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ASYMPTOTIC STABILITY OF MILD SOLUTIONS OF STOCHASTIC EVOLUTION EQUATIONS

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ABSTRACT: In this paper, we study the existence and stability problems associated with stochastic evolution equations in Hilbert spaces. To be precise, we first consider an existence result for a mild solution and then the exponential stability of the moments of such solutions and also of its sample paths. Such results are established employing the theory of a stochastic convolution integral and a comparison principle under less restrictive hypothesis than the Lipschitz condition on the nonlinear terms. Moreover, we consider asymptotic stability in probability of the sample paths of the solution process. The results obtained here generalize the corresponding main results from Taniguchi [18] and also the classical results of Ichikawa [7, 8] all established under the Lipschitz hypothesis.

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Dedicated to my mother Mrs. Suseela Sarangan
on her 70-th birthday.

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1. INTRODUCTION

In this paper, we consider a semilinear stochastic evolution equation in a Hilbert space

$$dx(t) = [Ax(t) + f(t, x(t))]dt + g(t, x(t))dw(t), \qquad t \geq 0, \tag{1}$$

$$x(0) = x_0;$$

and the equation to be made precise below. Stability of stochastic differential equations in infinite dimensions of the form (1) has been investigated by several authors, see Da Prato et al [1], Govindan [4], Ichikawa [7] and [8], Liu [12], Liu et al [13], Mao [14], Taniguchi [18] and Leha et al [11] and the references cited therein. Sufficient conditions for the exponential stability are given generally in terms of a Lyapunov functional with an application of Itô's formula in mind; and the existence and uniqueness of a solution are obtained using Lipschitz and linear growth conditions on the nonlinear terms, see Ichikawa [7] and [8], Liu [12] and Liu et al [13]. However, there are many examples where the nonlinear terms $f(t, x)$ and $g(t, x)$ do not satisfy the Lipschitz condition, see Rodkina [15], Taniguchi [16], Yamada [21], He [6], Gikhman et al [3] and Hasminskii [5] and the references cited therein. In order to handle such situations, we must look for conditions other the Lipschitz condition. Our goal here is to investigate the existence and stability problems without the Lipschitz hypothesis. To be precise, we shall use conditions weaker than the Lipschitz condition and use a nonlinear growth condition as in Taniguchi [16]. Such general conditions compel the use of a general method of successive approximations and a comparison principle Taniguchi [16]. At this point, it is worth mentioning that comparison principle technique has been exploited in the literature to study the existence and stability problems in Galcuk et al [2], Govindan [4], Ikeda et al [9], Taniguchi [16, 17], Yamada [20], He [6] and Ting [19], among others. In fact, the author made an attempt earlier with such a technique in Govindan [4]. But, the exponential stability condition obtained there depends upon the time, see Remark 1 below, which does not make sense the way one defines this stability concept; a possible reason being that the estimates of the integrals are not time-independent. To overcome this problem, a plausible way is to use the theory of a stochastic convolution integral, see Da Prato et al [1] and Taniguchi [18]. To do so, it is natural to work in a different framework as in Da Prato et al [1] and Taniguchi [18] as apposed to the one in Govindan [4]. In other words, our objective here is to work essentially within the framework of Taniguchi [18] but without using the Lipschitz condition. However, it is really interesting to observe that the exponential stability condition obtained here boils down to the one in Taniguchi [18]. Furthermore, we shall consider the asymptotic stability in probability of the sample paths, see Ichikawa [7] and [8] of the solution process again using a comparison principle. The results obtained here generalize the work, particularly, those of Ichikawa [7] and [8] and Taniguchi [18] in the sense of Remarks 1 and 2, see Section 5. In the process, we also rectify the aforementioned problem from Govindan [4], see Remark 1 for details.

In Section 2, we formulate the problem and state the comparison principle from Taniguchi [16] that shall play a fundamental and crucial role throughout the paper and also state a property of a stochastic convolution integral, see Da Prato et al [1]. For a general theory on equation (1), we again refer to Da Prato et al [1]. In Section 3, we shall consider the existence and uniqueness of a solution. In Section 4, assuming the semigroup to be exponentially stable, we first obtain an estimate of a mild solution in terms of a global solution of a scalar ordinary differential equation and use this later to establish the exponential stability in the quadratic mean of a trivial solution. In Section 5, we deduce the almost sure exponential stability of the sample paths of the solution process from Section 4 using the standard Borel-Cantelli Lemma arguments as in Ichikawa [7] and [8] and Taniguchi [18]. Lastly, we shall also consider the asymptotic stability in probability of the sample paths again by exploiting the comparison principle.

