(m)}, u_h\,) - \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{h_k h_j} |\varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,)| \\ &\quad + \frac{1}{2h_k} \partial_{p_k} f_h(P)\,, \\ \boldsymbol{C}_k^{(i,m)} &= \frac{1}{h_k^2} \varrho_{h\cdot kk}(\, t^{(i)}, x^{(m)}, u_h\,) - \sum_{\substack{j=1\\ j\neq k}}^{n} \frac{1}{h_k h_j} |\varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,)| \\ &\quad - \frac{1}{2h_k} \partial_{p_k} f_h(P)\,,\end{aligned}$$

$$\boldsymbol{D}_{k,j}^{(i,m)} = \frac{1}{2h_k h_j} |\varrho_{h\cdot kj}(\, t^{(i)}, x^{(m)}, u_h\,)|.$$

Let ω_h and $\widetilde{\omega}_h$ be given by

$$\omega_h^{(i)} = \max\big\{\, |\,\varepsilon_h^{(r,m)}| \,:\; (t^{(r)}, x^{(m)}) \in E_h^* \cap \big(\,[\,0, t^{(i)}] \times R^n\big)\,\big\},\qquad 0 \le i \le N_a, \tag{19}$$

$$\widetilde{\omega}_h^{(i)} = \max\big\{\, |\,\varepsilon_h^{(r,m)}| \,:\; (t^{(r)}, x^{(m)}) \in E_h^{+1} \cap \big(\,[\,0, t^{(i)}] \times R^n\big)\,\big\},\qquad 0 \le i \le N_a. \tag{20}$$

With this notation Λ_h can be estimated as follows

$$|\Lambda_h^{(i,m)}| \le \sigma(t^{(i)}, \widetilde{\omega}_h^{(i)})\,(1+nc_0) + \gamma(h) \qquad \text{on } E_h'. \tag{21}$$

We conclude from (12), (17), (18) and (21) that the functions ω_h and $\widetilde{\omega}_h$ satisfy the recursive inequalities

$$\omega_h^{(i+1)} \le \widetilde{\omega}_h^{(i)} + \bar{c}\, h_0\, \sigma\big(\, t^{(i)},\, \widetilde{\omega}_h^{(i)}\big) + h_0\gamma(h),$$

$$\widetilde{\omega}_h^{(i)} \le \omega_h^{(i)} + \sqrt{h_0}\, h_0 C_\varphi\, \widetilde{C}$$

for $0 \le i \le N_a - 1$ and with $\bar{c} = (1 + nc_0)$. Thus

$$\omega_h^{(i+1)} \le \omega_h^{(i)} + \bar{c}\, h_0\, \sigma\big(t^{(i)},\, \omega_h^{(i)} + \sqrt{h_0}\, h_0 C_\varphi\, \widetilde{C}\big) + h_0\big[\gamma(h) + \sqrt{h_0}\, C_\varphi\, \widetilde{C}\,\big]$$

for $0 \le i \le N_a - 1$ and

$$\omega_h^{(0)} \le \gamma_0(h).$$

Consider the differential equation

$$\eta'(t) = \bar{c}\, \sigma\big(t, \eta(t) + \sqrt{h_0}\, h_0 C_\varphi\, \widetilde{C}\big) + \big[\gamma(h) + \sqrt{h_0}\, C_\varphi\, \widetilde{C}\,\big] \tag{22}$$

with the initial condition

$$\eta(0) = \gamma_0(h) \tag{23}$$

and its solution η_h. It follows from (22), (23) and Assumption H$[\sigma]$ that

$$\lim_{h\to 0} \eta_h(\cdot) = 0\,.$$

Thus, because η_h is a convex function:

$$\eta_h^{(i+1)} \ge \eta_h^{(i)} + h_0\, \bar{c}\, \sigma\big(t, \eta_h^{(i)} + \sqrt{h_0}\, h_0 C_\varphi\, \widetilde{C}\big) + h_0 \alpha(h).$$

Using induction we prove that

$$\omega_h^{(i)} \le \eta_h^{(i)} \qquad \text{for } 0 \le i \le N_a.$$

This gives (14) and Theorem 1 is proved. □

3. DIFFERENCE METHOD FOR MIXED PROBLEM

We will need an interpolating operator $T_h : \mathfrak{F}(E_h^*, R) \to C(E, R)$. Let

$$S_+ = \{\, \xi = (\xi_1, \ldots, \xi_n) : \quad \xi_j = \{0, 1\}, \quad \text{for } 0 \le j \le n \,\}.$$

