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OSCILLATORY CRITERIA FOR EVEN ORDER HALF-LINEAR NEUTRAL EQUATION WITH DISTRIBUTED DEVIATING ARGUMENTS

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ABSTRACT: In this paper we investigate a class of even order half-linear neutral differential equation with distributed deviating arguments, and obtain some oscillatory criteria for the equation by employing the generalized Riccati technique and the integral averaging technique.

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1. INTRODUCTION

Recently, some results concerning the oscillation and asymptotic behavior of the solutions for half-linear differential equations has been investigated by many authors. For a survey of results, we can refer to the monograph Agarwal et al [2], papers Ladas and Sficas [9], Bainov and Mishev [3], Erbe et al [4], Zahariev and Bainov [13], Grace [5], Grace [6], Agarwal et al [1], Wang and Shi [11] and references cited therein. However, to the best of our knowledge, very little has been done for the works on half-linear equations

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with distributed deviating arguments (see Wang and Zhang [12]), in particular, the case of high order equations. The purpose of this paper is to study oscillation for the following even order half-linear neutral equations

$$\begin{aligned}\{|[x(t)+c(t)\,x(t-\tau)]^{(n-1)}|^{\alpha-1}[x(t)+c(t)x(t-\tau)]^{(n-1)}\}'\\ +\int_a^b q(t,\xi)\,|x[g(t,\xi)]|^{\alpha-1}\,x[g(t,\xi)]d\mu(\xi)=0,\end{aligned}\tag{1}$$

where n is an even, α and τ are positive constants.

By choosing appropriate function $H(t,s), h(t,s)$ and $\rho(s)$ and introducing a transformation, we can establish a series of explicit oscillation criteria.

We assume throughout this paper that the following conditions hold.

(H_1) $c(t)\in C([t_0,+\infty),R)$, $q(t,\xi)\in C([t_0,+\infty)\times[a,b],R_+)$;

(H_2) $g(t,\xi)\in C([t_0,+\infty)\times[a,b],R)$, $g(t,\xi)\le t, \xi\in[a,b]$. $g(t,\xi)$ is nondecreasing with respect to t and ξ respectively, and $\lim\limits_{t\to+\infty,\ \xi\in[a,b]}\min\{g(t,\xi)\}=+\infty$;

(H_3) $\sigma(\xi)\in([a,b],R)$ is nondecreasing, integral of equation (1) is a Stieltjes one.

We restrict our attention to a nontrivial solutions of equation (1), that is, to nonconstant solutions of existing on $[T,\infty]$ for $T\ge t_0$ and satisfying $\sup_{t\ge T}|x(t)|>0$. A nontrivial solution $x(t)$ of equation (1) is called oscillatory if it has arbitrarily large zeros; otherwise it is said to be nonoscillatory.

To obtain oscillatory criteria of equation (1), we first need the following lemmas.

Lemma 1.1. (see Kiguradze [8]) *Let $u(t)$ be a positive and n times differentiable function on R_+. If $u^{(n)}(t)$ is constant sign and not identically zero on any ray $[t_1,+\infty)$ for $t_1>0$, then there exists a $t_u\ge t_1$ and an integer l $(0\le l\le n)$, with $n+l$ even for $u(t)u^{(n)}(t)\ge 0$ or $n+l$ odd for $u(t)u^{(n)}(t)\le 0$; and for $t\ge t_u$, $u(t)u^{(k)}(t)>0$, $0\le k\le l$; $(-1)^{k-l}u(t)u^{(k)}(t)>0$, $l\le k\le n$.*

Lemma 1.2. (see Philos [10]) *Suppose that the conditions of Lemma 1.1 is satisfied, and*

$$u^{(n-1)}(t)u^{(n)}(t)\le 0,\qquad t\ge t_u,$$

then there exists a constant $\lambda\in(0,1)$ such that for sufficiently large t, there exists a constant $M>0$ satisfying

$$|u'(\lambda t)|\ge Mt^{n-2}|u^{(n-1)}(t)|.$$

Lemma 1.3. (see Hardy et al [7]) *If X and Y are nonnegative, then*

$$X^\lambda-\lambda XY^{\lambda-1}+(\lambda-1)Y^\lambda\ge 0,\qquad \lambda>1;$$

$$X^\lambda-\lambda XY^{\lambda-1}-(1-\lambda)Y^\lambda\le 0,\qquad 0<\lambda<1,$$

where the equality holds if and only if $X=Y$.