2. PRELIMINARIES

Let X, Y be real separable Hilbert spaces and $L(Y, X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notation $|.|$ to denote the norms in X, Y and $L(Y, X)$ and use $(.,.)$ to denote innerproduct of X and Y without any confusion. Let $\sigma_2(Y, X)$ denote the space of all Hilbert-Schmidt operators from Y to X with the Hilbert-Schmidt norm $|.|_2$. Let $(\Omega, B, P, \{B_t\}_{t\geq 0})$ be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub-σ-algebras of B. Let $\beta_n(t)$ $(n = 1, 2, 3, ...)$ be a sequence of real-valued standard Brownian motions mutually independent defined on this probability space. Set

$$w(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\beta_n(t)e_n, \qquad t \geq 0,$$

where $\lambda_n \geq 0$ $(n = 1, 2, 3, ...)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1, 2, 3, ...)$ is a complete orthonormal basis in Y. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$. The above Y-valued stochastic process $w(t)$ is called a Q-Wiener process. Now, we define a real-valued stochastic integral of Y-valued B_t-adapted predictable process $h(t)$ with respect to the Q-Wiener process $w(t)$.

Definition 1. Let $h(t)$ be a Y-valued B_t-adapted predictable process such that $E\int_0^t |h(t)|^2 dt < \infty$ for any $t \in [0, \infty)$. Then, we define the real-valued stochastic integral $\int_0^t < h(s), dw(s) >$ by

$$\int_0^t < h(s), dw(s) > = \sum_{n=1}^{\infty} \int_0^t (h(s), e_n) dw(s) e_n,$$

where $w(s)e_n = (w(s), e_n) = \sqrt{\lambda_n}\beta_n(s)$.

Definition 2. Let $h(t)$ be an $L(Y, X)$-valued function and let λ be a sequence $\{\sqrt{\lambda_1}, \sqrt{\lambda_2}, ...\}$. Then we define

$$|h(t)|_\lambda = \{\sum_{n=1}^{\infty} |\sqrt{\lambda_n} h(t) e_n|^2\}^{1/2}.$$

If $|h(t)|_\lambda^2 < \infty$, then $h(t)$ is called λ-Hilbert-Schmidt operator and let $\sigma(\lambda)(Y, X)$ denote the space of all λ-Hilbert-Schmidt operators from Y to X.

Lemma 1. (see Taniguchi [18]) *If trace $Q < \infty$ and $h(t) \in L(Y, X)$, then*

$$|h(t)|_\lambda^2 \leq (\operatorname{trace} Q)|h(t)|^2.$$

If $0 \leq \lambda_j \leq \mu$ for all $j \geq 1$ and $h(t) \in \sigma_2(Y, X)$, then

$$|h(t)|_\lambda^2 \leq \mu |h(t)|_2^2.$$

Next, we define the X-valued stochastic integral with respect to the Y-valued Q-Wiener process $w(t)$.

Definition 3. Let $\Phi : [0, \infty) \to \sigma(\lambda)(Y, X)$ be a predictable, B_t-adapted process. Then, for any Φ satisfying $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty$ we define the X-valued stochastic integral $\int_0^t \Phi(s) dw(s) \in X$ with respect to $w(t)$ by

$$(\int_0^t \Phi(s) dw(s), h) = \int_0^t < \Phi^*(s) h, dw(s) >, \qquad h \in X,$$

where Φ^* is the adjoint operator of Φ.

Let C_T be the space of continuous functions $x : [0, T] \to X (0 < T < +\infty)$ with the norm $\| x \|_T = \sup_{0 \leq s \leq T} |x(s)|$. Let S_T be the space of measurable $X-$ valued random processes $\phi(t, \omega)$ with almost sure (a.s.) continuous sample paths with the norm $\| x \|_{S_T} = (E \| \phi(., \omega) \|_T^2)^{1/2}$.

The following lemma is from Rodkina [15].

Lemma 2. *S_T is a Banach space.*

We now make the equation (1) precise: Let $A : X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ defined on X. Let the functions $f(t, x)$, and $g(t, x)$ be defined as follows: $f : R^+ \times X \to X$, where $R^+ = [0, \infty)$, and $g : R^+ \times X \to L(Y, X)$ are measurable functions; and x_0 is a B_0- measurable random variable satisfying $E|x_0|^2 < \infty$.

Next, we introduce the concept of a mild solution of equation (1).

Definition 4. A stochastic process $\{x(t), t \in [0, T]\}(0 < T < +\infty)$ is called a mild solution of equation (1) if:

i) $x(t)$ is B_t-adapted and predictable with $\int_0^T |x(t)|^2 dt < \infty$, a.s.,

ii) $x(t)$ satisfies the integral equation

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x(s))ds + \int_0^t S(t-s)g(s, x(s))dw(s), \quad \text{a.s., } t \in [0, T]. \tag{2}$$

Here the second integral is understood in the sense of Itô.