Suppose that $z \in \mathfrak{F}(E_h^*, R)$. For every $(t, x) \in E$ there is $(t^{(i)}, x^{(m)}) \in E_h$ such that $(\, t^{(i+1)}, x^{(m+1)}) \in E_h'$, where $m + 1 = (m_1 + 1, \ldots, m_n + 1)$ and $t^{(i)} \le t \le t^{(i+1)}$, $x^{(m)} \le x \le x^{(m+1)}$. Then we put

$$\big(T_h\, z\big)(t, x) = \frac{t - t^{(i)}}{h_0} \sum_{\xi \in S_+} z^{(i+1, m+\xi)} \Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi}$$
$$+ \Big(1 - \frac{t - t^{(i)}}{h_0}\Big) \sum_{\xi \in S_+} z^{(i, m+\xi)} \Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} \Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi},$$

where

$$\Big(\frac{x - x^{(m)}}{h'}\Big)^{\xi} = \prod_{j=1}^{n} \Big(\frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{\xi_j},$$

$$\Big(1 - \frac{x - x^{(m)}}{h'}\Big)^{1-\xi} = \prod_{j=1}^{n} \Big(1 - \frac{x_j - x_j^{(m_j)}}{h_j}\Big)^{1-\xi_j}.$$

In the above formulas we adopt the convention that $0^0 = 1$. For $h_0 N_a < t \le a$ we put

$$\big(T_h\, z\big)(t, x) = \big(T_h\, z\big)(h_0 N_a, x).$$

Lemma 2. *Suppose that:*

1) $z(t, \cdot) : [-b, b] \to R$ is of class C^2 for $t \in [0, a]$ and $z_h = z|_{E_h^*}$;

2) $\tilde{d}_2 \in R_+$ is such a constant that on E^*

$$|\, \partial_{x_j x_k} z(t, x)| \le \tilde{d}_2 \quad \textit{for } j, k = 1, \ldots, n;$$

3) there exists $\tilde{c} \in R_+$ such that $h_0 < \tilde{c} \|h'\|^2$;

4) there is $L \in R_+$ such that

$$|\, z(t, x) - z(\bar{t}, x)| \le L |t - \bar{t}\,|.$$

Then

$$\|\, T_h z_h - z \,\|_{E^*} \le C_0 \|h'\|^2, \tag{24}$$

where $C_0 = \tilde{d}_2 + 2L\tilde{c}$ and $\|h'\| = h_1 + \cdots + h_n$.

The proof of Lemma 2 is given in Ciarski [1].

Assumption H$[\varrho, f]$**.** *Suppose that:*

1) $\varrho : E' \times C(E,R) \to M_{n\times n}$ *and* $f : E \times C(E,R) \times R^n \to R$ *are continuous;*

2) there is $\sigma : [0,a] \times R_+ \to R_+$ *satisfying Assumption* $H[\sigma]$ *and such that*

$$\| \varrho(t,x,w) - \varrho(t,x,\bar{w})\| \le \sigma(t, \|w-\bar{w}\|_t),$$

$$| f(t,x,w) - f(t,x,\bar{w})| \le \sigma(t, \|w-\bar{w}\|_t);$$

3) $\partial_p f = (\partial_{p_1} f, \dots, \partial_{p_n} f)$ *exists on* $E\times C(E,R)\times R^n$ *and* $\partial_p f(t,x,w,\cdot) \in C(R^n, R^n)$*.*

Now we will approximate the solution of the functional differential problem (1), by the solution of the difference problem

$$\delta_0\, z^{(i,m)} = \sum_{k,\,j=1}^{n} \varrho_{h\cdot kj}(\,t^{(i)},\, x^{(m)},\, T_h z)\delta^{(2)}_{kj} z^{(i,m)} + f_h(\,t^{(i)}, x^{(m)}, T_h z, \delta z^{(i,m)}\,), \tag{25}$$

with Neumann initial boundary conditions

$$z^{(i,m)} = \varphi_0^{(i,m)} \qquad \text{on } \; E_{0\cdot h}, \tag{26}$$

$$z_h^{(i,m+\alpha)} - z_h^{(i,m-\alpha)} = 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \quad \text{for } \; (t^{(i)}, x^{(m)}) \in \partial_0 E_h \; \text{ and } \; \alpha \in \mathcal{A}^{(m)}. \tag{27}$$

Let Ω be an open set such that $E^* \subset \Omega$.