2. MAIN RESULTS

Theorem 2.1. *Suppose that the following conditions hold:*

(A_1) $0 \le c(t) \le 1$ *and* $\frac{d}{dt}g(t,a)$ *exists.*

If there exists a function $\varphi(t) \in C'([t_0,+\infty),(0,+\infty))$, *which is nondecreasing with respect to* t, *such that*

$$\int_{t_1}^{+\infty}\Bigg[\varphi(s)\int_a^b q(s,\xi)1-c[g(s,\xi)]^{\alpha}d\mu(\xi) - \lambda\varphi'(s)\left(\frac{\varphi'(s)}{M\varphi(s)[g(s,a)]^{n-2}g'(s,a)}\right)^{\alpha}\Bigg]ds = +\infty, \quad (2)$$

where $\lambda = \frac{\alpha+1}{\alpha}$, *then all solutions of equation (1) are oscillatory.*

Proof. Suppose to the contrary that there exists a nonoscillatory solution $x(t)$ of equation (1). Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. From (H_3), $\lim\limits_{t\to+\infty,\,\xi\in[a,b]}\min\{g(t,\xi)\} = +\infty$, there exists a $t_1 \ge t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$ and $x[g(t,\xi)] > 0$ for $t \ge t_1$ and $\xi \in [a,b]$. Letting

$$z(t) = x(t) + c(t)x(t-\tau), \quad (3)$$

then equation (1) can be written as

$$\Big[|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\Big]' + \int_a^b q(t,\xi)x^{\alpha}[g(t,\xi)]d\mu(\xi) = 0,$$

and from the assumption of $c(t)$ and $q(t,\xi)$, we have $z(t) \ge x(t) > 0$ and

$$\Big[|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)\Big]' \le 0. \quad (4)$$

Furthermore, we can prove $z^{(n-1)}(t) \ge 0$, $t \ge t_1$. In fact, suppose that $z^{(n-1)}(t) < 0, t \ge t_1$, then $|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) < 0$. From (4) we have that $|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t)$ is decreasing in t, and then

$$|z^{(n-1)}(t)|^{\alpha-1}z^{(n-1)}(t) \le |z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2), \qquad t \ge t_2 \ge t_1,$$

which implies that

$$-z^{(n-1)}(t) = |z^{(n-1)}(t)| \ge (|z^{(n-1)}(t_2)|^{\alpha-1}z^{(n-1)}(t_2))^{\frac{1}{\alpha}} = |z^{(n-1)}(t_2)|,$$

therefore, we have $z^{(n-1)}(t) \le -|z^{(n-1)}(t_2)|$. Integrating both sides of the above inequality from t_2 to t, we have

$$z^{(n-2)}(t) \le z^{(n-2)}(t_2) - \int_{t_2}^{t}|z^{(n-1)}(t_2)|ds = z^{(n-2)}(t_2) - |z^{(n-1)}(t_2)|(t-t_2). \quad (5)$$

Letting $t \to +\infty$, we have $\lim_{t\to+\infty} z^{(n-2)}(t) = -\infty$, and thus $\lim_{t\to+\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Thus $z^{(n-1)}(t) \geq 0$. Furthermore, from Lemma1.1, there exists a $t_3 \geq t_2$ and an odd number $l, 0 \leq l \leq n-1$, for $t \geq t_3$, such that

$$z^{(i)}(t) > 0, \quad 0 \leq i \leq l; \qquad (-1)^{i-1} z^{(i)}(t) > 0, \quad l \leq i \leq n-1.$$

By choosing $i = 1$, we have

$$z'(t) > 0. \tag{6}$$

From (3), equation (1) can be written as

$$\Big[|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)\Big]' + \int_a^b q(t,\xi)\{z[g(t,\xi)] - c[g(t,\xi)]x[g(t,\xi)-\tau]\}^{\alpha} d\mu(\xi) = 0.$$

Since $z(t) \geq x(t) > 0, z'(t) \geq 0$, we have $z[g(t,\xi)] \geq z[g(t,\xi)-\tau] \geq x[g(t,\xi)-\tau]$, and thus

$$\Big[|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)\Big]' + \int_a^b q(t,\xi) z^{\alpha}[g(t,\xi)]\{1 - c[g(t,\xi)]\}^{\alpha} d\mu(\xi) \leq 0. \tag{7}$$