Definition 5. A semigroup $\{S(t), t \geq 0\}$ is said to be exponentially stable if there exist positive constants M, a such that

$$\| S(t) \| \leq Me^{-at}, \qquad t \geq 0,$$

where $\| \,.\, \|$ denotes the operator norm in X. If $M = 1$, then the semigroup is said to be a contraction semigroup.

For convenience, we will state below two lemmas that we will be using in the sequel.

Lemma 3. (see Da Prato and Zabczyk [1, Theorem 6.10]) *Suppose A generates a contraction semigroup. Let $W_A^\Phi(t) = \int_0^t S(t-s)\Phi(s)dw(s), t \in [0, T]$. Then the process $W_A^\Phi(.)$ has a continuous modification and there exists a constant $\kappa > 0$ such that*

$$E \sup_{s \in [0,t]} |W_A^\Phi(s)|^2 \leq \kappa E \int_0^t |\Phi(s)|_\lambda^2 ds, \qquad t \in [0, T].$$

Lemma 4. (see Taniguchi [16]) *Suppose that a function $H(\tau, u) : R^+ \times R^+ \to R^+ = [0, \infty)$ is continuous, monotone nondecreasing with respect to $u \in R^+$ for each $\tau \in R^+$ and is locally integrable with respect to $\tau \in R^+$ for each fixed $u \in R^+$. If two continuous functions $a(t)$ and $b(t)$ defined on $[s, \theta)$ ($s \geq 0$ and θ may be ∞) satisfy the inequality*

$$a(t) - \int_s^t H(\tau, a(\tau))d\tau < b(t) - \int_s^t H(\tau, b(\tau))d\tau,$$

for all $t \in (s, \theta)$ and $a(s) < b(s)$, then $a(t) < b(t)$ for all $t \in [s, \theta)$.

3. AN EXISTENCE THEOREM

In this section, we consider an existence result making less restrictive assumptions than the Lipschitz and linear growth conditions. This result is proved mimicking arguments from Taniguchi [16].

Let us now make the following assumptions:

(A1) A is the infinitesimal generator of a strongly continuous semigroup $\{S(t), t \geq 0\}$ which is a contraction;

(A2) there exists a function $H : R^+ \times R^+ \to R^+$ that is integrable in $t \in R^+$ for each fixed $u \in R^+$ and is continuous and monotone nondecreasing in u for each fixed $t \in R^+$ such that

$$E|f(t,x)|^2 + E|g(t,x)|_\lambda^2 \leq H(t, E|x|^2),$$

for all $t \in R^+$ and all $x \in X$;

(A3) there exists a function $G : R^+ \times R^+ \to R^+$ that is monotone nondecreasing and continuous in $u \in R^+$ for each fixed $t \in R^+$ and is integrable in $t \in R^+$ for each fixed $u \in R^+$ such that $G(t,0) = 0$ and

$$E|f(t,x) - f(t,y)|^2 + E|g(t,x) - g(t,y)|_\lambda^2 \leq G(t, E|x-y|^2),$$

for all $t \in R^+$ and all $x, y \in X$;

(A4) the function $G(t,u)$ in (A2) satisfies a sufficient condition under which if a nonnegative, continuous function $z(t)$ satisfies that

$$z(t) \leq C \int_0^t G(s, z(s))ds, \qquad \text{for all} \qquad t \in R^+,$$

where $C > 0$ and if $z(0) = 0$, then $z(t) \equiv 0$ for all $t \in R^+$;

(A5) the scalar deterministic ordinary differential equation

$$\frac{du}{dt} = DH(t,u), \qquad t \in R^+, \tag{3}$$

$$u(0) = u_0;$$

where $D > 0$ has a global solution.

Note 1. Assumption (A1) is standard, see Da Prato et al [1] while Assumptions (A2)-(A6) are made as in Taniguchi [16]; and also that if $H(u)$ is a concave function, it satisfies the inequality in Assumption (A2), see Yamada [21]. We refer the reader to Taniguchi [16] for a motivation and discussion of these assumptions. In fact, in Taniguchi [16], under these assumptions, an existence and uniqueness result was considered for a classical Itô stochastic differential equations. See also He [6] for a study in the deterministic case for retarded equations using a comparison principle.

Let us now introduce the successive approximations to equation (1) (or (2)) as follows:

$$x_{n+1}(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x_n(s))ds$$

$$+ \int_0^t S(t-s)g(s, x_n(s))dw(s); \qquad n = 0, 1, 2, 3, ... \tag{4}$$

with the initial approximation as $x(0) = x_0$.

