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HIGHER ORDER FUZZY KOROVKIN THEORY VIA INEQUALITIES

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ABSTRACT: Here is studied with rates the fuzzy uniform and L_p, $p \geq 1$, convergence of a sequence of fuzzy positive linear operators to the fuzzy unit operator acting on spaces of fuzzy differentiable functions. This is done quantitatively via fuzzy Korovkin type inequalities involving the fuzzy modulus of continuity of a fuzzy derivative of the engaged function. From there we deduce general fuzzy Korovkin type theorems with high rate of convergence. The surprising fact is that basic real positive linear operator simple assumptions enforce here the fuzzy convergences. At the end we give applications. Our results are univariate and multivariate. The assumptions are minimal and natural fulfilled by almost all example—fuzzy positive linear operators.

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1. INTRODUCTION

Motivation for this work are Anastassiou [2], Anastassiou [4], Anastassiou [5], Anastassiou and Gal [7], Korovkin [14], Shisha and Mond [15]. In fact this is continuation of Anastassiou [4], Anastassiou [5]. Here first we translate the necessary measure theory approximation results from Anastassiou [2] into the language of real positive linear operators, then by combining the facts, e.g. use of Proposition 1, we transfer results to the fuzzy level.

Applications are on univariate and multivariate Bernstein operators. At the beginning we provide all necessary fuzzy terminology, definitions and theorems we use here. In that background section we prove some results

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that they stand by themselves, such as in positivity. The basic ingredient to establish our results it the bridge between real operators to fuzzy ones. It is the natural realization condition: Assumption 1, see (17). This is fulfilled by almost all example positive operators, in fact by all summation and integration operators: real and fuzzy. The concept of fuzzy positivity we use is the natural analog of the real positivity. The same thing with linearity. We hope we contribute to the development of Fuzzy Approximation Theory and Fuzzy Functional Analysis.

2. BACKGROUND

We start with the following definition.

Definition 1. (see Wu and Gong [17]) Let $\mu\colon \mathbb{R} \to [0,1]$ with the following properties:

(i) is *normal*, i.e., $\exists x_0 \in \mathbb{R}$; $\mu(x_0) = 1$.

(ii) $\mu(\lambda x + (1-\lambda)y) \geq \min\{\mu(x), \mu(y)\}$, $\forall x, y \in \mathbb{R}$, $\forall \lambda \in [0,1]$ (μ is called a convex fuzzy subset).

(iii) μ is *upper semicontinuous* on $\mathbb{R}$, i.e., $\forall x_0 \in \mathbb{R}$ and $\forall \varepsilon > 0$, $\exists$ neighborhood $V(x_0)$: $\mu(x) \leq \mu(x_0) + \varepsilon$, $\forall x \in V(x_0)$.

(iv) The set $\overline{\operatorname{supp}(\mu)}$ is compact in $\mathbb{R}$ (where $\operatorname{supp}(\mu) := \{x \in \mathbb{R};\ \mu(x) > 0\}$).

We call μ a *fuzzy real number*. Denote the set of all μ with $\mathbb{R}_{\mathcal{F}}$.

E.g., $\mathcal{X}_{\{x_0\}} \imath n \mathbb{R}_{\mathcal{F}}$, for any $x_0 \in \mathbb{R}$, where $\mathcal{X}_{\{x_0\}}$ is the characteristic function at x_0.

For $0 < r \leq 1$ and $\mu \in \mathbb{R}_{\mathcal{F}}$ define $[\mu]^r := \{x \in \mathbb{R}\colon\ \mu(x) \geq r\}$ and

$$[\mu]^0 := \overline{\{x \in \mathbb{R} : \mu(x) > 0\}}.$$

Then it is well known Goetschel Jr. and Voxman [10] that for each $r \in [0,1]$, $[\mu]^r$ is a closed and bounded interval of $\mathbb{R}$. For $u, v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, we define uniquely the sum $u \oplus v$ and the product $\lambda \odot u$ by

$$[u \oplus v]^r = [u]^r + [v]^r, \quad [\lambda \odot u]^r = \lambda [u]^r, \qquad \forall r \in [0,1],$$

where $[u]^r + [v]^r$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $\lambda[u]^r$ means the usual product between a scalar and a subset of $\mathbb{R}$ (see, e.g., Kaleva [11]). Notice $1 \odot u = u$ and it holds $u \oplus v = v \oplus u$, $\lambda \odot u = u \odot \lambda$. If $0 \leq r_1 \leq r_2 \leq 1$ then $[u]^{r_2} \subseteq [u]^{r_1}$. Actually $[u]^r = [u_-^{(r)}, u_+^{(r)}]$, where $u_-^{(r)} \leq u_+^{(r)}$, $u_-^{(r)}$, $u_+^{(r)} \in \mathbb{R}$, $\forall r \in [0,1]$. Based on Goetschel Jr. and Voxman [10] we can then identify any $u \in \mathbb{R}_{\mathcal{F}}$ with the parametrized representation $\{(u_-^{(r)}, u_+^{(r)}) \mid 0 \leq r \leq 1\}$. We denote $u \precsim v$ iff $u_-^{(r)} \leq v_-^{(r)}$ and $u_+^{(r)} \leq v_+^{(r)}$, all $r \in [0,1]$. Define

$$D\colon \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \to \mathbb{R}_+$$

by

$$D(u,v) := \sup_{r\in[0,1]} \max\{|u_-^{(r)} - v_-^{(r)}|, |u_+^{(r)} - v_+^{(r)}|\},$$

where $[v]^r = [v_-^{(r)}, v_+^{(r)}]$; $u, v \in \mathbb{R}_{\mathcal{F}}$. We have that D is a metric on $\mathbb{R}_{\mathcal{F}}$. Then $(\mathbb{R}_{\mathcal{F}}, D)$ is a complete metric space, see Wu and Ma [16], with the properties

$$\begin{aligned} D(u\oplus w, v\oplus w) &= D(u,v), \quad \forall u,v,w \in \mathbb{R}_{\mathcal{F}},\\ D(k\odot u, k\odot v) &= |k| D(u,v), \quad \forall u,v \in \mathbb{R}_{\mathcal{F}},\ \forall k\in\mathbb{R},\\ D(u\oplus v, w\oplus e) &\le D(u,w) + D(v,e), \quad \forall u,v,w,e \in \mathbb{R}_{\mathcal{F}}. \end{aligned}$$

Let $f, g\colon U \to \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M,d)$ metric space, be *fuzzy real number valued functions*. The distance between f, g is defined by

$$D^*(f,g) := \sup_{x\in U} D(f(x), g(x)).$$

Denote $[f]^r = [f_-^{(r)}, f_+^{(r)}]$ for $[f(x)]^r = [f_-^{(r)}(x), f_+^{(r)}(x)]$, $\forall x \in U$, $r \in [0,1]$.

We need the following lemmas.

Lemma 1. (see Anastassiou and Gal [6]) *For any $a, b \in \mathbb{R}\colon ab \ge 0$ and any $u \in \mathbb{R}_{\mathcal{F}}$ we have*

$$D(a\odot u, b\odot u) \le |a-b| \cdot D(u, \tilde{o}),$$

where $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is defined by $\tilde{o} := \mathcal{X}_{\{0\}}$.

Lemma 2. (see Anastassiou and Gal [6])

(i) *If we denote $\tilde{o} := \mathcal{X}_{\{0\}}$, then $\tilde{o} \in \mathbb{R}_{\mathcal{F}}$ is the neutral element with respect to $\oplus$, i.e., $u \oplus \tilde{o} = \tilde{o} \oplus u = u$, $\forall u \in \mathbb{R}_{\mathcal{F}}$.*

(ii) *With respect to $\tilde{o}$, none of $u \in \mathbb{R}_{\mathcal{F}}$, $u \ne \tilde{o}$ has opposite in $\mathbb{R}_{\mathcal{F}}$.*

(iii) *Let $a, b \in \mathbb{R}\colon a\cdot b \ge 0$, and any $u \in \mathbb{R}_{\mathcal{F}}$, we have $(a+b)\odot u = a\odot u \oplus b\odot u$. For general $a, b \in \mathbb{R}$, the above property is false.*

(iv) *For any $\lambda \in \mathbb{R}$ and any $u, v \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda\odot(u\oplus v) = \lambda\odot u \oplus \lambda\odot v$.*

(v) *For any $\lambda, \mu \in \mathbb{R}$ and $u \in \mathbb{R}_{\mathcal{F}}$, we have $\lambda\odot(\mu\odot u) = (\lambda\cdot\mu)\odot u$.*

(vi) *If we denote $\|u\|_{\mathcal{F}} := D(u, \tilde{o})$, $\forall u \in \mathbb{R}_{\mathcal{F}}$, then $\|\cdot\|_{\mathcal{F}}$ has the properties of a usual norm on $\mathbb{R}_{\mathcal{F}}$, i.e.,*

$$\begin{aligned} \|u\|_{\mathcal{F}} &= 0 \text{ iff } u = \tilde{o}, \|\lambda\odot u\|_{\mathcal{F}} = |\lambda|\cdot\|u\|_{\mathcal{F}},\\ \|u\oplus v\|_{\mathcal{F}} &\le \|u\|_{\mathcal{F}} + \|v\|_{\mathcal{F}},\ \|u\|_{\mathcal{F}} - \|v\|_{\mathcal{F}} \le D(u,v). \end{aligned}$$

Notice that $(\mathbb{R}_{\mathcal{F}}, \oplus, \odot)$ is *not* a linear space over $\mathbb{R}$, and consequently $(\mathbb{R}_{\mathcal{F}}, \|\cdot\|_{\mathcal{F}})$ is not a normed space. Here $\sum^*$ denotes the fuzzy summation.

We use the following definition.

Definition 2. Let $U \subseteq (M,d)$ be a metric space and let $f\colon U \to \mathbb{R}_{\mathcal{F}}$. We define the (*first*) *fuzzy modulus of continuity* of f by

$$\omega_1^{(\mathcal{F})}(f;\delta) := \sup_{\substack{x,y\in U\\ d(x,y)\le\delta}} D(f(x), f(y))$$

for $0 < \delta \leq \text{diameter}(U)$. If $\delta > \text{diam}(U)$ then we define

$$\omega_1^{(\mathcal{F})}(f;\delta) := \omega_1^{(\mathcal{F})}(f;\text{diam}(U)).$$

Proposition 1. (see Anastassiou [5]) *Let $U \subseteq (M,d)$ be a metric space and $f\colon U \to \mathbb{R}_{\mathcal{F}}$. Assume that*

$$\omega_1^{(\mathcal{F})}(f,\delta), \quad \omega_1(f_-^{(r)},\delta), \quad \omega_1(f_+^{(r)},\delta)$$

are finite for any $\delta > 0$. Here ω_1 is the usual real modulus of continuity, i.e. for $g\colon U \to \mathbb{R}$ we define

$$\omega_1(g;\delta) := \sup_{\substack{x,y\in U \\ d(x,y)\leq\delta}} |g(x) - g(y)|,$$

etc. Then

$$\omega_1^{(\mathcal{F})}(f;\delta) = \sup_{r\in[0,1]} \max\bigl\{\omega_1(f_-^{(r)},\delta), \omega_1(f_+^{(r)},\delta)\bigr\}.$$

We need the following definitions.