Since that $g(t,\xi)$ is nondecreasing in ξ, we have $g(t,a) \leq g(t,\xi), t > t_0, \xi \in [a,b]$, thus $z[g(t,a)] \leq z[g(t,\xi)]$. Then (7) can be written as

$$\Big[|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)\Big]' + z^{\alpha}[g(t,a)] \int_a^b q(t,\xi)\{1 - c[g(t,\xi)]\}^{\alpha} d\mu(\xi) \leq 0, \tag{8}$$

where $t \geq t_1$. Letting

$$w(t) = \varphi(t) \frac{|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)}{z[g(t,a)]^{\alpha}},$$

then $w(t) \geq 0$, for $t \geq t_1$, and

$$w'(t) = \frac{\varphi'(t)}{\varphi(t)} w(t) + \frac{\varphi(t)[|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)]'}{z^{\alpha}[g(t,a)]} - \frac{\varphi(t)|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t) z'[g(t,a)] g'(t,a) \alpha z^{\alpha-1}[g(t,a)]}{z^{2\alpha}[g(t,a)]}.$$

From $\big[|z^{(n-1)}(t)|^{\alpha-1} z^{(n-1)}(t)\big]' \leq 0$, $z^{(n-1)}(t) \geq 0$, we conclude that

$$(z^{(n-1)}(t))^{\alpha-1} z^{(n)}(t) \leq 0,$$

which implies that $z^{(n)}(t) \le 0$. According to Lemma 1.2, we obtain

$$z'[g(t,a)] \ge M[g(t,a)]^{n-2} z^{(n-1)}(t). \tag{9}$$

Thus

$$\begin{aligned} w'(t) \le & \frac{\varphi'(t)}{\varphi(t)} w(t) - \varphi(t) \int_a^b q(t,\xi)\{1 - c[g(t,\xi)]\}^\alpha d\mu(\xi) \\ & - \frac{\alpha\varphi(t)[z^{(n-1)}(t)]^{\alpha+1} M[g(t,a)]^{n-2} g'(t,a)}{z^{\alpha+1}[g(t,a)]} \\ = & -\varphi(t) \int_a^b q(t,\xi)\{1 - c[g(t,\xi)]\}^\alpha d\mu(\xi) + \frac{\varphi'(t)}{\varphi(t)} w(t) \\ & - \frac{\alpha M[g(t,a)]^{n-2} g'(t,a) w^{\frac{\alpha+1}{\alpha}}(t)}{[\varphi(t)]^{\frac{1}{\alpha}}}. \end{aligned} \tag{10}$$

Taking

$$X = \frac{(\alpha M[g(t,a)]^{n-2} g'(t,a))^{\frac{\alpha}{\alpha+1}} w(t)}{[\varphi(t)]^{\frac{1}{\alpha+1}}}, \qquad \lambda = \frac{\alpha+1}{\alpha},$$

$$Y = \left(\frac{\alpha}{\alpha+1}\right)^\alpha \left[\frac{\varphi'(t)}{\varphi(t)} [\varphi(t)]^{\frac{1}{\alpha+1}} (\alpha M[g(t,a)]^{n-2} g'(t,a))^{-\frac{\alpha}{\alpha+1}}\right]^\alpha.$$

According to Lemma 1.3, we obtain

$$\begin{aligned} & \frac{\varphi'(t)}{\varphi(t)} w(t) - \alpha M[g(t,a)]^{n-2} g'(t,a) [\varphi(t)]^{-\frac{1}{\alpha}} w(t)^{\frac{\alpha+1}{\alpha}} \\ & \le (\frac{1}{\alpha})(\frac{\alpha}{\alpha+1})^{\alpha+1} \varphi(t) (\frac{\varphi'(t)}{\varphi(t)})^{\alpha+1} (\alpha M[g(t,a)]^{n-2} g'(t,a))^{-\alpha} \\ & \le \lambda \varphi(t) (\frac{\varphi'(t)}{\varphi(t)})^{\alpha+1} (M[g(t,a)]^{n-2} g'(t,a))^{-\alpha}, \end{aligned}$$

thus

$$\begin{aligned} w'(t) \le -\varphi(t) \Bigg[& \int_a^b q(t,\xi)\{1 - c[g(t,\xi)]\}^\alpha d\mu(\xi) \\ & - \frac{\lambda \varphi'(t)}{\varphi(t)} \left(\frac{\varphi'(t)}{M[g(t,a)]^{n-2} g'(t,a) \varphi(t)}\right)^\alpha \Bigg]. \end{aligned} \tag{11}$$