Theorem 1. *Let the Assumptions (A1)-(A5) hold. Then equation (1) has a unique mild solution $x \in S_T$ and*

$$E[\sup_{0\le t\le T} |x_n(t) - x(t)|^2] \to 0 \qquad \textit{as} \qquad n \to \infty,$$

where $\{x_n(t)\}_{n\ge 1}$ are the successive approximations (4).

Proof. (Sketch) Let T be an arbitrary positive number $0 < T < +\infty$ and $x_0 \in S_T$ be a fixed initial approximation to (4). Since the proof follows as in Taniguchi [16], we shall only sketch it. From (4) for $n = 0$, we have

$$x_1(t) = S(t)x_0 + \int_0^t S(t-s)f(s, x_0)ds + \int_0^t S(t-s)g(s, x_0)dw(s).$$

By Lemma 3 and Assumption (A2), we have

$$E|x_1(t)|^2 \le 3E|x_0|^2 + 3T \int_0^t E|f(s, x_0)|^2 ds + 3\kappa \int_0^t E|g(s, x_0)|^2_\lambda ds.$$

$$\le 3E|x_0|^2 + 3(T+\kappa) \int_0^t H(s, E|x_0|^2)ds.$$

On the other hand, let $u(t)$ be a global solution of the scalar equation (3) with $D = 3(T + \kappa)$ given by

$$u(t) = u_0 + 3(T+\kappa) \int_0^t H(s, u(s))ds, \qquad t \ge 0.$$

Taking $u_0 \in R^+$ such that $u_0 > 3E|x_0|^2$, it follows that

$$E|x_1(t)|^2 - 3(T+\kappa) \int_0^t H(s, E|x_0|^2)ds$$

$$< u(t) - 3(T+\kappa) \int_0^t H(s, u(s))ds.$$

By Lemma 4, we get

$$E|x_1(t)|^2 < u(t), \qquad t \in [0, T].$$

Continuing the proof by mathematical induction, one can show that

$$E|x_n(t)|^2 < u(t), \qquad n = 1, 2, 3, ..., \tag{5}$$

for all $t \in [0, T]$. Since $u(t)$ is continuous on $[0, T]$, there exists $R > 0$ such that $E|x_n|^2 < R$ for all $t \in [0, T]$ and every $n = 1, 2, 3, ...$

Next, we show that $\{x_n\}$ is Cauchy in S_T. Define the functions $b_{mn}(t)$ and $b_n(t)$ on $[0, T]$ for all integers $m \geq n \geq 0$ as

$$b_{mn}(t) = E|x_m(t) - x_n(t)|^2,$$

$$b_n(t) = \sup\{b_{pq}(t) : p \geq q \geq n\}.$$

Arguing as in Taniguchi [16], one gets for some positive constants η, Q:

$$0 \leq b_n(t) < \eta \qquad \text{and} \qquad |b_n(t) - b_n(s)| \leq Q|\xi(t) - \xi(s)|^{1/2},$$

for all integers $n \geq 0$ and $t, s \in [0, T]$, where $\xi(t) = \int_0^t H(s, u(s))ds$. Therefore, by the Ascoli-Arzela theorem there exists a subsequence $\{b_{n(k)}(t)\}$ which converges uniformly to some continuous function $b(t)$ defined on $[0, T]$. Hence, for $m \geq n \geq n(k+1)$:

$$E(\sup_{t\in[0,T]} |x_m(t) - x_n(t)|^2) \leq 2(T + \kappa) \int_0^T G(s, E|x_{m-1}(s) - x_{n-1}(s)|^2)ds$$

$$\leq 2(T + \kappa) \int_0^T G(s, b_{n(k)}(s))ds \to 0$$

as $k \to \infty$ implying that $\{x_n\}$ is Cauchy in S_T. Therefore, there exists a stochastic process $x(t)$ on $[0, T]$ such that

$$E(\sup_{t\in[0,T]} |x_n(t) - x(t)|^2) \to 0$$

as $n \to \infty$ and $x(t)$ is indeed a unique mild solution on $t \geq 0$ since T is an arbitrary positive number. $\square$

4. EXPONENTIAL STABILITY IN THE QUADRATIC MEAN

In this section, we study the exponential stability of the second moment of a mild solution of equation (1). Assume from now on that $f(t, 0) \equiv 0$ and $g(t, 0) \equiv 0$ a.e. t so that equation (1) admits a trivial solution. Let $x(t) = x(t; x_0)$ be a mild solution of equation (1) where x_0 is any initial value.