Theorem 3. *Suppose that assumption* $H\,[\varrho, f]$ *holds and:*

1) $h \in \Delta$ *and*

$$1 - 2h_0 \sum_{k=1}^{n} \frac{1}{h_k^2}\varrho_{kk}(t,x,w) + h_0 \sum_{\substack{k,j=1\\ j\ne k}}^{n} \frac{1}{h_k h_j}|\varrho_{kj}(t,x,w)| \ge 0,$$

$$\frac{1}{h_k}\varrho_{kk}(t,x,w) - \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{1}{h_j}|\varrho_{kj}(t,x,w)| - \frac{1}{2}\,|\partial_{p_k} f(t,x,w,p)| \ge 0, \quad 1 \le k \le n;$$

2) there exists $c_\star \in R_+$ *such that* $h_k \le c_\star h_j$ *for* $1 \le k, j \le n$*;*

3) there exist $\widetilde{C}, \tilde{c} \in R_+$ *such that* $\widetilde{C}^{-1}\|h'\|^2 \le h_0 \le \tilde{c}\|h'\|^2$*;*

4) $u_h : E_h^{+1} \to R$ *is a solution of* (25)-(27)*;*

5) $v : \Omega \to R$ *is a solution of* (1) *on* E^* *and* $v_h = v|_{E_h^*}$, $\varphi_{j\cdot h} = \varphi_j|_{\partial_0 E_h}$, $1 \le j \le n$*;*

6) there exists $\gamma_0 : \Delta \to R_+$ *such that*

$$| \varphi_0^{(i,m)} - \varphi_{0\cdot h}^{(i,m)} | \le \gamma_0(h) \quad \text{on } E_{0\cdot h} \quad \text{with } \lim_{h\to 0} \gamma_0(h) = 0; \tag{28}$$

7) $v(\cdot, x)$ *is of class* C^1 *and* $v(t, \cdot)$ *is of class* C^3 *.*

Then, there is $\eta : \Delta \to R_+$ *such that on* E_h

$$| u_h^{(i,m)} - v_h^{(i,m)} | \le \eta(h) \tag{29}$$

and

$$\lim_{h\to 0} \eta(h) = 0.$$

Proof. We apply Theorem 1 to get the above assertion. Write

$$\begin{aligned} \psi_h^{(i,m)} &= \delta_0 v_h^{(i,m)} - \sum_{k,\,j=1}^{n} \varrho_{kj}(\, t^{(i)}, x^{(m)}, T_h v_h\,)\, \delta_{kj}^{(2)} v_h^{(i,m)} \\ &\quad - f(\, t^{(i)}, x^{(m)}, T_h v_h, \delta v_h{}^{(i,m)}\,), \\ \xi_h^{(i,m)} &= v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)}. \end{aligned} \tag{30}$$

We see at once that on E_h'

$$\begin{aligned} \psi_h^{(i,m)} =\; & \delta_0 v_h^{(i,m)} - \partial_t v(t^{(i)}, x^{(m)}) \\ &+ \sum_{k,\,j=1}^{n} \Big[\, \varrho_{kj}(\, t^{(i)}, x^{(m)}, v\,) - \varrho_{kj}(\, t^{(i)}, x^{(m)}, T_h v_h\,)\Big]\, \delta_{kj}^{(2)} v_h^{(i,m)} \\ &+ \sum_{k,\,j=1}^{n} \varrho_{kj}(\, t^{(i)}, x^{(m)}, v\,)\Big[\, \partial_{x_k x_j}^{(2)} v^{(i,m)} - \delta_{kj}^{(2)} v_h^{(i,m)}\,\Big] \\ &+ f\big(\, t^{(i)},\, x^{(m)},\, v, \partial_x v(t^{(i)}, x^{(m)})\,\big) - f\big(\, t^{(i)},\, x^{(m)},\, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})\,\big) \\ &+ f\big(\, t^{(i)},\, x^{(m)},\, T_h v_h, \partial_x v(t^{(i)}, x^{(m)})\,\big) - f\big(\, t^{(i)},\, x^{(m)},\, T_h v_h, \delta v_h^{(i,m)}\,\big). \end{aligned}$$