Integrating both sides from t_1 to t, we have

$$\begin{aligned} w(t) \le w(t_1) - \int_{t_1}^t \Bigg[& \varphi(s) \int_a^b q(s,\xi)\{1 - c[g(s,\xi)]\}^\alpha d\mu(\xi) \\ & - \lambda \varphi'(s) \left(\frac{\varphi'(s)}{M[g(s,a)]^{n-2} g'(s,a) \varphi(s)}\right)^\alpha \Bigg] ds. \end{aligned}$$

Letting $t \to +\infty$, from (2), we have $\lim\limits_{t\to+\infty} w(t) = -\infty$, which leads to a contradiction with $w(t) \geq 0$. This completes the proof of Theorem 2.1. $\square$

Theorem 2.2. *Suppose that conditions* (A_1) *holds. If there exist function* $\varphi(t), \rho(s) \in C'([t_0, +\infty), (0, +\infty))$, *in which* $\varphi(t)$ *is nondecreasing with respect to* t. *Letting function* $H(t, s), h(t, s) \in C'(D, R)$, *in which* $D = \{(t, s)|t \geq s \geq t_0\}$, *such that:*

(H_4*)* $H(t, t) = 0, t \geq t_0; H(t, s) > 0, t > s \geq t_0$.

(H_5*)* $H'_t(t, s) \geq 0, H'_s(t, s) \leq 0$.

(H_6*)* $-\frac{\partial[H(t,s)\rho(s)]}{\partial s} - H(t, s)\rho(s)\frac{\varphi'(s)}{\varphi(s)} = h(t, s)$.

If

$$\limsup_{t\to+\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[H(t, s)\rho(s)\varphi(s) \int_a^b q(s, \xi)\{1 - c[g(s, \xi)]\}^\alpha d\mu(\xi) - \frac{\beta\varphi(s)|h(t, s)|^{\alpha+1}}{(MH(t, s)\rho(s)[g(s, a)]^{n-2}g'(s, a))^\alpha} \right] ds = +\infty, \quad (12)$$

where $\beta = (\frac{1}{\alpha+1})^{\alpha+1}$. *Then all solutions of equation (1) are oscillatory.*

Proof. Suppose to the contrary that there exists a nonoscillatory solution $x(t)$ of equation (1). Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Then, proceeding as for Theorem 2.1, we have

$$w'(t) \leq -\varphi(t) \int_a^b q(t, \xi)\{1 - c[g(t, \xi)]\}^\alpha d\mu(\xi) + \frac{\varphi'(t)}{\varphi(t)} w(t) - \alpha \frac{M[g(t, a)]^{n-2}g'(t, a)w^{\frac{\alpha+1}{\alpha}}(t)}{[\varphi(t)]^{\frac{1}{\alpha}}}.$$

Multiplying the above inequality by $H(t, s)\rho(s)$, for $t \geq s \geq T$, and integrating from T to t

$$\begin{aligned}\int_T^t w'(s)H(t, s)\rho(s)ds \leq &- \int_T^t H(t, s)\rho(s)\varphi(s) \int_a^b q(s, \xi)\{1 - c[g(s, \xi)]\}^\alpha d\mu(\xi)ds \\ &+ \int_T^t \frac{\varphi'(s)}{\varphi(s)} w(s)H(t, s)\rho(s)ds \\ &- \alpha M \int_T^t H(t, s)\rho(s) \frac{[g(s, a)]^{n-2}g'(s, a)w^{\frac{\alpha+1}{\alpha}}(s)}{[\varphi(s)]^{\frac{1}{\alpha}}} ds. \quad (13)\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi)ds\\
&\le H(t,T)\rho(T)w(T) - \int_T^t \left[-\frac{\partial[H(t,s)\rho(s)]}{\partial s} - H(t,s)\rho(s)\frac{\varphi'(s)}{\varphi(s)}\right] w(s)ds\\
&\qquad - \alpha M\int_T^t H(t,s)\rho(s)\frac{[g(s,a)]^{n-2}g'(s,a)w^{\frac{\alpha+1}{\alpha}}(s)}{[\varphi(s)]^{\frac{1}{\alpha}}}ds\\
&\qquad\le H(t,T)\rho(T)w(T) + \int_T^t |h(t,s)w(s)|ds\\
&\qquad\quad - \alpha M\int_T^t H(t,s)\rho(s)\frac{[g(s,a)]^{n-2}g'(s,a)w^{\frac{\alpha+1}{\alpha}}(s)}{[\varphi(s)]^{\frac{1}{\alpha}}}ds. \qquad (14)
\end{aligned}$$