Definition 6. The trivial solution of equation (1) or equation (1) itself is said to be exponentially stable in the quadratic mean if there exist positive constants K, ν such that

$$E|x(t;x_0)|^2 \leq KE|x_0|^2 \exp(-\nu t), \qquad t \geq 0.$$

The following lemmas are needed to consider the main results.

Lemma 5. (see Taniguchi [18]) *Let the semigroup $\{S(t), t \geq 0\}$ be exponentially stable. Then, for any stochastic process $F : [0,\infty) \to X$ which is strongly measurable with $\int_0^T E|F(t)|^2 dt < \infty, 0 < T \leq \infty$, the following inequality holds for $0 < t \leq T$:*

$$E|\int_0^t S(t-s)F(s)ds|^2 \leq \frac{M^2}{a}\int_0^t e^{-a(t-s)}E|F(s)|^2 ds.$$

Lemma 6. (see Taniguchi [18]) *Let the semigroup $\{S(t), t \geq 0\}$ be exponentially stable. Then, for any B_t-adapted predictable process $\Phi : [0,\infty) \to \sigma(\lambda)(Y, X)$ with $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty, t \geq 0$, the following inequality holds:*

$$E|\int_0^t S(t-s)\Phi(s)dw(s)|^2 \leq M^2 \int_0^t e^{-a(t-s)}E|\Phi(s)|_\lambda^2 ds.$$

We first consider a result that gives an estimate for the solution of equation (1). For this, we need the following assumption made as in Taniguchi [17].

(A6) the function $H(t,u)$ in (A2) satisfies

$$\delta H(t,u) \leq H(t, \delta u), \qquad \delta > 1,$$

for all $u \in R^+$ and $t \geq 0$.

Theorem 2. *Let the hypothesis of Theorem 1 hold. Assume further that the condition (A6) is satisfied and the semigroup $\{S(t), t \geq 0\}$ is exponentially stable. Then the mild solution of equation (1) satisfies*

$$e^{at}E|x(t)|^2 < u(t), \qquad t \geq 0.$$

Proof. Consider the mild solution of equation (1):

$$x(t) = S(t)x_0 + \int_0^t S(t-s)f(s,x(s))ds + \int_0^t S(t-s)g(s,x(s))dw(s).$$

Applying Lemma 5 and Lemma 6, one gets

$$E|x(t)|^2 \leq 3e^{-2at}E|x_0|^2 + \frac{3}{a}\int_0^t e^{-a(t-s)}E|f(s,x(s))|^2 ds$$

$$+3\int_0^t e^{-a(t-s)}E|g(s,x(s))|_\lambda^2 ds.$$

Thus,

$$e^{at}E|x(t)|^2 \leq 3E|x_0|^2 + 3(1/a+1)\int_0^t H(s, e^{as}E|x(s)|^2)ds.$$

Let $u(t)$ be a global solution of (3) with $D = 3(1/a+1)$:

$$u(t) = u_0 + 3(1/a+1)\int_0^t H(s, u(s))ds, \qquad t \geq 0. \tag{6}$$

Choosing $u_0 > 3E|x_0|^2$, we obtain as before that

$$e^{at}E|x(t)|^2 - 3(1/a+1)\int_0^t H(s, e^{as}E|x(s)|^2)ds$$

$$< u(t) - 3(1/a+1)\int_0^t H(s, u(s))ds, \qquad t \geq 0.$$

Lemma 4 then yields

$$e^{at}E|x(t)|^2 < u(t), \qquad t \geq 0. \qquad \square$$

We need a further assumption to consider the main result of this section.

(A7) the function $H(t, u)$ in (A2) satisfies a linear growth condition:

$$H(t, u) \leq Lu, \qquad \text{for all} \qquad u \in R^+, \qquad t \geq 0,$$

where $L > 0$ constant.

Theorem 3. *Let the hypothesis of Theorem 2 hold except for the Assumption (A6) which is replaced by (A7). Then equation (1) is exponentially stable in the quadratic mean provided* $3L(1/a+1) < a$.

Proof. It follows from (6) and Assumption (A7) that

$$u(t) \leq u_0 + 3L(1/a+1)\int_0^t u(s)ds.$$

By Gronwall's Lemma, we get

$$u(t) \leq u_0 e^{3L(1/a+1)t}, \qquad t \geq 0.$$

Now, if $E|x_0|^2 > 0$, set $k(\varepsilon) = 3 + \varepsilon$ and if $E|x_0|^2 = 0$, then set $k(\varepsilon) = \varepsilon$. Hence, set $u_0 = k(\varepsilon)E|x_0|^2$.