Definition 3. Let U be an open or compact $\subseteq (M,d)$ metric space and $f\colon U \to \mathbb{R}_{\mathcal{F}}$. We say that f is *fuzzy continuous at* $x_0 \in U$ iff whenever $x_n \to x_0$, then $D(f(x_n), f(x_0)) \to 0$. If f is continuous for every $x_0 \in U$, we then call f a *fuzzy continuous real number valued function.* We denote the related space by $C_{\mathcal{F}}(U)$. Similarly one defines $C_{\mathcal{F}}([a,b])$, $[a,b] \subseteq \mathbb{R}$, etc.

Definition 4. Let $f\colon K \to \mathbb{R}_{\mathcal{F}}$, and K be an open or compact $\subseteq (M,d)$ metric space. We call f a *fuzzy uniformly continuous real number valued function*, iff $\forall\varepsilon > 0$, $\exists\delta > 0$: whenever $d(x,y) \leq \delta$, $x,y \in K$, implies that $D(f(x), f(y)) \leq \varepsilon$. We denote the related space by $C_{\mathcal{F}}^U(K)$.

Definition 5. Let $f\colon U \to \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M,d)$ be a metric space. If $D(f(x),\tilde{o}) \leq M$, $\forall x \in U$, $M \geq 0$, we call f a *fuzzy bounded real number valued function.*

In particular if $f \in C_{\mathcal{F}}([a,b])$, $[a,b] \subseteq \mathbb{R}$, then f is a fuzzy bounded function, also $\omega_1^{(\mathcal{F})}(f;\delta) < \infty$ for any $0 < \delta \leq b-a$, etc. Also notice that $C_{\mathcal{F}}^U(K) = C_{\mathcal{F}}(K)$, for K compact $\subseteq (V, \|\cdot\|)$ real normed vector space.

We use the next proposition.

Proposition 2. (see Anastassiou [5]) *Let $K \subseteq (V, \|\cdot\|)$ be a real normed vector space and*

$$\omega_1^{(\mathcal{F})}(f;\delta) = \sup_{\substack{x,y\in K \\ \|x-y\|\leq\delta}} D(f(x), f(y)), \quad \delta > 0,$$

the fuzzy modulus of continuity for $f\colon K \to \mathbb{R}_{\mathcal{F}}$. Then:

(1) *If $f \in C_{\mathcal{F}}^U(K)$, K open convex or compact convex $\subseteq (V, \|\cdot\|)$, then $\omega_1^{(\mathcal{F})}(f;\delta) < \infty$, $\forall\delta > 0$.*

(2) *Assume that K is open convex or compact convex $\subseteq (V, \|\cdot\|)$, then $\omega_1^{(\mathcal{F})}(f;\delta)$ is continuous on $\mathbb{R}_+$ in δ for $f \in C_{\mathcal{F}}^U(K)$.*

(3) *Assume that K is convex, then*

$$\omega_1^{(\mathcal{F})}(f, t_1 + t_2) \leq \omega_1^{(\mathcal{F})}(f, t_1) + \omega_1^{(\mathcal{F})}(f, t_2), \quad t_1, t_2 \geq 0,$$

that is the subadditivity property is true. Also it holds

$$\omega_1^{(\mathcal{F})}(f, n\delta) \leq n\omega_1^{(\mathcal{F})}(f, \delta),$$

and

$$\omega_1^{(\mathcal{F})}(f, \lambda\delta) \leq \lceil\lambda\rceil\omega_1^{(\mathcal{F})}(f, \delta) \leq (\lambda + 1)\omega_1^{(\mathcal{F})}(f, \delta),$$

where $n \in \mathbb{N}$, $\lambda > 0$, $\delta > 0$, $\lceil\cdot\rceil$ is the ceiling of the number.

(4) *Clearly in general $\omega_1^{(\mathcal{F})}(f;\delta) \geq 0$ and is increasing in $\delta > 0$ and $\omega_1^{(\mathcal{F})}(f;0) = 0$.*

(5) *Let K be open or compact $\subseteq (V, \|\cdot\|)$. Then $\omega_1^{(\mathcal{F})}(f;\delta) \to 0$ as $\delta \downarrow 0$ iff $f \in C_{\mathcal{F}}^U(K)$.*

(6) *It holds*

$$\omega_1^{(\mathcal{F})}(f \oplus g;\delta) \leq \omega_1^{(\mathcal{F})}(f;\delta) + \omega_1^{(\mathcal{F})}(g;\delta),$$

for $\delta > 0$, any $f, g\colon K \to \mathbb{R}_{\mathcal{F}}$, $K \subseteq (V, \|\cdot\|)$ is arbitrary.

We also need the next results.

Lemma 3. (see Anastassiou [5]) *Let K be a compact subset of the real normed vector space $(V, \|\cdot\|)$ and $f \in C_{\mathcal{F}}(K)$. Then f is a fuzzy bounded function.*

Proposition 3. (see Anastassiou [5]) *Let U be an open of compact $\subseteq (M, d)$ metric space, $f \in C_{\mathcal{F}}(U)$. Then $f_{\pm}^{(r)}$ are equicontinuous with respect to $r \in [0,1]$ over U, respectively in $\pm$.*

For the reverse we have:

Proposition 4. (see Anastassiou [5]) *Let $f_-^{(r)}$, $f_+^{(r)}$ be equicontinuous with respect to $r \in [0,1]$ on U-open or compact $\subseteq (M, d)$-metric space, respectively in $\pm$, then $f \in C_{\mathcal{F}}(U)$.*

We mention:

Definition 6. (see Wu and Gong [17]) Let $x, y \in \mathbb{R}_{\mathcal{F}}$. If there exists a $z \in \mathbb{R}_{\mathcal{F}}$, such that $x = y + z$, then, we call z the H-difference of x and y, denoted by $z := x - y$.

Definition 7. (see Wu and Gong [17]) Let $T := [x_0, x_0 + \beta] \subset \mathbb{R}$, with $\beta > 0$. A function $f\colon T \to \mathbb{R}_{\mathcal{F}}$ is differentiable at $x \in T$, if there exists a

$f'(x) \in \mathbb{R}_{\mathcal{F}}$, such that the limits

$$\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}, \quad \lim_{h \to 0^+} \frac{f(x) - f(x-h)}{h},$$

exist and are equal to $f'(x)$. We call f' the derivative of f at x. If f is differentiable at any $x \in T$, we call f differentiable and it has derivative over T, the function f'.

Similarly we define higher order fuzzy derivatives. Regarding functions of several variables one can define the same way partial derivatives in the fuzzy sense.

Let Q be a compact convex subset of $\mathbb{R}^k$, $k > 1$ and $n \in \mathbb{N}$. By $C^n_{\mathcal{F}}(Q)$ we mean all the functions from Q into $\mathbb{R}_{\mathcal{F}}$ that are n-times continuously differentiable in the fuzzy sense.

We use:

Theorem 1. (see Kaleva [11]) *Let $f\colon [a,b] \subseteq \mathbb{R} \to \mathbb{R}_{\mathcal{F}}$ be fuzzy differentiable. Let $t \in [a,b]$, $0 \le r \le 1$. Clearly*

$$[f(t)]^r = \big[(f(t))^{(r)}_-, (f(t))^{(r)}_+\big] \subseteq \mathbb{R}. \tag{1}$$

Then $(f(t))^{(r)}_\pm$ are differentiable and

$$[f'(t)]^r = \big[((f(t))^{(r)}_-)', ((f(t))^{(r)}_+)'\big], \tag{2}$$

i.e.

$$(f')^{(r)}_\pm = (f^{(r)}_\pm)', \quad \textit{for any } r \in [0,1]. \tag{3}$$

We make:

Remark 1. 1) Let $f \in C^n_{\mathcal{F}}([a,b])$. Then by Theorem 1 and Proposition 3 we obtain $f^{(r)}_\pm \in C^n([a,b])$ and

$$[f^{(i)}(t)]^r = \big[((f(t))^{(r)}_-)^{(i)}, ((f(t))^{(r)}_+)^{(i)}\big], \tag{4}$$

for $i = 0, 1, 2, \dots, n$, and, in particular, we have that

$$(f^{(i)})^{(r)}_\pm = (f^{(r)}_\pm)^{(i)}, \tag{5}$$

for any $r \in [0,1]$.

2) Let $f \in C^n_{\mathcal{F}}(Q)$, denote $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$ and $0 < |\alpha| := \sum_{i=1}^k \alpha_i \le n$, $n > 1$. Then we get by Theorem 1 that

$$(f^{(r)}_\pm)_\alpha = (f_\alpha)^{(r)}_\pm, \tag{6}$$

for any $r \in [0,1]$ and any $\alpha\colon |\alpha| \le n$. Here $f^{(r)}_\pm \in C^n(Q)$.

We also make:

Remark 2. (see Anastassiou [3], Remark 3) Let $r \in [0,1]$, $x_i^{(r)}$, $y_i^{(r)} \in \mathbb{R}$, $i = 1, \ldots, m \in \mathbb{N}$. Assume

$$\sup_{r\in[0,1]} \max(x_i^{(r)}, y_i^{(r)}) \in \mathbb{R}, \quad i = 1, \ldots, m.$$

Then

$$\sup_{r\in[0,1]} \max\left(\sum_{i=1}^{m} x_i^{(r)}, \sum_{i=1}^{m} y_i^{(r)}\right) \leq \sum_{i=1}^{m} \sup_{r\in[0,1]} \max(x_i^{(r)}, y_i^{(r)}). \tag{7}$$

We use:

Lemma 4. *Let $r \in [0,1]$, $\alpha \in I$, and let I be a finite index set, and $x_\alpha^{(r)}, y_\alpha^{(r)} \in \mathbb{R}$. Assume*

$$A := \max_{\alpha\in I} \sup_{r\in[0,1]} \max\{x_\alpha^{(r)}, y_\alpha^{(r)}\} \in \mathbb{R}$$

and

$$B := \sup_{r\in[0,1]} \max\left\{\max_{\alpha\in I} x_\alpha^{(r)}, \max_{\alpha\in I} y_\alpha^{(r)}\right\} \in \mathbb{R}. \tag{8}$$

It holds

$$A = B. \tag{9}$$

Proof. 1) First we prove $B \leq A$. We see that

$$x_\alpha^{(r)}, y_\alpha^{(r)} \leq A, \quad \forall r \in [0,1] \text{ and } \forall \alpha \in I.$$