Taking

$$\begin{aligned}
X &= (\alpha M H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\frac{\alpha}{\alpha+1}}\frac{w(s)}{[\varphi(s)]^{\frac{1}{\alpha+1}}};\\
Y &= \left(\frac{\alpha}{\alpha+1}\right)^\alpha \frac{\varphi(s)]^{\frac{\alpha}{\alpha+1}}|h(t,s)|^\alpha}{(\alpha M H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\frac{\alpha^2}{\alpha+1}}},
\end{aligned}$$

in which $\lambda = \frac{\alpha+1}{\alpha}$. According to Lemma 1.3, we obtain for $t > T \ge t_0$

$$\begin{aligned}
&|h(t,s)w(s)| - \alpha M H(t,s)\rho(s)\frac{[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}}\\
&\qquad\le \frac{1}{\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\varphi(s)|h(t,s)|^{\alpha+1}}{(\alpha M H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^\alpha}\\
&\qquad\qquad = \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^\alpha}, \qquad (15)
\end{aligned}$$

where $\beta = (\frac{1}{\alpha+1})^{\alpha+1}$. From (14) and (15), we have

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi)ds\\
&\qquad - \int_T^t \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^\alpha}ds\\
&\qquad\le H(t,T)\rho(T)w(T) \le H(t,t_0)\rho(T)w(T), \quad t > T \ge t_0. \qquad (16)
\end{aligned}$$

Thus

$$\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi) - \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}}\Bigg]ds$$
$$= \frac{1}{H(t,t_0)}\left[\int_{t_0}^{T}+\int_{T}^{t}\right]\Bigg[H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi) - \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}}\Bigg]ds$$
$$\le \frac{1}{H(t,t_0)}\int_{t_0}^{T}H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi)ds+\rho(T)w(T)$$
$$\le \int_{t_0}^{T}\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi)ds+\rho(T)w(T).$$

Letting $t \to +\infty$, we have

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\Bigg[H(t,s)\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi) - \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}}\Bigg]ds$$
$$\le \int_{t_0}^{T}\rho(s)\varphi(s)\int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^{\alpha}d\mu(\xi)ds+\rho(T)w(T) < +\infty,$$

which leads to a contradiction with (12). This completes the proof of Theorem 2.2. □

Theorem 2.3. *Suppose that conditions (A_1) holds. Letting functions $H(t,s)$, $h(t,s)$, $\varphi(s)$ and $\rho(s)$ be the same as in Theorem 2.2. Moreover, suppose that*

$$0 < \inf_{s\ge t_0}\left\{\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right\} \le +\infty, \tag{17}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\varphi(s)|h(t,s)|^{\alpha+1}}{(H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}}ds < +\infty. \tag{18}$$

If there exists a function $A(t) \in C([t_0,+\infty),R)$ such that

$$\limsup_{t\to+\infty}\int_{t_0}^{t}\frac{(A_+(s))^{\frac{\alpha+1}{\alpha}}[g(s,a)]^{n-2}g'(s,a)}{[\rho(s)\varphi(s)]^{\frac{1}{\alpha}}}ds = +\infty, \tag{19}$$

and for every $T \geq t_0$

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)\varphi(s) \int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi) - \frac{\beta\varphi(s)|h(t,s)|^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^\alpha} \right] ds \geq A(T), \quad (20)$$

where $A_+(s) = \max\{A(s), 0\}$. Then all solutions of equation (1) are oscillatory.