By Theorem 2, we have

$$\begin{aligned} E|x(t)|^2 &< e^{-at}u(t) \\ &< e^{-at}u_0 e^{3L(1/a+1)t} \\ &< e^{-at}k(\varepsilon)E|x_0|^2 e^{3L(1/a+1)t}, \qquad t \geq 0, \end{aligned}$$

from which letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} E|x(t)|^2 &\le 3E|x_0|^2 e^{-(a-3L(1/a+1))t} \\ &\le 3E|x_0|^2 e^{-\nu t}, \qquad t \ge 0, \end{aligned}$$

where $\nu = a - 3L(1/a+1)$. □

Definition 7. The trivial solution of equation (3) is said to be exponentially stable if there exist positive constants K_1, γ such that

$$|u(t;u_0)| \le K_1 |u_0| \exp(-\gamma t), \qquad t \ge 0.$$

Theorem 4. *Let the hypothesis of Theorem 2 hold. Then the exponential stability of equation (3) implies likewise the exponential stability of equation (1).*

Proof. By Theorem 2, we have

$$\begin{aligned} E|x(t)|^2 &< e^{-at} u(t) \\ &< e^{-at} K_1 u_0 e^{-\gamma t} \\ &\le K_1 E|x_0|^2 e^{-\nu t}, \qquad t \ge 0, \end{aligned}$$

where $\nu = a + \gamma$ arguing as before. □

5. ASYMPTOTIC STABILITY OF SAMPLE PATHS

In this section, we study sample path asymptotic stability, see Ichikawa [7] and [8], of the solution process of equation (1). We begin with the almost sure exponential stability in Section 5.1. In Section 5.2, we consider the asymptotic stability in probability.

5.1. ALMOST SURE EXPONENTIAL STABILITY

The main result of this sub-section can be easily deduced from Theorem 3 by exploiting the Borel-Cantelli lemma. For this, we need a lemma.

Lemma 7. (see Taniguchi [18]) *Let A be the infinitesimal generator of a contraction semigroup. Let $\Phi : [0,\infty) \to \sigma(\lambda)(Y,X)$ be a predictable, B_t-adapted process such that $\int_0^t E|\Phi(s)|_\lambda^2 ds < \infty$, for any $t \ge 0$. Then, there exists a constant $c_0 > 0$, independent of N, such that for any fixed natural number $N > 0$*

$$E\{\sup_{N \le t \le N+1} |\int_N^t S(t-s)\Phi(s)dw(s)|^2\} \le c_0 \int_N^{N+1} E|\Phi(s)|_\lambda^2 ds.$$

Theorem 5. *Suppose that all the conditions of Theorem 3 hold. Then the mild solution of equation (1) satisfies*

$$\limsup_{t\to\infty} (1/t)\log|x(t)| \le -\nu/4, \qquad a.s.$$

Proof. Let N be a sufficiently large positive integer. Let $N \le t \le N+1$. Then,

$$x(t) = S(t)x(N) + \int_N^t S(t-s)f(s,x(s))ds + \int_N^t S(t-s)g(s,x(s))dw(s).$$

Thus, for any fixed $\varepsilon_N > 0$, we obtain

$$P\{\sup_{N\le t\le N+1} |x(t)| > \varepsilon_N\} \le P\{\sup_{N\le t\le N+1} |S(t)x(N)| > \varepsilon_N/3\}$$

$$+P\{\sup_{N\le t\le N+1} |\int_N^t S(t-s)f(s,x(s))ds| > \varepsilon_N/3\}$$

$$+P\{\sup_{N\le t\le N+1} |\int_N^t S(t-s)g(s,x(s))dw(s)| > \varepsilon_N/3\}$$

$$\le (3/\varepsilon_N)^2 E[\sup_{N\le t\le N+1} |S(t)x(N)|^2]$$

$$+(3/\varepsilon_N)^2 E[\sup_{N\le t\le N+1} |\int_N^t S(t-s)f(s,x(s))ds|^2]$$

$$+(3/\varepsilon_N)^2 E[\sup_{N\le t\le N+1} |\int_N^t S(t-s)g(s,x(s))dw(s)|^2]$$

$$= I_1 + I_2 + I_3, \quad \text{say.}$$

Thus by Theorem 3, we have that

$$I_1 \le (3/\varepsilon_N)^2 E[|x(N)|^2]$$

$$\le (3/\varepsilon_N)^2 3E|x_0|^2 \exp(-\nu N),$$

and

$$I_2 \le (3/\varepsilon_N)^2 \sup_{N\le t\le N+1} \int_N^t E[\| S(t-s) \| \, |f(s,x(s))|]^2 ds$$