Then

$$A_1 := \max_{\alpha\in I} x_\alpha^{(r)} \leq A, \quad A_2 := \max_{\alpha\in I} y_\alpha^{(r)} \leq A$$

and

$$\max(A_1, A_2) \leq A.$$

Thus

$$\sup_{r\in[0,1]} \max(A_1, A_2) \leq A.$$

2) We last prove $A \leq B$. We notice

$$x_\alpha^{(r)} \leq \max_{\alpha\in I} x_\alpha^{(r)}, \quad \forall r \in [0,1], \quad \forall \alpha \in I$$

and

$$y_\alpha^{(r)} \leq \max_{\alpha\in I} y_\alpha^{(r)}, \quad \forall r \in [0,1], \quad \forall \alpha \in I.$$

Then

$$\max\{x_\alpha^{(r)}, y_\alpha^{(r)}\} \leq \max\left\{\max_{\alpha\in I} x_\alpha^{(r)}, \max_{\alpha\in I} y_\alpha^{(r)}\right\}, \quad \forall r \in [0,1],\ \forall \alpha \in I.$$

Furthermore

$$\sup_{r\in[0,1]} \max\{x_\alpha^{(r)}, y_\alpha^{(r)}\} \le \sup_{r\in[0,1]} \max\left\{\max_{\alpha\in I} x_\alpha^{(r)}, \max_{\alpha\in I} y_\alpha^{(r)}\right\} = B, \quad \forall \alpha \in I,$$

and finally

$$\max_{\alpha\in I} \sup_{r\in[0,1]} \max\{x_\alpha^{(r)}, y_\alpha^{(r)}\} \le B. \qquad \square$$

We mention:

Definition 8. Let $L\colon C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$, where U is open or compact $\subseteq (M,d)$ metric space, such that

$$L(c_1 f + c_2 g) = c_1 L(f) + c_2 L(g), \quad \forall c_1, c_2 \in \mathbb{R}. \tag{10}$$

We call L a *fuzzy linear operator.*

We give the following example of a fuzzy linear operator, etc.

Definition 9. Let $f\colon [0,1] \to \mathbb{R}_{\mathcal{F}}$ be a fuzzy real function. The fuzzy algebraic polynomial defined by

$$B_N^{(\mathcal{F})}(f)(x) = \sum_{k=0}^{N}{}^{*} \binom{N}{k} x^k (1-x)^{N-k} \odot f\left(\frac{k}{N}\right), \quad \forall x \in [0,1],\ N \in \mathbb{N}, \tag{11}$$

will be called the *fuzzy Bernstein operator.*

We do have:

Theorem 2. (see Gal [9], p. 642) *If $f \in C_{\mathcal{F}}([0,1])$, then*

$$D\big(B_N^{(\mathcal{F})}(f)(x), f(x)\big) \le \frac{3}{2}\omega_1^{(\mathcal{F})}\left(f; \frac{1}{\sqrt{N}}\right), \quad N \in \mathbb{N},\ \forall x \in [0,1], \tag{12}$$

i.e.,

$$\lim_{N\to+\infty} D^*\big(B_N^{(\mathcal{F})}(f), f\big) = 0, \tag{13}$$

that is $B_N^{(\mathcal{F})} f \to_{n\to+\infty} f$, fuzzy uniform convergence.

We also need:

Definition 10. Let $f, g\colon U \to \mathbb{R}_{\mathcal{F}}$, $U \subseteq (M,d)$ metric space. We denote $f \succsim g$, iff $f(x) \succsim g(x)$, $\forall x \in U$, iff $f_+^{(r)}(x) \ge g_+^{(r)}(x)$ and $f_-^{(r)}(x) \ge g_-^{(r)}(x)$, $\forall x \in U$, $\forall r \in [0,1]$, iff $f_+^{(r)} \ge g_+^{(r)}$ and $f_-^{(r)} \ge g_-^{(r)}$, $\forall r \in [0,1]$.

We give:

Definition 11. Let $L\colon C_{\mathcal{F}}(U) \hookrightarrow C_{\mathcal{F}}(U)$ be a fuzzy linear operator, U open or compact $\subseteq (M,d)$ metric space. We say that L is *positive*, iff whenever $f, g \in C_{\mathcal{F}}(U)$ are such that $f \succsim g$ then $L(f) \succsim L(g)$, iff

$$(L(f))_+^{(r)} \ge (L(g))_+^{(r)} \tag{14}$$

and

$$(L(f))_-^{(r)} \geq (L(g))_-^{(r)}, \quad \forall r \in [0,1]. \tag{15}$$

Here we denote

$$[L(f)]^r = \big[(L(f))_-^{(r)}, (L(f))_+^{(r)}\big], \quad \forall r \in [0,1]. \tag{16}$$

An example of a fuzzy positive linear operator is the fuzzy Bernstein operator on the domain $[0,1]$, etc.

We will use this type of supposition.

Assumption 1. Let L be a fuzzy positive linear operator from $C_{\mathcal{F}}(K)$, K compact $\subseteq (M,d)$ metric space, into itself. Here we *assume* that there exists a positive linear operator $\tilde{L}$ from $C(K)$ into itself with the property

$$(Lf)_\pm^{(r)} = \tilde{L}(f_\pm^{(r)}), \tag{17}$$

respectively, for all $r \in [0,1]$, $\forall f \in C_{\mathcal{F}}(K)$.

As an example again we mention the fuzzy Bernstein operator and the real Bernstein operator fulfilling the above assumption on $[0,1]$, etc.

We will use

Theorem A. *Let (X,O) be a linearly ordered vector space, (O denotes the order) and G a majorant subspace of X (i.e. for any x in X there exist z and y in G such that $zO \times Oy$). If $f\colon G \to \mathbb{R}$ is a positive linear map, then there exists (a not necessarily unique) $F\colon X \to \mathbb{R}$ positive linear map, such that $F(x) = f(x)$, $\forall x \in G$.*

Above Theorem A is a special case of the famous Kantorovich Extension Theorem, due to Kantorovich [12]. For a proof see Aliprantis and Burkinshaw [1], Theorem 2.8, p. 26, see also Cristescu [8], p. 72, Proposition 1.

We need:

Lemma 5. *Let $\tilde{L}\colon C^n(Q) \to C(Q)$ be a positive linear operator, $n \in \mathbb{N}$, Q compact $\subseteq \mathbb{R}^k$, $k \geq 1$. Then there exists a unique finite Borel measure μ_x, $x \in Q$, such that*

$$(\tilde{L}(f))(x) = \int_Q f(t)d\mu_x(t), \quad \forall f \in C^n(Q).$$

Notice $(\tilde{L}(1))(x) = \mu_x(Q) < \infty$.

Proof. Clearly $(\tilde{L}(\cdot))(x)$ is a positive linear functional on $C^n(Q)$. Also since $1 \in C^n(Q)$, $C^n(Q)$ is a majorant subspace of $C(Q)$. Thus, by Theorem A there exists a positive linear functional $M\colon C(Q) \to \mathbb{R}$ extending $\tilde{L}$, i.e.

$\tilde{L} = M\big|_{C^n(Q)}$. Consequently, by Riesz Representation theorem there exists a unique finite Borel measure μ_x for this M, such that

$$M(f) = \int_Q f(t)d\mu_x(t), \quad \forall f \in C(Q).$$

Therefore

$$\tilde{L}(f)(x) = \int_Q f(t)d\mu_x(t), \quad \forall f \in C^n(Q).$$

We prove uniqueness μ_x regarding $\tilde{L}(\cdot)(x)$. Let us assume that there exists another finite Borel measure ν such that

$$\int_Q f(t)d\mu_x(t) = \int_Q f(t)d\nu(t), \quad \forall f \in C^n(Q).$$

In particular, we have

$$\int_Q p(t)d\mu_x(t) = \int_Q p(t)d\nu(t),$$

for all polynomials p. Since by the Stone–Weierstrass Approximation Theorem the polynomials are uniformly dense in $C(Q)$, it follows for any $f \in C(Q)$ that there exists a sequence $\{p_n\}_{n\in\mathbb{N}}$ of polynomials that converges uniformly to f on Q. In particular $\{p_n\}$ is uniformly bounded. Now from

$$\int_Q p_n(t)d\mu_x(t) = \int_Q p_n(t)d\nu(t)$$

and the Lebesgue Dominated Convergence Theorem, by taking the limits in the last equation, we find

$$\int_Q f(t)d\mu_x(t) = \int_Q f(t)d\nu(t).$$

Hence, by the uniqueness of the measure in the Riesz Representation Theorem, we get indeed that $\mu_x = \nu$. $\square$

We give:

Remark 3. We recall from Anastassiou [2], p. 210–211 the function

$$\phi_n(x) := \int_0^{|x|} \left\lceil \frac{t}{h} \right\rceil \frac{(|x| - t)^{n-1}}{(n-1)!} dt, \tag{18}$$

$x \in \mathbb{R}$, $n \in \mathbb{N}$, where $\lceil\cdot\rceil$ is the ceiling of the number.

We have

$$\phi_n(x) = \int_0^{|x|} \int_0^{x_1} \cdots \left(\int_0^{x_{n-1}} \left\lceil \frac{x_n}{h} \right\rceil dx_n \right) \cdots dx_1, \tag{19}$$

and

$$\phi_n(x) = \frac{1}{n!}\left(\sum_{j=0}^{\infty}(|x| - jh)_+^n\right), \quad x \in \mathbb{R}. \tag{20}$$

Also it holds

$$\phi_n(x) \le \left(\frac{|x|^{n+1}}{(n+1)!h} + \frac{|x|^n}{2n!} + \frac{h|x|^{n-1}}{8(n-1)!}\right), \tag{21}$$

and

$$\phi_n(x) = \int_0^x \phi_{n-1}(t)dt, \quad x \in \mathbb{R}_+, \quad n \in \mathbb{N}. \tag{22}$$

And from Anastassiou [2], p. 217 we have

$$\phi_n(x) \le \frac{|x|^n}{n!}\left(1 + \frac{|x|}{(n+1)h}\right), \quad x \in \mathbb{R}, \quad n \in \mathbb{N}. \tag{23}$$

3. UNIVARIATE RESULTS

We need to mention:

Theorem 3. (by Corollary 7.2.2, p. 219-220 in Anastassiou [2] and Geometric Moment Theory, Kemperman [13]) *Consider the positive linear operator*

$$\tilde{L}\colon C([a,b]) \to C([a,b]). \tag{24}$$

Let

$$\begin{aligned} c_k(x) &:= \tilde{L}((t-x)^k, x), \quad k = 0, 1, \ldots, n, \ n \in \mathbb{N}; \\ d_n(x) &:= \left[\tilde{L}(|t-x|^n, x)\right]^{1/n}; \\ c(x) &:= \max(x-a, b-x) \ \ (c(x) \ge (b-a)/2). \end{aligned} \tag{25}$$

Let $g \in C^n([a,b])$ *such that* $\omega_1(g^{(n)}, h) \le w$, *where* $w, h > 0$, $0 < h \le b-a$. *Then*

$$\begin{aligned} |\tilde{L}(g,x) - g(x)| \ &\le \ |g(x)|\,|c_0(x) - 1| + \sum_{k=1}^{n} \frac{|g^{(k)}(x)|}{k!}|c_k(x)| \\ &\quad + w\phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n, \quad \forall x \in [a,b]. \end{aligned} \tag{26}$$