Proof. Suppose to the contrary that there exists a nonoscillatory solution $x(t)$ of equation (1). Without loss of generality, we may suppose that $x(t)$ is an eventually positive solution. Then, proceeding as for Theorem 2.2, we have (14) and (16), then for $t > T \geq t_0$, we obtain

$$\limsup_{t\to+\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)\varphi(s) \int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi) - \frac{\beta\varphi(s)h(t,s)^{\alpha+1}}{(MH(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^\alpha} \right] ds \leq \rho(T)w(T).$$

Thus by (20) we have

$$A(T) \leq \rho(T)w(T), \qquad T \geq t_0, \quad (21)$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\rho(s)\varphi(s) \int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi) ds \geq A(t_0). \quad (22)$$

Letting

$$F(t) = \frac{1}{H(t,t_0)} \int_{t_0}^t |h(t,s)w(s)| ds\,,$$
$$G(t) = \frac{\alpha M}{H(t,t_0)} \int_{t_0}^t \frac{H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}} ds$$

for $t > t_0$. Then by (14) and (22), we see that

$$\liminf_{t\to+\infty}[G(t) - F(t)] \leq \rho(t_0)w(t_0) - \limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H(t,s)\rho(s)\varphi(s) \int_a^b q(s,\xi)\{1-c[g(s,\xi)]\}^\alpha d\mu(\xi) ds \leq \rho(t_0)w(t_0) - A(t_0) < +\infty. \quad (23)$$

Now, we claim that

$$\int_{t_0}^{+\infty} \frac{\rho(s)[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}} ds < +\infty. \quad (24)$$

Suppose to the contrary that

$$\int_{t_0}^{+\infty} \frac{\rho(s)[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}}ds = +\infty, \tag{25}$$

by (17), there exists a positive constant η satisfying

$$\inf_{s\geq t_0}\left\{\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right\} > \eta > 0. \tag{26}$$

On the other hand, by (25) for any positive number μ there exists a $t_1 > t_0$ such that

$$\int_{t_0}^{t} \frac{\rho(s)[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}}ds \geq \frac{\mu}{\alpha M\eta}, \qquad t\geq t_1,$$

then for all $t \geq t_1$

$$\begin{aligned}
G(t) &= \frac{\alpha M}{H(t,t_0)}\int_{t_0}^{t}\frac{H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a)w(s)^{\frac{\alpha+1}{\alpha}}}{[\varphi(s)]^{\frac{1}{\alpha}}}ds\\
&= \frac{\alpha M}{H(t,t_0)}\int_{t_0}^{t}H(t,s)d\left[\int_{t_0}^{s}\frac{\rho(u)[g(u,a)]^{n-2}g'(u,a)w(u)^{\frac{\alpha+1}{\alpha}}}{[\varphi(u)]^{\frac{1}{\alpha}}}du\right]\\
&\geq \frac{\alpha M}{H(t,t_0)}\int_{t_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)\left[\int_{t_0}^{s}\frac{\rho(u)[g(u,a)]^{n-2}g'(u,a)w(u)^{\frac{\alpha+1}{\alpha}}}{[\varphi(u)]^{\frac{1}{\alpha}}}du\right]ds\\
&\geq \frac{\mu}{\alpha M\eta}\frac{\alpha M}{H(t,t_0)}\int_{t_1}^{t}\left(-\frac{\partial H(t,s)}{\partial s}\right)ds = \frac{\mu}{\eta}\frac{H(t,t_1)}{H(t,t_0)}.
\end{aligned} \tag{27}$$

From (26) we have

$$\liminf_{t\to+\infty}\frac{H(t,t_1)}{H(t,t_0)} > \eta > 0,$$

then there exists $t_2 \geq t_1$, such that $\frac{H(t,t_1)}{H(t,t_0)} \geq \eta$ for all $t \geq t_2$. Therefore by (27), $G(t) \geq \mu$ for $t \geq t_2$, and since μ is arbitrary constant, we conclude that

$$\lim_{t\to+\infty} G(t) = +\infty. \tag{28}$$

Then, consider a sequence $\{t_n\}_{n=1}^{+\infty}$, $t_n \in (t_0, +\infty)$ and $\lim\limits_{t\to+\infty} t_n = +\infty$, such that

$$\lim_{n\to+\infty}[G(t_n) - F(t_n)] = \liminf_{t\to+\infty}[G(t) - F(t)].$$

According to (27), there exists a constant M, such that for all sufficiently large n

$$G(t_n) - F(t_n) \leq M.$$

It follows from (28) that $\lim_{n\to+\infty} G(t_n) = +\infty$, and thus

$$\lim_{n\to+\infty} F(t_n) = +\infty. \tag{29}$$

Then, we have for n large enough $\frac{F(t_n)}{G(t_n)} - 1 \geq -\frac{M}{G(t_n)} > -\frac{1}{2}$, that is $\frac{F(t_n)}{G(t_n)} > \frac{1}{2}$ for all n large enough. This and (29) imply that

$$\lim_{n\to+\infty} \frac{F^{\alpha+1}(t_n)}{G^{\alpha}(t_n)} = \lim_{n\to+\infty} \left[\frac{F(t_n)}{G(t_n)}\right]^{\alpha} F(t_n) = +\infty. \tag{30}$$