It is easily seen, that there is $\gamma : \Delta \to R_+$ such that

$$|\psi_h^{(i,m)}| \le \gamma(h) \quad \text{on } E'_h \quad \text{and} \quad \lim_{h\to 0} \gamma(h) = 0.$$

Now on ∂E_h we have

$$\xi_h^{(i,m)} = v_h^{(i,m+\alpha)} - v_h^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)}.$$

By Taylor formula we get

$$\begin{aligned}
v^{(i,m+\alpha)} &- v^{(i,m-\alpha)} - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \\
=& v^{(i,m)} + \sum_{j=1}^{n} \alpha_j h_j \partial_{x_j} v^{(i,m)} + \frac{1}{2}\sum_{k,\,j=1}^{n} \alpha_j \alpha_k h_j h_k \partial^{(2)}_{x_j x_k} v^{(i,m)} \\
&+ \frac{1}{6}\sum_{j,\,k,\,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \partial^{(3)}_{x_j x_k x_l} v(P) \\
&- \Big[v^{(i,m)} - \sum_{j=1}^{n} \alpha_j h_j \partial_{x_j} v^{(i,m)} + \frac{1}{2}\sum_{k,\,j=1}^{n} \alpha_j \alpha_k h_j h_k \partial^{(2)}_{x_j x_k} v^{(i,m)} \\
&- \frac{1}{6}\sum_{j,\,k,\,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \partial^{(3)}_{x_j x_k x_l} v(Q) \Big] - 2\sum_{j=1}^{n} \alpha_j h_j \varphi_{j\cdot h}^{(i,m)} \\
=& \frac{1}{6}\sum_{j,\,k,\,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \big[\partial^{(3)}_{x_j x_k x_l} v(P) - \partial^{(3)}_{x_j x_k x_l} v(Q) \big]
\end{aligned}$$

and finally

$$|\xi_h^{(i,m)}| \le \tilde{d}_3 \Big| \sum_{j,\,k,\,l=1}^{n} \alpha_j \alpha_k \alpha_l h_j h_k h_l \Big| \le \tilde{d}_3 \, \|h'\|^3 ,$$

where $\tilde{d}_3 \in R_+$. Thus, all the assumptions of Theorem 1 are satisfied and assertion (29) follows from (14). □

4. NUMERICAL EXAMPLE

For $n = 2$ we put

$$E = [0,1] \times [-1,1] \times [-1,1],$$

Let z be the unknown function of the variables (t, x, y) and consider the differential integral equation

$$\begin{aligned}\partial_t z(t,x,y) = {} & 2\,\partial_{xx} z(t,x,y) + 2\,\partial_{yy} z(t,x,y) \\ & + xy \int_{-1}^{1}\int_{-1}^{1} z(t,\eta,\xi)\, d\xi d\eta\, \partial_{xy} z(t,x,y) \\ & + x^4y^2 + x^3y - 4tx^4 - t(24x^2y^2 - 12xy) \\ & - \frac{4}{15} t^2 xy(8x^3y + 3x^2)\end{aligned} \tag{31}$$

with Neumann initial boundary conditions

$$z(t,x,y) = 0 \qquad \text{for } (t,x,y) \in E_0 \tag{32}$$

and

$$\partial_x z(t,x,y) = t(4x^3y^2 + 3x^2y) \quad \text{for } (t,x,y) \in \partial_0 E \text{ and } x = 1 \text{ or } x = -1, \tag{33}$$

$$\partial_y z(t,x,y) = t(2x^4y + x^3) \quad \text{for } (t,x,y) \in \partial_0 E \text{ and } y = 1 \text{ or } y = -1, \tag{34}$$

where

$$E_0 = \{0\} \times [-1,1] \times [-1,1],$$

$$\partial_0 E = (0,1] \times \Big[\big([-1,1]\times[-1,1]\big) \setminus \big((-1,1)\times(-1,1)\big)\Big].$$

For the above problem we apply difference method (5)-(9) and (4).

The function $v(t,x,y) = t(x^4y^2 + x^3y)$ is the solution of (31)-(34). Let $u_h : E_h^* \to R$ be the solution of the corresponding difference equations and $\varepsilon_h = u_h - v$. The values $\varepsilon_h(0.6, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.7, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.8, x^{(j)}, y^{(k)})$, $\varepsilon_h(0.9, x^{(j)}, y^{(k)})$ are listed in the Table 1 for $h_0 = 0.00001$ and $h_1 = h_2 = 0.02$.