Inequality (26) is sharp. It is attained in a certain sense by $w\phi_n((t-x)_+)$ *and a measure* $\tilde{\mu}_x$ *supported by* $\{x, b\}$ *when* $x - a \le b - x$, *also attained by* $w\phi_n((x-t)_+)$ *and a measure* $\tilde{\mu}_x$ *supported by* $\{x, a\}$ *when* $x - a \ge b - x$: *in each case with masses* $c_0(x) - \left(\frac{d_n(x)}{c(x)}\right)^n$ *and* $\left(\frac{d_n(x)}{c(x)}\right)^n$, *respectively.*

Note 1. Assuming $\omega_1(g^{(n)}, h) > 0$ for $h > 0$ and all as in the context of Theorem 3. We prefer to write

$$|\tilde{L}(g,x) - g(x)| \leq |g(x)|\,|c_0(x) - 1| + \sum_{k=1}^{n} \frac{|g^{(k)}(x)|}{k!}|c_k(x)| + \phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n \omega_1(g^{(n)}, h), \quad \forall x \in [a,b]. \tag{27}$$

We need also:

Theorem 4. (by Theorem 7.3.5, p. 231–232 in Anastassiou [2] and Lemma 5) *Consider the positive linear operator*

$$\tilde{L}\colon C^n([a,b]) \to C([a,b]), \quad n \in \mathbb{N}.$$

Assume that

$$\tilde{L}(1,x) > 0,$$

and

$$\tilde{L}\big(|t-x|^{n+1}, x\big) > 0, \quad x \in [a,b]. \tag{28}$$

Consider also $\rho > 0$. *Consider* $g \in C^n([a,b])$, $n \geq 1$, *with* $\omega_1(g^{(n)}, \delta) > 0$ *for any* $\delta > 0$. *Then*

$$|\tilde{L}(g,x) - g(x)| \leq |g(x)|\,|\tilde{L}(1,x) - 1| + \sum_{k=1}^{n} \frac{|g^{(k)}(x)|}{k!}\big|\tilde{L}((t-x)^k, x)\big| + \left[\frac{n\rho^2}{8} + \frac{\rho}{2} + \frac{1}{(n+1)}\right]\frac{\big(\tilde{L}(1,x)\big)^{1/(n+1)}}{\rho n!}\big(\tilde{L}(|t-x|^{n+1}, x)\big)^{n/(n+1)} \times \omega_1\left(g^{(n)}, \rho\left(\frac{\tilde{L}(|t-x|^{n+1}, x)}{\tilde{L}(1,x)}\right)^{\frac{1}{(n+1)}}\right), \ x \in [a,b], \ n \in \mathbb{N}. \tag{29}$$

We will use also:

Theorem 5. (by Theorem 7.2.2, p. 216–217, in Anastassiou [2] and Lemma 5) *Consider the positive linear operator*

$$\tilde{L}\colon C^n([a,b]) \to C([a,b]), \quad n \in \mathbb{N}.$$

Assume that

$$\tilde{L}(1,x) > 0, \tag{30}$$

and

$$\tilde{L}(|t-x|^{n+1}, x) > 0, \quad x \in [a,b].$$

Consider $g \in C^n([a,b])$, $n \geq 1$, *with* $\omega_1(g^{(n)}, \delta) > 0$ *for any* $\delta > 0$. *Then*

$$|\tilde{L}(g,x) - g(x)| \leq |g(x)|\,|\tilde{L}(1,x) - 1| + \sum_{k=1}^{n} \frac{|g^{(k)}(x)|}{k!}|\tilde{L}((t-x)^k, x)|$$

$$+\frac{\big(\tilde{L}(|t-x|^{n+1},x)\big)^{n/(n+1)}}{n!}\bigg((\tilde{L}(1)(x))^{\frac{1}{(n+1)}}$$

$$+\frac{1}{(n+1)}\bigg)\omega_1\big(g^{(n)},(\tilde{L}(|t-x|^{n+1},x))^{\frac{1}{n+1}}\big),$$

$$x\in[a,b],\ n\in\mathbb{N}. \tag{31}$$

Remark 4. 1) If $\|\tilde{L}(1)\|_\infty=0$, then $\tilde{L}=0$ the trivial operator, therefore without loss of generality we may assume $\tilde{L}\not\equiv 0$, and as a result we have $\|\tilde{L}(1)\|_\infty>0$.

2) By using Hölder's inequality and Riesz Representation Theorem, for $\tilde{L}$ as in (24), we easily derive that

$$|\tilde{L}((t-x)^k,x)|\le\big(\tilde{L}(1,x)\big)^{1-\frac{k}{n}}\big(\tilde{L}(|t-x|^n,x)\big)^{\frac{k}{n}}, \tag{32}$$

under the assumption $\tilde{L}(1,x)>0$, $0<k\le n$. And it holds

$$\|\tilde{L}((t-x)^k,x)\|_\infty\le\|\tilde{L}(1)\|_\infty^{1-\frac{k}{n}}\|\tilde{L}(|t-x|^n,x)\|_\infty^{\frac{k}{n}},$$

$$0<k\le n,\ \tilde{L}\not\equiv 0. \tag{33}$$

Similarly, by Lemma 5, for $\tilde{L}\colon C^n([a,b])\to C([a,b])$ positive linear operator, it holds

$$|\tilde{L}((t-x)^k,x)|\le\big(\tilde{L}(1,x)\big)^{1-\frac{k}{(n+1)}}\big(\tilde{L}(|t-x|^{n+1},x)\big)^{\frac{k}{(n+1)}},$$

$$0<k<n+1. \tag{34}$$

And also it holds

$$\|\tilde{L}((t-x)^k,x)\|_\infty\le\|\tilde{L}(1)\|_\infty^{1-\frac{k}{(n+1)}}\|\tilde{L}(|t-x|^{n+1},x)\|_\infty^{\frac{k}{(n+1)}},$$

$$0<k<n+1. \tag{35}$$

3) If $\tilde{L}(1,x)=0$, $x\in[a,b]$, then easily we get that $\tilde{L}(g,x)=0$, $\forall g\in C^n([a,b])$. So that inequalities (26), (27) and (31) hold trivially.

4) If $d_n(x)=0$, $x\in[a,b]$, then we get that $c_k(x)=0$, $k=0,1,\dots,n$, since the associated measure μ_x is concentrated at $\{x\}$ only, etc., and inequalities (26) and (27) hold again, in fact as equalities.

5) Similarly, if $\tilde{L}(|t-x|^{n+1},x)=0$, $x\in[a,b]$, then we get again $\tilde{L}((t-x)^k,x)=0$, $k=1,\dots,n$, since the associated measure μ_x is concentrated at $\{x\}$ only, etc., and inequalities (29) and (31) hold again, in fact as equalities.

6) If $\omega_1(g^{(n)},h)=0$ for some $h>0$ then $g^{(n)}$ is the constant function, furthermore inequalities (26), (27), (29) and (31) are again valid.

We give our first main fuzzy result.

Theorem 6. *Consider the fuzzy positive linear operator*

$$L\colon C^n_{\mathcal{F}}([a,b]) \to C_{\mathcal{F}}([a,b]), \quad n \in \mathbb{N}, \tag{36}$$

with the property

$$(Lf)^{(r)}_{\pm} = \tilde{L}(f^{(r)}_{\pm}), \tag{37}$$

respectively, for all $r \in [0,1]$, $\forall f \in C^n_{\mathcal{F}}([a,b])$. *Here* $\tilde{L}$ *is a positive linear operator such that*

$$\tilde{L}\colon C([a,b]) \to C([a,b]). \tag{38}$$

Let

$$\begin{aligned} c_k(x) &:= \tilde{L}\big((t-x)^k, x\big), \quad k = 0,1,\dots,n; \\ d_n(x) &:= \big(\tilde{L}(|t-x|^n, x)\big)^{1/n}; \\ c(x) &:= \max(x-a, b-x). \end{aligned} \tag{39}$$

Let $f \in C^n_{\mathcal{F}}([a,b])$ *such that* $\omega_1^{(\mathcal{F})}(f^{(n)}, h) > 0$ *for any* $h > 0$. *Then:*

1)

$$D\big((Lf)(x), f(x)\big) \le |c_0(x) - 1| D(f(x), \tilde{o}) + \sum_{k=1}^{n} \frac{|c_k(x)|}{k!} D(f^{(k)}(x), \tilde{o})$$
$$+\ \phi_n(c(x)) \left(\frac{d_n(x)}{c(x)}\right)^n \omega_1^{(\mathcal{F})}(f^{(n)}, h), \quad x \in [a,b], \tag{40}$$

and

2)

$$D^*(Lf, f) \le \|\tilde{L}1 - 1\|_\infty D^*(f, \tilde{o}) + \sum_{k=1}^{n} \frac{\|\tilde{L}((t-x)^k, x)\|_\infty}{k!} D^*(f^{(k)}, \tilde{o})$$
$$+\ \|\phi_n(c(x))\|_\infty \left\| \frac{\tilde{L}(|t-x|^n, x)}{c(x)^n} \right\|_\infty \omega_1^{(\mathcal{F})}(f^{(n)}, h). \tag{41}$$

Proof. We have the following

$$D((Lf)(x), f(x))$$
$$= \sup_{r\in[0,1]} \max\big\{|(Lf)^{(r)}_{-}(x) - f^{(r)}_{-}(x)|, |(Lf)^{(r)}_{+}(x) - f^{(r)}_{+}(x)|\big\}$$
$$= \sup_{r\in[0,1]} \max\big\{|(\tilde{L}(f^{(r)}_{-}))(x) - f^{(r)}_{-}(x)|, |(\tilde{L}(f^{(r)}_{+}))(x) - f^{(r)}_{+}(x)|\big\}$$

(by Remark 1 (1) and (27))

$$\le \sup_{r\in[0,1]} \max\Bigg\{ |f^{(r)}_{-}(x)|\,|c_0(x) - 1| + \sum_{k=1}^{n} \frac{|(f^{(k)})^{(r)}_{-}(x)|}{k!} |c_k(x)|$$

$$+\ \phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n \omega_1\big((f^{(n)})_-^{(r)},h\big), |f_+^{(r)}(x)|\,|c_0(x)-1|$$

$$+\ \sum_{k=1}^{n}\frac{|(f^{(k)})_+^{(r)}(x)|}{k!}|c_k(x)| + \phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n \omega_1\big((f^{(n)})_+^{(r)},h\big)\Bigg\}$$

$$\overset{(7)}{\le}\ |c_0(x)-1| \sup_{r\in[0,1]} \max\big\{|f_-^{(r)}(x)|, |f_+^{(r)}(x)|\big\}$$

$$+\ \sum_{k=1}^{n}\frac{|c_k(x)|}{k!}\sup_{r\in[0,1]}\max\big\{|(f^{(k)})_-^{(r)}(x)|, |(f^{(k)})_+^{(r)}(x)|\big\}$$

$$+\ \phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n \sup_{r\in[0,1]}\max\big\{\omega_1((f^{(n)})_-^{(r)},h), \omega_1((f^{(n)})_+^{(r)},h)\big\}$$

$$\overset{\text{(by Proposition 1)}}{=}\ |c_0(x)-1|D(f(x),\tilde{o})$$

$$+\ \sum_{k=1}^{n}\frac{|c_k(x)|}{k!}D(f^{(k)}(x),\tilde{o}) + \phi_n(c(x))\left(\frac{d_n(x)}{c(x)}\right)^n \omega_1^{(\mathcal{F})}(f^{(n)},h).$$

That is proving (40). □

We proceed with our next main result.