On the other hand, by Hölder's inequality, we have

$$\begin{aligned}
F(t_n) &= \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} |h(t_n,s)w(s)|ds \\
&\leq \left\{ \frac{\alpha M}{H(t_n,t_0)} \int_{t_0}^{t_n} \frac{H(t_n,s)\rho(s)[g(s,a)]^{n-2} g'(s,a) w^{\frac{\alpha+1}{\alpha}}(s)}{[[\varphi(s)]^{\frac{1}{\alpha}}} ds \right\}^{\frac{\alpha}{\alpha+1}} \\
&\times \left\{ \frac{1}{(\alpha M)^{\alpha} H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\varphi(s)|h(t_n,s)|^{\alpha+1}}{(H(t_n,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}} ds \right\}^{\frac{1}{\alpha+1}} \\
&\qquad = [G(t_n)]^{\frac{\alpha}{\alpha+1}} \\
&\times \left\{ \frac{1}{(\alpha M)^{\alpha} H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\varphi(s)|h(t_n,s)|^{\alpha+1}}{(H(t_n,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}} ds \right\}^{\frac{1}{\alpha+1}},
\end{aligned}$$

and therefore

$$\frac{F^{\alpha+1}(t_n)}{G^{\alpha}(t_n)} \leq \frac{1}{(\alpha M)^{\alpha} H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\varphi(s)|h(t_n,s)|^{\alpha+1}}{(H(t_n,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}} ds$$

for all large n. It follows from (30) that

$$\lim_{n\to+\infty} \frac{1}{H(t_n,t_0)} \int_{t_0}^{t_n} \frac{\varphi(s)|h(t_n,s)|^{\alpha+1}}{(H(t_n,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}} ds = +\infty,$$

which implies that

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{\varphi(s)|h(t,s)|^{\alpha+1}}{(H(t,s)\rho(s)[g(s,a)]^{n-2}g'(s,a))^{\alpha}} ds = +\infty$$

which contradicts to (18). Hence (24) holds. Then, it follows from (21) that

$$\begin{aligned}\int_{t_0}^{+\infty}&\frac{(A_+(s))^{\frac{\alpha+1}{\alpha}}[g(s,a)]^{n-2}g'(s,a)}{[\rho(s)\varphi(s)]^{\frac{1}{\alpha}}}ds\\ &\leq\int_{t_0}^{+\infty}\frac{[\rho(s)w(s)]^{\frac{\alpha+1}{\alpha}}[g(s,a)]^{n-2}g'(s,a)}{[\rho(s)\varphi(s)]^{\frac{1}{\alpha}}}ds\\ &=\int_{t_0}^{+\infty}\frac{\rho(s)[g(s,a)]^{n-2}g'(s,a)w^{\frac{\alpha+1}{\alpha}}(s)}{[\varphi(s)]^{\frac{1}{\alpha}}}ds<+\infty,\end{aligned}$$

which contradicts to (19). This completes the proof of Theorem 2.3. □

The following examples illustrate our theory.

Example 2.1. Consider the 4-order equation

$$\begin{aligned}\Big\{\big|[x(t)+(1-e^{-t/\alpha})x(t-\tau)]^{(3)}\big|^{\alpha-1}[x(t)+(1-e^{-t/\alpha})x(t-\tau)]^{(3)}\Big\}'&\\ +\int_{-1}^{0}e^{2t+2\xi}|x(t+\xi)|^{\alpha-1}x(t+\xi)d\xi=0,&\end{aligned}\tag{31}$$

in which $a=-1, b=0, c(t)=1-e^{-t/\alpha}, g(t,\xi)=t+\xi, q(t,\xi)=e^{2t+2\xi}$ and $\mu(\xi)=\xi$. Choosing $\varphi(t)=t$. Then the conditions of (A_1) holds, and we have

$$\begin{aligned}&\int_{t_1}^{+\infty}\left[s\int_{-1}^{0}e^{2s+2\xi}\{1-(1-e^{-(s+\xi)/\alpha})\}^{\alpha}d\xi-\lambda\frac{1}{(Ms(s-1)^2)^{\alpha}}\right]ds\\ &=\int_{t_1}^{+\infty}se^{s}ds-\int_{t_1}^{+\infty}se^{s-1}ds-\frac{\lambda}{M^{\alpha}}\int_{t_1}^{+\infty}\frac{1}{(s(s-1)^2)^{\alpha}}ds=+\infty.\end{aligned}$$

Therefore, all solution of equation (31) are oscillatory by Theorem 2.1.