Let ε_{max} be the biggest and ε_{mean} mean value of all ε_h for a given $t^{(i)}$. The values are listed in Table 2.

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		$t^{(i)} = 0.6$	$t^{(i)} = 0.7$	$t^{(i)} = 0.8$	$t^{(i)} = 0.9$
$x^{(j)}$	$y^{(k)}$	ε_h	ε_h	ε_h	ε_h
-0.5	-0.5	$4.077\ 10^{-4}$	$5.374\ 10^{-4}$	$7.067\ 10^{-4}$	$9.311\ 10^{-4}$
-0.5	0.0	$2.767\ 10^{-4}$	$3.767\ 10^{-4}$	$5.053\ 10^{-4}$	$6.732\ 10^{-4}$
-0.5	0.5	$2.146\ 10^{-4}$	$3.052\ 10^{-4}$	$4.218\ 10^{-4}$	$5.737\ 10^{-4}$
0.0	-0.5	$2.420\ 10^{-4}$	$3.344\ 10^{-4}$	$4.543\ 10^{-4}$	$6.121\ 10^{-4}$
0.0	0.0	$2.377\ 10^{-4}$	$3.278\ 10^{-4}$	$4.439\ 10^{-4}$	$5.955\ 10^{-4}$
0.0	0.5	$2.420\ 10^{-4}$	$3.344\ 10^{-4}$	$4.543\ 10^{-4}$	$6.121\ 10^{-4}$
0.5	-0.5	$2.146\ 10^{-4}$	$3.052\ 10^{-4}$	$4.218\ 10^{-4}$	$5.737\ 10^{-4}$
0.5	0.0	$2.767\ 10^{-4}$	$3.767\ 10^{-4}$	$5.053\ 10^{-4}$	$6.732\ 10^{-4}$
0.5	0.5	$4.077\ 10^{-4}$	$5.374\ 10^{-4}$	$7.067\ 10^{-4}$	$9.311\ 10^{-4}$

TABLE 1

$t^{(i)}$	ε_{max}	ε_{mean}
0.0	$2.736\ 10^{-4}$	$1.922\ 10^{-5}$
0.1	$3.645\ 10^{-4}$	$3.563\ 10^{-5}$
0.2	$5.023\ 10^{-4}$	$7.045\ 10^{-5}$
0.3	$6.334\ 10^{-4}$	$1.173\ 10^{-4}$
0.4	$7.790\ 10^{-4}$	$1.749\ 10^{-4}$
0.5	$9.510\ 10^{-4}$	$2.467\ 10^{-4}$
0.6	$1.162\ 10^{-3}$	$3.371\ 10^{-4}$
0.7	$1.427\ 10^{-3}$	$4.527\ 10^{-4}$
0.8	$1.768\ 10^{-3}$	$6.024\ 10^{-4}$
0.9	$2.217\ 10^{-3}$	$7.993\ 10^{-4}$
1.0	$2.820\ 10^{-3}$	$1.063\ 10^{-3}$

TABLE 2

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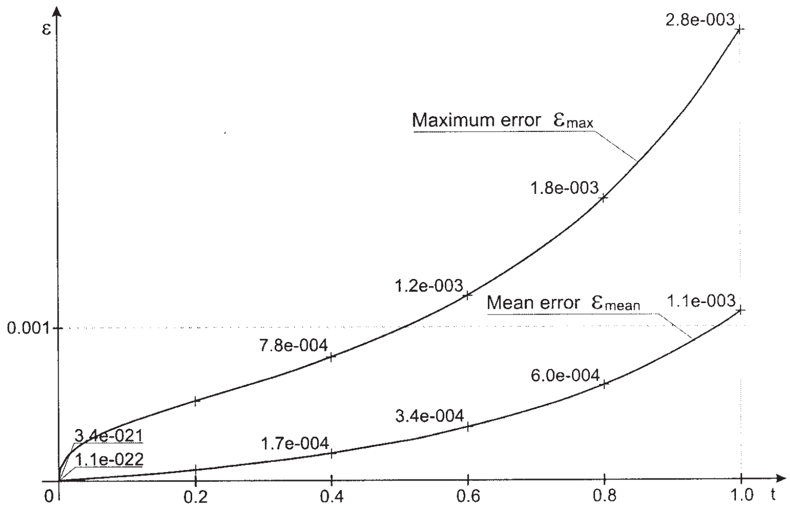

Figure 4. Mean and maximum error graphs.

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