Theorem 7. *Consider the fuzzy positive linear operator*

$$L\colon C_{\mathcal{F}}^n([a,b]) \to C_{\mathcal{F}}([a,b]),\quad n\in\mathbb{N},$$

with the property

$$(Lf)_\pm^{(r)} = \tilde{L}(f_\pm^{(r)}),$$

respectively, for all $r\in[0,1]$, $\forall f\in C_{\mathcal{F}}^n([a,b])$. *Here* $\tilde{L}$ *is a positive linear operator from* $C^n([a,b])$ *into* $C([a,b])$. *Additionally assume that* $\tilde{L}(1,x)>0$, $x\in[a,b]$ *and consider* $\rho>0$. *Let* $f\in C_{\mathcal{F}}^n([a,b])$ *such that* $\omega_1^{(\mathcal{F})}(f^{(n)},h)>0$ *for any* $h>0$. *Then:*

1)

$$D((Lf)(x),f(x)) \le |\tilde{L}(1,x)-1|D(f(x),\tilde{o})$$

$$+\sum_{k=1}^{n}\frac{|\tilde{L}((t-x)^k,x)|}{k!}D(f^{(k)}(x),\tilde{o}) + \left[\frac{n\rho^2}{8}+\frac{\rho}{2}+\frac{1}{(n+1)}\right]$$

$$\times\ \frac{\big(\tilde{L}(1,x)\big)^{\frac{1}{(n+1)}}}{\rho n!}\big(\tilde{L}(|t-x|^{n+1},x)\big)^{\frac{n}{(n+1)}}$$

$$\times\ \omega_1^{(\mathcal{F})}\left(f^{(n)}, \rho\left(\frac{\tilde{L}(|t-x|^{n+1},x)}{\tilde{L}(1,x)}\right)^{\frac{1}{(n+1)}}\right),\ n\in\mathbb{N},\ x\in[a,b]. \quad (42)$$

2) *And by assuming* $\tilde{L}(1,x) > 0$, $\forall x \in [a,b]$, *it holds*

$$D^*(Lf, f) \le \|\tilde{L}1 - 1\|_\infty D^*(f, \tilde{o}) + \sum_{k=1}^{n} \frac{\|\tilde{L}((t-x)^k, x)\|_\infty}{k!} D^*(f^{(k)}, \tilde{o})$$
$$+ \left[\frac{n\rho^2}{8} + \frac{\rho}{2} + \frac{1}{(n+1)}\right] \frac{\|\tilde{L}(1,x)\|_\infty^{\frac{1}{(n+1)}}}{\rho n!} \|\tilde{L}(|t-x|^{n+1}, x)\|_\infty^{\frac{n}{(n+1)}}$$
$$\times \omega_1^{(\mathcal{F})}\left(f^{(n)}, \rho \left\|\frac{\tilde{L}(|t-x|^{n+1}, x)}{\tilde{L}(1,x)}\right\|_\infty^{\frac{1}{(n+1)}}\right). \quad (43)$$

Proof. Here we are using (29), see also Remark 4(5). The proof is similar to the proof of Theorem 6 and is omitted. □

We present:

Theorem 8. *Consider the fuzzy positive linear operator*

$$L\colon C^n_{\mathcal{F}}([a,b]) \to C_{\mathcal{F}}([a,b]), \quad n \in \mathbb{N}, \quad (44)$$

with the property

$$(Lf)^{(r)}_{\pm} = \tilde{L}(f^{(r)}_{\pm}), \quad (45)$$

respectively, for all $r \in [0,1]$, $\forall f \in C^n_{\mathcal{F}}([a,b])$. *Here* $\tilde{L}$ *is a positive linear operator such that*

$$\tilde{L}\colon C^n([a,b]) \to C([a,b]). \quad (46)$$

Let $f \in C^n_{\mathcal{F}}([a,b])$ *such that* $\omega_1^{(\mathcal{F})}(f^{(n)}, h) > 0$ *for any* $h > 0$. *Then*

1)

$$D((Lf)(x), f(x)) \le |\tilde{L}(1,x) - 1| D(f(x), \tilde{o})$$
$$+ \sum_{k=1}^{n} \frac{|(\tilde{L}((\cdot - x)^k))(x)|}{k!} D(f^{(k)}, (x), \tilde{o})$$
$$+ \frac{\left(\tilde{L}(|\cdot - x|^{n+1})(x)\right)^{\frac{n}{(n+1)}}}{n!} \left(\left(\tilde{L}(1,x)\right)^{\frac{1}{(n+1)}} + \frac{1}{(n+1)}\right)$$
$$\times \omega_1^{(\mathcal{F})}\left(f^{(n)}, (\tilde{L}(|\cdot - x|^{n+1})(x))^{\frac{1}{(n+1)}}\right), \quad \forall x \in [a,b], \quad (47)$$

and also it holds

2)

$$D^*(Lf, f) \le \|\tilde{L}1 - 1\|_\infty D^*(f, \tilde{o}) + \sum_{k=1}^{n} \frac{\|(\tilde{L}((\cdot - x)^k))(x)\|_\infty}{k!} D^*(f^{(k)}, \tilde{o})$$

$$+\ \frac{\big\|(\tilde{L}(|\cdot-x|^{n+1}))(x)\big\|_{\infty}^{\frac{n}{(n+1)}}}{n!}\left\|\big(\tilde{L}(1)\big)^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right\|_{\infty}$$

$$\times\ \omega_1^{(\mathcal{F})}\big(f^{(n)},\big\|(\tilde{L}(|\cdot-x|^{n+1}))(x)\|_{\infty}^{\frac{1}{n+1}}\big). \quad (48)$$

Proof. Here we use (31) and see also Remark 4 (3) and (5). We do have

$$\begin{aligned}
&D((Lf)(x),f(x))\\
&=\sup_{r\in[0,1]}\max\big\{|(Lf)_-^{(r)}(x)-f_-^{(r)}(x)|,|(Lf)_+^{(r)}(x)-f_+^{(r)}(x)|\big\}\\
&\quad=\sup_{r\in[0,1]}\max\big\{|\tilde{L}(f_-^{(r)})(x)-f_-^{(r)}(x)|,|\tilde{L}(f_+^{(r)})(x)-f_+^{(r)}(x)|\big\}\\
&\quad\le\sup_{r\in[0,1]}\max\Bigg\{|f_-^{(r)}(x)|\,|\tilde{L}(1,x)-1|\\
&\qquad+\sum_{k=1}^{n}\frac{|(f^{(k)})_-^{(r)}(x)|}{k!}|(\tilde{L}((\cdot-x)^k))(x)|\\
&\qquad+\frac{((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{n}{(n+1)}}}{n!}\left((\tilde{L}(1,x))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right)\\
&\qquad\times\omega_1\big((f^{(n)})_-^{(r)},((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{1}{(n+1)}}\big),|f_+^{(r)}(x)|\,|\tilde{L}(1,x)-1|\\
&\qquad+\sum_{k=1}^{n}\frac{|(f^{(k)})_+^{(r)}(x)|}{k!}|(\tilde{L}((\cdot-x)^k))(x)|\\
&\qquad+\frac{((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{n}{(n+1)}}}{n!}\left((\tilde{L}(1,x))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right)\\
&\qquad\times\omega_1\left((f^{(n)})_+^{(r)},((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{1}{(n+1)}}\right)\Bigg\}\\
&\quad\le|\tilde{L}(1,x)-1|\sup_{r\in[0,1]}\max\big\{|f_-^{(r)}(x)|,|f_+^{(r)}(x)|\big\}\\
&\qquad+\sum_{k=1}^{n}\frac{|(\tilde{L}((\cdot-x)^k))(x)|}{k!}\sup_{r\in[0,1]}\max\big\{|(f^{(k)})_-^{(r)}(x)|,|(f^{(k)})_+^{(r)}(x)|\big\}\\
&\qquad+\frac{((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{n}{(n+1)}}}{n!}\left((\tilde{L}(1,x))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right)\\
&\qquad\times\sup_{r\in[0,1]}\max\big\{\omega_1\big((f^{(n)})_-^{(r)},((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{1}{(n+1)}}\big),\\
&\qquad\times\omega_1\big((f^{(n)})_+^{(r)},((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{1}{(n+1)}}\big)\big\}
\end{aligned}$$

$$= |\tilde{L}(1,x)-1|D(f(x),\tilde{o})+\sum_{k=1}^{n}\frac{|(\tilde{L}((\cdot-x)^k))(x)|}{k!}D(f^{(k)}(x),\tilde{o})$$

$$+\ \frac{(\tilde{L}(|\cdot-x|^{n+1})(x))^{\frac{n}{(n+1)}}}{n!}\left((\tilde{L}(1,x))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right)$$

$$\times\ \omega_1^{(\mathcal{F})}\big(f^{(n)},((\tilde{L}(|\cdot-x|^{n+1}))(x))^{\frac{1}{(n+1)}}\big).$$

Note 2. If $\omega_1^{(\mathcal{F})}(f^{(n)},h)=0$ for some $h>0$, then by Proposition 1 we get $\omega_1((f^{(n)})_-^{(r)},h)$, $\omega_1((f^{(n)})_+^{(r)},h)=0$, $\forall r\in[0,1]$, i.e. $(f_\pm^{(r)})^{(n)}=(f^{(n)})_\pm^{(r)}$ are constant real valued functions, $\forall r\in[0,1]$. Consequently (see Remark 4(6) and repeat proofs) inequalities (40), (41), (42), (43), (47) and (48) are again valid.

We give the following fuzzy Korovkin Type Theorem (Korovkin [14]).