$$\le (3/\varepsilon_N)^2 L \int_N^{N+1} E|x(s)|^2 ds$$

$$\leq (3/\varepsilon_N)^2 L \int_N^{N+1} 3E|x_0|^2 \exp(-\nu s) ds$$

$$\leq (3/\varepsilon_N)^2 3LE|x_0|^2 \exp(-\nu N).$$

Finally, by using Lemma 7:

$$I_3 \leq (3/\varepsilon_N)^2 c_0 \int_N^{N+1} E|g(s, x(s))|_\lambda^2 ds$$

$$\leq (3/\varepsilon_N)^2 3Lc_0 E|x_0|^2 \exp(-\nu N).$$

Therefore

$$P\{\sup_{N \leq t \leq N+1} |x(t)| \geq \varepsilon_N\} \leq \Lambda/\varepsilon_N^2 E|x_0|^2 \exp(-\nu N),$$

where $\Lambda = 27[1 + L(1 + c_0)]$. Hence, if we set $\varepsilon_N = (E|x_0|^2)^{1/2} \exp(-\nu N/4)$, then

$$P\{\sup_{N \leq t \leq N+1} |x(t)| > (E|x_0|^2)^{1/2} \exp(-\nu N/4)\}$$

$$\leq \Lambda \exp(-\nu N/2).$$

Thus by the Borel-Cantelli Lemma we conclude that there exists a random time $0 < T(\omega) < \infty$ such that

$$|x(t)|^2 \leq \vartheta E|x_0|^2 \exp(-\nu t/2) \qquad \text{a.s.} \qquad \text{for} \qquad t > T(\omega),$$

where $\vartheta = \exp(\nu/2)$. □

Remark 1. Taniguchi [18] established Theorems 3 and 5 using the Lipschitz condition on the nonlinear terms $f(t, x)$ and $g(t, x)$. In fact, the exponential stability of the quadratic moments as well as of the sample paths of a mild solution were shown under the condition that $3L(1/a + 1) < a$, which agrees with ours. As pointed out earlier, in Govindan [4] the exponential stability was shown under the condition $3L(T + \operatorname{tr} W) < a$, that is, when $T < a/3L - \operatorname{tr} W$ is finite contrary to the definition.

5.2. ASYMPTOTIC STABILITY IN PROBABILITY

Assume that x_0 is nonrandom.

Definition 8. The trivial solution of equation (1) is said to be:

i) stable in probability if for any $\varepsilon, \varepsilon' > 0$, there exists a $\delta > 0$ such that

$$P\{\sup_{t \geq 0} |x(t; x_0)| > \varepsilon'\} < \varepsilon \qquad \text{whenever} \qquad |x_0| < \delta;$$

ii) asymptotically stable in probability if it is stable in probability and if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\lim_{T\to\infty} P\{\sup_{t\geq T} |x(t;x_0)| > \varepsilon\} = 0 \qquad \text{whenever} \qquad |x_0| < \delta.$$

We shall establish the above stability notions of equation (1) again by the comparison principle. For this, let us recall the following corresponding stability notions for the scalar differential equation (3) from Lakshmikantham et al [10, Chapter 2].

Let $H(t,0) \equiv 0$ a.e. t so that equation (3) admits a trivial solution. Let $u(t) = u(t,u_0)$ be a solution of equation (3) where u_0 is any initial value.

Definition 9. The trivial solution of equation (3) is said to be:

i) stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $|u_0| < \delta$ implies that $|u(t)| < \varepsilon, t \geq 0$;
ii) asymptotically stable if it is stable and if there exists a $\delta > 0$ such that $|u_0| < \delta$ implies that $u(t) \to 0$ as $t \to \infty$.

Theorem 6. *Let the hypothesis of Theorem 1 hold. Then the stability or asymptotic stability of the trivial solution of the scalar equation (3) implies likewise the stability in probability or asymptotic stability in probability of the trivial solution of equation (1).*

Proof. Let T be an arbitrary positive number $0 < T < +\infty$. Then for $t \in [0,T]$,

$$E \parallel x \parallel_t^2 \leq 3|x_0|^2 + 3T\int_0^t E|f(s,x(s))|^2 ds + 3\kappa \int_0^t E|g(s,x(s))|_\lambda^2 ds$$

$$\leq 3|x_0|^2 + 3(T+\kappa)\int_0^t H(s, E\parallel x \parallel_s^2) ds.$$

Now, let $u(t;u_0)$ be a global solution of the scalar equation (3) when $D = 3(T+\kappa)$. Taking $u_0 \in R^+$ such that $u_0 > 3|x_0|^2$, it follows from Lemma 4 that

$$E \parallel x \parallel_t^2 < u(t;u_0), \qquad t \geq 0,$$

since T is an arbitrary positive number $0 < T < +\infty$.