Example 2.2. Consider the high-order equation for $n=m+2$

$$\begin{aligned}\left\{\left|[x(t)+\left(1-\frac{1}{t}\right)x(t-\tau)]^{(m+1)}\right|^{\alpha-1}[x(t)+(1-\frac{1}{t})x(t-\tau)]^{(m+1)}\right\}'&\\ +\int_{\frac{1}{2}}^{1}(t^2\xi)^{\alpha}|x(t\xi)|^{\alpha-1}x(t\xi)d\xi=0,&\end{aligned}\tag{32}$$

in which m is an even, $\alpha>2$. $a=\frac{1}{2}$, $b=1$, $c(t)=1-\frac{1}{t}$, $g(t,\xi)=t\xi$, $q(t,\xi)=(t^2\xi)^{\alpha}$ and $\mu(\xi)=\xi$. Then the conditions of (A_1) holds. Moreover, taking $\varphi(t)=t^2$, $\rho(t)=\frac{1}{t^2}$, $H(t,s)=(t-s)^2$ for $t\geq s\geq t_0$. Then the conditions of $(H_4),(H_5)$ in Theorem 2.2 are satisfied, and we have $h(t,s)=$

$\frac{2(t-s)}{s^2}$. Thus we conclude that

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)} \int_{t_0}^{t} (t-s)^2 \int_{\frac{1}{2}}^{1} (s^2\xi)^\alpha (\frac{1}{s\xi})^\alpha d\xi$$

$$= \limsup_{t\to+\infty} \frac{1}{(t-t_0)} \int_{t_0}^{t} (t-s)^2 \int_{\frac{1}{2}}^{1} s^\alpha d\xi ds$$

$$= \limsup_{t\to+\infty} \frac{1}{2(t-t_0)} \left\{ t^{\alpha+3} \left(\frac{1}{\alpha+1} + \frac{1}{\alpha+3} - \frac{2}{\alpha+2} \right) - \left(\frac{t^2 t_0^{\alpha+1}}{\alpha+1} + \frac{t_0^{\alpha+3}}{\alpha+3} + \frac{2tt_0^{\alpha+2}}{\alpha+2} \right) \right\} = +\infty.$$

On the other hand

$$\int_{t_0}^{t} \frac{\beta s^2 [\frac{2(t-s)}{s^2}]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2} (\frac{s}{2})^m \frac{1}{2}\right)^\alpha} ds = \frac{2^{(m+2)\alpha+1}\beta}{M^\alpha} \int_{t_0}^{t} (t-s)^{1-\alpha} s^{-m\alpha} ds$$

$$\leq \frac{2^{(m+2)\alpha+1}\beta}{M^\alpha} \times \frac{(t-t_0)^{2-\alpha}}{\alpha-2} \times \frac{(t^{1-m\alpha} - t_0^{1-m\alpha})}{1-m\alpha}.$$

When $\alpha > 2$, we have that

$$\limsup_{t\to+\infty} \int_{t_0}^{t} \frac{\beta s^2 [\frac{2(t-s)}{s^2}]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2} (\frac{s}{2})^m \frac{1}{2}\right)^\alpha} ds = 0.$$

Therefore

$$\limsup_{t\to+\infty} \frac{1}{(t-t_0)} \int_{t_0}^{t} \left[(t-s)^2 \int_{\frac{1}{2}}^{1} (s^2\xi)^\alpha \left(\frac{1}{s\xi}\right)^\alpha d\xi - \frac{\beta s^2 [\frac{2(t-s)}{s^2}]^{\alpha+1}}{\left(M(t-s)^2 \frac{1}{s^2} (\frac{s}{2})^m \frac{1}{2}\right)^\alpha} \right] ds = +\infty,$$

that is (12) is satisfied. Thus, all solutions of equation (32) are oscillatory by Theorem 2.2.

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