Theorem 9. *Consider the sequence of fuzzy positive linear operators*

$$L_N\colon C_{\mathcal{F}}^n([a,b])\to C_{\mathcal{F}}([a,b]),\quad n\in\mathbb{N},\ \forall N\in\mathbb{N}, \tag{49}$$

with the property

$$(L_Nf)_\pm^{(r)}=\tilde{L}_N(f_\pm^{(r)}), \tag{50}$$

respectively, for all $r\in[0,1]$, $\forall f\in C_{\mathcal{F}}^n([a,b])$, $\forall N\in\mathbb{N}$. Here $\{\tilde{L}_N\}_{N\in\mathbb{N}}$ is a sequence of positive linear operators such that

$$\tilde{L}_N\colon C^n([a,b])\to C([a,b]).$$

Assume that $\|\tilde{L}_N(1)\|_\infty\le\gamma$, $\forall N\in\mathbb{N}$, for some $\gamma>0$. Furthermore assume that $\tilde{L}_N1\overset{u}{\to}1$ and $\|(\tilde{L}_N(|\cdot-x|^{n+1}))(x)\|_\infty\to 0$, as $N\to\infty$. Then $D^*(L_Nf,f)\to 0$ as $N\to\infty$, $\forall f\in C_{\mathcal{F}}^n([a,b])$, i.e. $L_N\to I$, as $N\to\infty$, fuzzy and uniformly, where I is the fuzzy unit operator.

Proof. From (48) we get

$$D^*(L_Nf,f)\le\|\tilde{L}_N1-1\|_\infty D^*(f,\tilde{o})+\sum_{k=1}^{n}\frac{\|(\tilde{L}_N((\cdot-x)^k))(x)\|_\infty}{k!}D^*(f^{(k)},\tilde{o})$$

$$+\ \frac{\|(\tilde{L}_N(|\cdot-x|^{n+1}))(x)\|_\infty^{\frac{n}{(n+1)}}}{n!}\left\|(\tilde{L}_N(1))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right\|_\infty$$

$$\times\ \omega_1^{(\mathcal{F})}\big(f^{(n)},\|(\tilde{L}_N(|\cdot-x|^{n+1}))(x)\|_\infty^{\frac{1}{(n+1)}}\big),\quad\forall N\in\mathbb{N}. \tag{51}$$

Also by (35) we have

$$\|\tilde{L}_N((t-x)^k,x)\|_\infty\le\|\tilde{L}_N(1)\|_\infty^{1-\frac{k}{(n+1)}}\|\tilde{L}_N(|t-x|^{n+1},x)\|_\infty^{\frac{k}{(n+1)}}, \tag{52}$$

any $0<k<n+1$, $\forall N\in\mathbb{N}$.

Now by using (52), (51) and the assumptions of the theorem we conclude that $D^*(L_N f, f) \to 0$, as $N \to \infty$. □

Note 3. Inequality (51), proof of Theorem 9, gives the convergence of $L_N \to I$, quantitatively, and at higher rate, reflecting the higher order fuzzy differentiability of f.

4. MULTIDIMENSIONAL RESULTS

We need:

Theorem 10. (by Theorem 7.4.1, p. 236 of Anastassiou [2] and the Riesz Representation Theorem) *Take* $Q := \{x \in \mathbb{R}^k \colon \ \|x\|_{\ell_1} \le 1\}$, $k \ge 1$, $x \in Q$. *Let* $\tilde{L} \colon C(Q) \to C(Q)$ *positive linear operator with* $\tilde{L}(1,x) = 1$ *and* $f \in C^n(Q)$, $n \in \mathbb{N}$. *Here* $f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}$, *where* $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$, *and* $0 < |\alpha| := \sum_{i=1}^{k} \alpha_i \le n$. *Assume for* $h > 0$ *we have* $w := \max_{|\alpha|=n} \omega_1(f_\alpha, h) > 0$, *where* ω_1 *is the usual modulus of continuity with respect to* $\|\cdot\|_{\ell_1}$ *relative to* Q. *Then*

$$|\tilde{L}(f,x) - f(x)| \le \sum_{j=1}^{n} \left(\sum_{|\alpha|=j} \left(\frac{|f_\alpha(x)|}{\alpha_1! \cdots \alpha_k!} \left| \tilde{L}\left(\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right), x \right) \right| \right) \right)$$
$$+ (\tilde{L}(\|z - x\|_{\ell_1}, x)) \frac{\phi_n(1 + \|x\|_{\ell_1})}{(1 + \|x\|_{\ell_1})} w. \tag{53}$$

Proof. We add the following, in order to transfer here Theorem 7.4.1, p. 236 of Anastassiou [2]. Let $g_z(t) := f(x + t(z - x))$, $x, z \in Q$, all $0 \le t \le 1$. For $j = 1, \dots, n$ we have

$$g_z^{(j)}(t) = \left[\left(\sum_{i=1}^{k} (z_i - x_i) \frac{\partial}{\partial x_i} \right)^j f \right] (x + t(z - x))$$

with

$$g_z^{(j)}(0) = \left[\left(\sum_{i=1}^{k} (z_i - x_i) \frac{\partial}{\partial x_i} \right)^j f \right] (x), \quad \forall x, z \in Q.$$

More precisely, we get

$$\frac{g_z^{(j)}(0)}{j!} = \sum_{|\alpha|=j} \frac{\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i}\right)}{\prod_{i=1}^{k} \alpha_i!} f_\alpha(x). \tag{54}$$

We will use:

Theorem 11. (By Theorem 7.4.2, p. 237 of Anastassiou [2]) *Let μ be a measure of mass $m > 0$ on $Q \subseteq \mathbb{R}^k$, $k \geq 1$ compact and convex. Assume*

$$\frac{1}{(n+1)} \left(\frac{1}{m} \int_Q \|x - x_0\|_{\ell_1}^{n+1} \mu(dx) \right)^{\frac{1}{(n+1)}} =: h > 0, \tag{55}$$

where x_0 is a fixed point of Q. Also, let $f \in C^n(Q)$: $\max_{|\alpha|=n} \omega_1(f_\alpha, h) \leq w$, where $w > 0$. Then

$$\left| \int_Q f\, d\mu - f(x_0) \right| \leq |m-1|\,|f(x_0)| + \left| \sum_{j=1}^{n} \frac{1}{j!} \int_Q g_x^{(j)}(0) \mu(dx) \right| + mwh^n \left[\frac{3}{2} \frac{(n+1)^n}{n!} + \frac{(n+1)^{n-1}}{8(n-1)!} \right], \tag{56}$$

where $g_x(t) := f(x_0 + t(x - x_0))$, $t \geq 0$. Above inequality (56) is trivially true if $h = 0$, or if $w = 0$ with $h > 0$.

Translating last Theorem 11 into the terminology of positive linear operators and by expanding we have:

Theorem 12. *Let $Q \subseteq \mathbb{R}^k$, $k \geq 1$ compact and convex. Let $\tilde{L}$ be a positive linear operator from $C(Q)$ into $C(Q)$. Assume $\tilde{L}(1,x) > 0$, $\forall x \in Q$. Consider $f \in C^n(Q)$, $n \in \mathbb{N}$. Then*

$$\begin{aligned} |\tilde{L}(f)(x) - f(x)| &\leq |\tilde{L}(1)(x) - 1|\,|f(x)| \\ &+ \sum_{j=1}^{n} \left\{ \sum_{|\alpha|=j} \left[\frac{|f_\alpha(x)|}{\prod_{i=1}^{k} \alpha_i!} \left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right| \right] \right\} \\ &+ \left(\tilde{L}(1)(x)\right)^{\frac{1}{(n+1)}} \left(\tilde{L}(\|z - x\|_{\ell_1}^{n+1})(x) \right)^{\frac{n}{(n+1)}} \left[\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right] \\ &\times \max_{|\alpha|=n} \omega_1 \left(f_\alpha, \frac{1}{(n+1)} \left(\frac{(\tilde{L}(\|z - x\|_{\ell_1}^{n+1}))(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \right), \quad \forall x \in Q. \end{aligned} \tag{57}$$

We have:

Corollary 1. (see Theorem 12) *All as in Theorem 12 with $\tilde{L}(1,x) = 1$, $\forall x \in Q$. Then*

$$\begin{aligned} |\tilde{L}(f)(x) - f(x)| &\leq \sum_{j=1}^{n} \left\{ \sum_{|\alpha|=j} \left[\frac{|f_\alpha(x)|}{\prod_{i=1}^{k} \alpha_i!} \left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right| \right] \right\} \\ &+ \left(\tilde{L}(\|z - x\|_{\ell_1}^{n+1})(x) \right)^{\frac{n}{(n+1)}} \left[\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right] \end{aligned}$$

$$\times \max_{|\alpha|=n} \omega_1\left(f_\alpha, \frac{1}{(n+1)}\left((\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)\right)^{\frac{1}{(n+1)}}\right),$$
$$\forall x \in Q,\ \forall f \in C^n(Q). \quad (58)$$

We further give:

Theorem 13. *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \geq 1$, and let $x_0 := (x_{01}, \ldots, x_{0k}) \in Q$ be fixed and let μ be a measure on Q of mass $m \geq 0$. Consider $f \in C^n(Q)$, $n \in \mathbb{N}$, and suppose that each nth order partial derivative $f_\alpha := \frac{\partial^\alpha f}{\partial x^\alpha}$, where $\alpha := (\alpha_1, \ldots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \ldots, k$ and $|\alpha| := \sum_{i=1}^n \alpha_i = n$, has relative to Q and the $\|\cdot\|_{\ell_1}$, a modulus of continuity $\omega_1(f_\alpha, h) \leq w$. Here we take*

$$h := \left(\int_Q \|z-x_0\|_{\ell_1}^{n+1} d\mu(z)\right)^{\frac{1}{(n+1)}}. \quad (59)$$

Then

$$\left|\int_Q f\, d\mu - f(x_0)\right| \leq |m-1|\,|f(x_0)| + \left|\sum_{j=1}^n \frac{1}{j!}\int_Q g_x^{(j)}(0)\mu(dx)\right| + \frac{wh^n}{n!}\left(m^{\frac{1}{(n+1)}} + \frac{1}{(n+1)}\right),$$

where $g_x(t) := f(x_0 + t(x-x_0))$, $t \geq 0$, $x \in Q$.