On the other hand

$$P\{\sup_{t\geq 0} |x(t;x_0)| > \varepsilon'\} \leq \frac{1}{\varepsilon'^2} E \parallel x \parallel_t^2$$

$$< \frac{1}{\varepsilon'^2} u(t;u_0)$$

$$\leq \frac{1}{\varepsilon'^2} u(t; 3|x_0|^2), \qquad t \geq 0,$$

where the last inequality is obtained arguing as in the proof of Theorem 3. Since the trivial solution of the scalar equation (3) is stable, given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\frac{1}{\varepsilon'^2} u(t; 3|x_0|^2) < \varepsilon$ whenever $|x_0| < \sqrt{\delta/3}$. This proves at once the stability in probability of equation (1). The asymptotic stability in probability follows similarly. $\square$

Remark 2. In his pioneering work, Ichikawa [7] and [8] used the Lipschitz condition on the nonlinear terms together with a Lyapunov functional and Itô's formula to establish the stability results considered here, where as, we use less restrictive conditions than the Lipschitz condition and use the global solution $u(t)$ of equation (3) instead. It is interesting to observe that $u(t)$ plays the role of a Lyapunov functional in this approach. Quite recently, Liu et al [13] studied such stability issues by constructing an appropriate Lyapunov functional. As a matter of coincidence, the construction of $u(t)$ too appears to be nontrivial. See also Leha et al [11].

REFERENCES

[1] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge Univ. Press, Cambridge, 1992.

[2] L.I. Galcuk and M.H.A. Davis, A note on a comparison theorem for equations with different diffusions, *Stochastics*, **6** (1982), 147-149.

[3] I.I. Gikhman and A.V. Skorokhod, *Stochastic Differential Equations*, Springer-Verlag, 1972.

[4] T.E. Govindan, Existence and stability of solutions of stochastic semilinear evolution equations, *Comm. Applied Anal.*, **7** (2003), 101-114.

[5] R.Z. Hasminskii, *Stochastic Stability of Differential Equations*, Sijthoff and Noordhoff, 1980.

[6] M. He, Global existence and stability of solutions for reaction diffusion functional differential equations, *J. Math. Anal. Appl.*, **199** (1996), 842-858.

[7] A. Ichikawa, Stability of semilinear stochastic evolution equations, *J. Math. Anal. Appl.*, **90** (1982), 12-44.

[8] A. Ichikawa, Semilinear stochastic evolution equations: boundedness, stability and invariant measures, *Stochastics*, **12** (1984), 1–39.

[9] N. Ikeda and S. Watanabe, A comparison theorem for solutions of stochastic differential equations and its applications, *Osaka J. Math.*, **14** (1977), 619-633.

[10] V. Laksmikantham and S. Leela, *Differential and Integral Inequalities: Theory and Applications*, Vols. I & II, Academic Press, NY, 1969.

[11] G. Leha, B. Maslowski, and G. Ritter, Stability of solutions to semilinear stochastic evolution equations, *Stochastic Anal. Appl.*, **17** (1999), 1009-1051.

[12] K. Liu, *Stability of Stochastic Differential Equations in Infinite Dimensions*, to appear.

[13] R. Liu and V. Mandrekar, Stochastic semilinear evolution equations: Lyapunov function, stability and ultimate boundedness, *J. Math. Anal. Appl.*, **212** (1997), 537–553.

[14] X. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, NY, 1994.

[15] A.E. Rodkina, On existence and uniqueness of solution of stochastic differential equations with heredity, *Stochastics*, **12** (1984), 187–200.

[16] T. Taniguchi, Successive approximations to solutions of stochastic differential equations, *J. Differential Eqns.*, **96** (1992), 152–169.

[17] T. Taniguchi, On the estimate of solutions of perturbed linear differential equations, *J. Math. Anal. Appl.*, **153** (1990), 288-300.

[18] T. Taniguchi, Asymptotic stability theorems of semilinear stochastic evolution equations in Hilbert spaces, *Stoch. Stoch. Reports*, **53** (1995), 41-52.

[19] Y. Ting, Comparison method of stability analysis of semilinear time-delay stochastic evolution equation, *Stochastic Anal. Appl.*, **22** (2004), 403-427.

[20] T. Yamada, On comparison theorem for solutions of stochastic differential equations and its applications, *J. Math. Kyoto Univ.*, **13** (1973), 497-521.

[21] T. Yamada, On the successive approximations of solutions of stochastic differential equations, *J. Math. Kyoto Univ.*, **21** (1981), 501-515.