Proof. Here we have

$$f(z_1, \ldots, z_k) = g_z(1) = \sum_{j=0}^n \frac{g_z^{(j)}(0)}{j!} + R_n(z, 0),$$

where

$$R_n(z,0) := \int_0^1 \left(\int_0^{t_1} \cdots \left(\int_0^{t_{n-1}} (g_z^{(n)}(t_n) - g_z^{(n)}(0))dt_n\right)\cdots\right)dt_1, \quad z \in Q.$$

In Anastassiou [2], p. 236 we got that (see 7.4.4 there)

$$|R_n(z,0)| \leq w\phi_n(\|z-x_0\|_{\ell_1}), \quad \forall z \in Q.$$

But by (23) we have

$$w\phi_n(\|z-x_0\|_{\ell_1}) \leq w\frac{\|z-x_0\|_{\ell_1}^n}{n!}\left(1 + \frac{\|z-x_0\|_{\ell_1}}{(n+1)h}\right)$$
$$= \frac{w}{n!}\left(\|z-x_0\|_{\ell_1}^n + \frac{\|z-x_0\|_{\ell_1}^{n+1}}{(n+1)h}\right).$$

Hence

$$
\begin{aligned}
\int_Q |R_n(z,0)|\, d\mu(z) &\le w \int_Q \phi_n \|z-x_0\|_{\ell_1}) d\mu(z) \\
&\le \frac{w}{n!}\left[\int_Q \|z-x_0\|_{\ell_1}^n d\mu(z) + \frac{1}{(n+1)h}\int_Q \|z-x_0\|_{\ell_1}^{n+1} d\mu(z)\right] \\
&\quad \text{(not to have a trivial case we take } \mu(Q)=m>0) \\
&\le \frac{w}{n!}\Bigg[m^{\frac{1}{(n+1)}} \left(\int_Q \|z-x_0\|_{\ell_1}^{n+1} d\mu(z)\right)^{\frac{n}{(n+1)}} \\
&\quad + \frac{1}{h}\left(\frac{1}{(n+1)}\int_Q \|z-x_0\|_{\ell_1}^{n+1} d\mu(z)\right)\Bigg]
\end{aligned}
$$

(we choose

$$
h := \left(\int_Q \|z-x_0\|_{\ell_1}^{n+1} d\mu(z)\right)^{\frac{1}{(n+1)}} > 0,
$$

the case $h=0$ is trivial and not discussed here).

$$
= \frac{wh^n}{n!}\left(m^{\frac{1}{(n+1)}} + \frac{1}{(n+1)}\right),
$$

i.e. we got that

$$
\int_Q |R_n(z,0)| d\mu(z) \le \frac{wh^n}{n!}\left(m^{\frac{1}{(n+1)}} + \frac{1}{(n+1)}\right).
$$

The validity of (59) is now clear. □

By using Riesz Representation Theorem, Theorem 13 and expanding we obtain:

Theorem 14. *Let Q be a compact and convex subset of $\mathbb{R}^k$, $k \ge 1$. Let $\tilde{L}$ be a positive linear operator from $C(Q)$ into itself. Consider $f \in C^n(Q)$, $n \in \mathbb{N}$. Then*

$$
\begin{aligned}
|\tilde{L}(f)(x) - f(x)| &\le |\tilde{1}(x) - 1|\, |f(x)| \\
&+ \sum_{j=1}^{n}\left\{ \sum_{|\alpha|=j}\left[\frac{|f_\alpha(x)|}{\prod_{i=1}^{k}\alpha_i!}\left|\left(\tilde{L}\left(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\right)\right)(x)\right|\right]\right\} \\
&+ \frac{\left((\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)\right)^{\frac{n}{(n+1)}}}{n!}\left[\left((\tilde{L}(1))(x)\right)^{\frac{1}{(n+1)}} + \frac{1}{(n+1)}\right] \\
&\times \max_{|\alpha|=n} \omega_1\left(f_\alpha, \left((\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)\right)^{\frac{1}{(n+1)}}\right), \quad \forall x \in Q. \qquad (60)
\end{aligned}
$$

Next we give our fuzzy multidimensional related results.

Theorem 15. *Let* $Q := \{x \in \mathbb{R}^k \colon \|x\|_{\ell_1} \le 1\}$, $k \ge 1$. *Consider the fuzzy positive linear operator*

$$L\colon C^n_{\mathcal{F}}(Q) \to C(Q), \quad n \in \mathbb{N}, \tag{61}$$

with the property

$$(Lf)^{(r)}_{\pm} = \tilde{L}(f^{(r)}_{\pm}), \tag{62}$$

respectively, for all $r \in [0,1]$, $\forall f \in C^n_{\mathcal{F}}(Q)$. *Here* $\tilde{L}$ *is a positive linear operator such that*

$$\tilde{L}\colon C(Q) \to C(Q), \tag{63}$$

with $\tilde{L}(1,x) = 1$. *Also* $f_\alpha = \frac{\partial^\alpha f}{\partial x^\alpha}$ *is the fuzzy partial derivative of* f, *where* $\alpha := (\alpha_1, \dots, \alpha_k)$, $\alpha_i \in \mathbb{Z}^+$, $i = 1, \dots, k$, *and* $0 < |\alpha| := \sum_{i=1}^k \alpha_i \le n$. *Assume for* $h > 0$ *we have that* $w^{(\mathcal{F})} := \max_{|\alpha|=n} \omega_1^{(\mathcal{F})}(f_\alpha, h) > 0$, *where* $\omega_1^{(\mathcal{F})}$ *is the fuzzy modulus of continuity with respect to* $\|\cdot\|_{\ell_1}$ *relative to* Q. *Then*

1)

$$\begin{aligned} &D((Lf)(x), f(x)) \\ &\le \sum_{j=1}^n \left\{ \sum_{|\alpha|=j} \left[\frac{\left|\left(\tilde{L}\left(\left(\prod_{i=1}^k (z_i - x_i)^{\alpha_i}\right)\right)(x)\right|}{\prod_{i=1}^k \alpha_i!} D(f_\alpha(x), \tilde{o}) \right] \right\} \\ &\quad + \left((\tilde{L}(\|z - x\|_{\ell_1}))(x)\right) \frac{\phi_n(1 + \|x\|_{\ell_1})}{(1 + \|x\|_{\ell_1})} w^{(\mathcal{F})}, \quad \forall x \in Q. \end{aligned} \tag{64}$$

2)

$$\begin{aligned} D^*(Lf, f) \;\le\; &\sum_{j=1}^n \left\{ \sum_{|\alpha|=j} \left[\frac{\left\|\left(\tilde{L}\left(\left(\prod_{i=1}^k (z_i - x_i)^{\alpha_i}\right)\right)(x)\right\|_\infty}{\prod_{i=1}^k \alpha_i!} D^*(f_\alpha, \tilde{o}) \right] \right\} \\ &+ \left\|(\tilde{L}(\|z - x\|_{\ell_1}))(x)\right\|_\infty \left\| \frac{\phi_n(1 + \|x\|_{\ell_1})}{(1 + \|x\|_{\ell_1})} \right\|_\infty w^{(\mathcal{F})}. \end{aligned} \tag{65}$$

Proof. We have the following

$$D((Lf)(x), f(x)) = \sup_{r\in[0,1]} \max\left\{|(Lf)^{(r)}_- - f^{(r)}_-(x)|, |(Lf)^{(r)}_+ - f^{(r)}_+(x)|\right\}$$

$$\overset{(62)}{=} \sup_{r\in[0,1]} \max\left\{|\tilde{L}(f^{(r)}_-) - f^{(r)}_-(x)|, |\tilde{L}(f^{(r)}_+) - f^{(r)}_+(x)|\right\}$$

(by Remark 1 (2), (6) and (53))

$$\leq \sup_{r\in[0,1]} \max\Bigg\{\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{|(f_\alpha)_-^{(r)}(x)|}{\prod_{i=1}^{k}\alpha_i!}\Bigg|\Bigg(\tilde{L}\Bigg(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\Bigg)\Bigg)(x)\Bigg|\Bigg]$$

$$+(\tilde{L}(\|z-x\|_{\ell_1})(x))\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\max_{|\alpha|=n}\omega_1((f_\alpha)_-^{(r)},h),$$

$$\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{|(f_\alpha)_+^{(r)}(x)|}{\prod_{i=1}^{k}\alpha_i}\Bigg|\Bigg(\tilde{L}\Bigg(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\Bigg)\Bigg)(x)\Bigg|\Bigg)\Bigg]$$

$$+(\tilde{L}(\|z-x\|_{\ell_1})(x))\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\max_{|\alpha|=n}\omega_1((f_\alpha)_+^{(r)},h)\Bigg\}$$

$$\overset{(7)}{\leq}\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{D(f_\alpha(x),\tilde{o})}{\prod_{i=1}^{k}\alpha_i!}\Bigg|\Bigg(\tilde{L}\Bigg(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\Bigg)\Bigg)(x)\Bigg|\Bigg)\Bigg]$$

$$+(\tilde{L}(\|z-x\|_{\ell_1})(x))\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\sup_{r\in[0,1]}$$

$$\times\max\Bigg\{\max_{|\alpha|=n}\omega_1((f_\alpha)_-^{(r)},h),\max_{|\alpha|=n}\omega_1((f_\alpha)_+^{(r)},h)\Bigg\}$$

$$\overset{\text{(by Lemma 4)}}{=}\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{D(f_\alpha(x),\tilde{o})}{\prod_{i=1}^{k}\alpha_i!}\Bigg|\Bigg(\tilde{L}\Bigg(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\Bigg)\Bigg)(x)\Bigg|\Bigg)\Bigg]$$

$$+(\tilde{L}(\|z-x\|_{\ell_1})(x))\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\max_{|\alpha|=n}\sup_{r\in[0,1]}$$

$$\times\max\{\omega_1((f_\alpha)_-^{(r)},h),\omega_1((f_\alpha)_+^{(r)},h)\}$$

$$\overset{\text{(by Proposition 1)}}{=}$$

$$\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{D(f_\alpha(x),\tilde{o})}{\prod_{i=1}^{k}\alpha_i!}\Bigg|\Bigg(\tilde{L}\Bigg(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\Bigg)\Bigg)(x)\Bigg|\Bigg)\Bigg]$$

$$+(\tilde{L}(\|z-x\|_{\ell_1}(x))\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\max_{|\alpha|=n}\omega_1^{(\mathcal{F})}(f_\alpha,h).$$

Inequality (64) is established. □

The next result follows.

Theorem 16. *Let $Q \subseteq \mathbb{R}^k$, $k \geq 1$ be a compact convex subset. Consider the fuzzy positive linear operator*

$$L\colon C^n_{\mathcal{F}}(Q) \to C(Q), \quad n \in \mathbb{N},$$

with the property

$$(Lf)^{(r)}_{\pm} = \tilde{L}(f^{(r)}_{\pm}),$$

respectively, for all $r \in [0,1]$, $\forall f \in C^n_{\mathcal{F}}(Q)$. Here $\tilde{L}$ is a positive linear operator such that $\tilde{L}\colon C(Q) \to C(Q)$, with $\tilde{L}(1,x) > 0$, $\forall x \in Q$. Consider $f \in C^n_{\mathcal{F}}(Q)$. Then:

1)

$$\begin{aligned} D((Lf)(x), f(x)) &\leq |\tilde{L}(1)(x) - 1| D(f(x), \tilde{o}) \\ &\quad + \sum_{j=1}^{n} \left[\sum_{|\alpha|=j} \left(\frac{\left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right|}{\prod_{i=1}^{k} \alpha_i!} D(f_\alpha(x), \tilde{o}) \right) \right] \\ &\quad + \left(\tilde{L}(1)(x) \right)^{\frac{1}{(n+1)}} \left(\tilde{L}(\|z - x\|^{n+1}_{\ell_1})(x) \right)^{\frac{n}{(n+1)}} \left(\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right) \\ &\quad \times \max_{|\alpha|=n} \omega_1^{(\mathcal{F})} \left(f_\alpha, \frac{1}{(n+1)} \left(\frac{(\tilde{L}(\|z - x\|^{n+1}_{\ell_1}))(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \right), \ \forall x \in Q, \end{aligned} \tag{66}$$

and

2)

$$\begin{aligned} D^*(Lf, f) &\leq \|\tilde{L}(1) - 1\|_\infty D^*(f, \tilde{o}) \\ &\quad + \sum_{j=1}^{n} \left[\sum_{|\alpha|=j} \left(\frac{\left\| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right\|_\infty}{\prod_{i=1}^{k} \alpha_i!} D^*(f_\alpha, \tilde{o}) \right) \right] \\ &\quad + \|\tilde{L}(1)\|_\infty^{\frac{1}{(n+1)}} \left(\left\| (\tilde{L}(\|z - x\|^{n+1}_{\ell_1}))(x) \right\|_\infty^{\frac{n}{(n+1)}} \left(\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right) \right. \\ &\quad \times \max_{|\alpha|=n} \omega_1^{(\mathcal{F})} \left(f_\alpha, \frac{1}{(n+1)} \left\| \frac{(\tilde{L}(\|z - x\|^{n+1}_{\ell_1}))(x)}{\tilde{L}(1)(x)} \right\|_\infty^{\frac{1}{(n+1)}} \right). \end{aligned} \tag{67}$$

Proof. We observe the following:

$$\begin{aligned} &D((Lf)(x), f(x)) \\ &= \sup_{r \in [0,1]} \max \left\{ |(Lf)^{(r)}_{-}(x) - f^{(r)}_{-}(x)|, |(Lf)^{(r)}_{+}(x) - f^{(r)}_{+}(x)| \right\} \end{aligned}$$

$$= \sup_{r\in[0,1]} \max\left\{|\tilde{L}(f_-^{(r)})(x) - f_-^{(r)}(x)|, |\tilde{L}(f_+^{(r)})(x) - f_+^{(r)}(x)|\right\}$$

$$\overset{(57)}{\leq} \sup_{r\in[0,1]} \max\Bigg\{ |\tilde{L}(1)(x) - 1|\,|f_-^{(r)}(x)|$$

$$+ \sum_{j=1}^{n} \left[\sum_{|\alpha|=j} \left(\frac{|(f_\alpha)_-^{(r)}(x)|}{\prod_{i=1}^{k} \alpha_i!} \left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right| \right) \right]$$

$$+ (\tilde{L}(1)(x))^{\frac{1}{(n+1)}} \left(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x) \right)^{\frac{n}{(n+1)}} \left[\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right]$$

$$\times \max_{|\alpha|=n} \omega_1 \left((f_\alpha)_-^{(r)}, \frac{1}{(n+1)} \left(\frac{\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \right),$$

$$|\tilde{L}(1)(x) - 1|\,|f_+^{(r)}(x)|$$

$$+ \sum_{j=1}^{n} \left[\sum_{|\alpha|=j} \left(\frac{|(f_\alpha)_+^{(r)}(x)|}{\prod_{i=1}^{k} \alpha_i!} \left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right| \right) \right]$$

$$+ (\tilde{L}(1)(x))^{\frac{1}{(n+1)}} \left(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x) \right)^{\frac{n}{(n+1)}} \left[\frac{3}{2n!} + \frac{1}{8(n-1)!(n+1)} \right]$$

$$\times \max_{|\alpha|=n} \omega_1 \left((f_\alpha)_+^{(r)} \frac{1}{(n+1)} \left(\frac{\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \right) \Bigg\}$$

$$\leq |\tilde{L}(1)(x) - 1| D(f(x), \tilde{o})$$

$$+ \sum_{j=1}^{n} \left[\sum_{|\alpha|=j} \left(\frac{\left| \left(\tilde{L}\left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right)(x) \right| \right.}{\prod_{i=1}^{k} \alpha_i!} D(f_\alpha(x), \tilde{o}) \right) \right]$$

$$+ (\tilde{L}(1)(x))^{\frac{1}{(n+1)}} \left(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x) \right)^{\frac{n}{(n+1)}} \left[\frac{3}{2n!} \right.$$

$$\left. + \frac{1}{8(n-1)!(n+1)} \right] \sup_{r\in[0,1]} \max\Bigg\{ \max_{|\alpha|=n} \omega_1 \Bigg((f_\alpha)_-^{(r)},$$

$$\frac{1}{(n+1)} \left(\frac{\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \Bigg),$$

$$\max_{|\alpha|=n} \omega_1 \left((f_\alpha)_+^{(r)}, \frac{1}{(n+1)} \left(\frac{\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x)}{\tilde{L}(1)(x)} \right)^{\frac{1}{(n+1)}} \right) \Bigg\}$$

$$\overset{\text{(by Lemma 4 and Proposition 1)}}{=} \quad |\tilde{L}(1)(x)-1|D(f(x),\tilde{o})$$

$$+\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{\big|\big(\tilde{L}\big(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\big)(x)\big)\big|}{\prod_{i=1}^{k}\alpha_i!}\Bigg)D(f_\alpha(x),\tilde{o})\Bigg)\Bigg]$$

$$+\,(\tilde{L}(1)(x))^{\frac{1}{(n+1)}}\big(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x)\big)^{\frac{n}{(n+1)}}\left[\frac{3}{2n!}+\frac{1}{8(n-1)!(n+1)}\right]$$

$$\times\max_{|\alpha|=n}\omega_1^{(\mathcal{F})}\Bigg(f_\alpha,\frac{1}{(n+1)}\Bigg(\frac{(\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)}{\tilde{L}(1)(x)}\Bigg)^{\frac{1}{(n+1)}}\Bigg).$$

Inequality (66) is proved. □

Using Theorem 14 and working similarly as in the proof of Theorem 16 we obtain the very important:

Theorem 17. *Let $Q\subseteq\mathbb{R}^k$, $k\geq 1$ be a compact convex subset. Consider the fuzzy positive linear operator*

$$L\colon C^n_{\mathcal{F}}(Q)\to C(Q),\quad n\in\mathbb{N},$$

with the property

$$(Lf)^{(r)}_{\pm}=\tilde{L}(f^{(r)}_{\pm}),$$

respectively, for all $r\in[0,1]$, $\forall f\in C^n_{\mathcal{F}}(Q)$. Here $\tilde{L}$ is a positive linear operator from $C(Q)$ into itself. Consider $f\in C^n_{\mathcal{F}}(Q)$. Then:

1)

$$D((Lf)(x),f(x))\leq|\tilde{L}(1)(x)-1|D(f(x),\tilde{o})$$

$$+\sum_{j=1}^{n}\Bigg\{\sum_{|\alpha|=j}\Bigg[\frac{\big|\big(\tilde{L}\big(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\big)\big)(x)\big|}{\prod_{i=1}^{k}\alpha_i!}D(f_\alpha(x),\tilde{o})\Bigg]\Bigg\}$$

$$+\frac{(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x))^{\frac{n}{(n+1)}}}{n!}\left[((\tilde{L}(1))(x))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right]$$

$$\times\max_{|\alpha|=n}\omega_1^{(\mathcal{F})}\big(f_\alpha,((\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x))^{\frac{1}{(n+1)}}\big),\quad\forall x\in Q,\quad(68)$$

and

2)

$$D^*(Lf,f)\quad\leq\quad\|\tilde{L}1-1\|_\infty D^*(f,\tilde{o})$$

$$+\sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Big(\frac{\big\|\big(\tilde{L}(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i})\big)(x)\big\|_\infty}{\prod_{i=1}^{k}\alpha_i!}D^*(f_\alpha,\tilde{o})\Big)\Bigg]$$

$$+\frac{\big\|(\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)\big\|_\infty^{\frac{n}{(n+1)}}}{n!}\Bigg\|(\tilde{L}(1))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\Bigg\|_\infty$$

$$\times\max_{|\alpha|=n}\omega_1^{(\mathcal{F})}\big(f_\alpha,\big\|(\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)\big\|_\infty^{\frac{1}{n+1}}\big). \tag{69}$$

Next we give a fuzzy multivariate Korovkin type result.

Theorem 18. *Let $Q\subseteq\mathbb{R}^k$, $k\geq 1$ be a compact convex subset. Consider the sequence of fuzzy positive linear operators*

$$L_N\colon C^n_{\mathcal{F}}(Q)\to C(Q),\quad n\geq 1,\quad \forall N\in\mathbb{N}$$

with the property

$$(L_N f)_{\pm}^{(r)}=\tilde{L}_N(f_{\pm}^{(r)}),$$

respectively, for all $r\in[0,1]$, $\forall f\in C^n_{\mathcal{F}}(Q)$. Here $\tilde{L}_N$ is a sequence of positive linear operators from $C(Q)$ into itself, $\forall N\in\mathbb{N}$. Assume $\|\tilde{L}_N(1)\|\leq\gamma$, $\gamma>0$, $\forall N\in\mathbb{N}$, and $\tilde{L}_N(1)\overset{u}{\to}1$,

$$\|(\tilde{L}_N(\|z-x\|^{n+1}))(x)\|_\infty\to 0,\quad \text{as } N\to\infty.$$

Then $D^*(L_Nf,f)\to 0$, as $N\to\infty$, $\forall f\in C^n_{\mathcal{F}}(Q)$ at higher rate, i.e. $L_N\to I$, as $N\to\infty$, fuzzy and uniformly.

Proof. We use (69) and the following.

By Hölder's inequality and Riesz Representation Theorem we obtain that

$$\big\|(\tilde{L}_N(\|z-x\|_{\ell_1}^{j}))(x)\big\|_\infty\leq\gamma^{1-\frac{j}{n+1}}\big\|(\tilde{L}_N(\|z-x\|_{\ell_1}^{n+1}))(x)\big\|_\infty^{\frac{j}{(n+1)}}$$

for $j=1,\dots,n$. Therefore we get

$$\big\|(\tilde{L}_N(\|z-x\|_{\ell_1}^{j}))(x)\big\|_\infty\to 0,\quad 1\leq j\leq n,$$

as $N\to\infty$. Notice that

$$(\tilde{L}_N(\|z-x\|_{\ell_1}^{j}))(x)=\sum_{|\alpha|=j}\frac{j!}{\prod_{i=1}^{k}\alpha_i!}\Bigg(\tilde{L}_N\Bigg(\prod_{i=1}^{k}|z_i-x_i|^{\alpha_i}\Bigg)\Bigg)(x).$$

Hence, since in the last equality all parts are nonnegative, we have

$$\frac{j!}{\prod_{i=1}^{k}\alpha_i!}\Bigg(\tilde{L}_N\Bigg(\prod_{i=1}^{k}|z_i-x_i|^{\alpha_i}\Bigg)\Bigg)(x)\leq\big(\tilde{L}_N(\|z-x\|_{\ell_1}^{j})\big)(x).$$

Consequently we find that

$$\left\| \left(\tilde{L}_N \left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right\|_\infty \le \frac{\prod_{i=1}^{k} \alpha_i!}{j!} \left\| (\tilde{L}_N(\|z - x\|_{\ell_1}^j))(x) \right\|_\infty .$$

Thus

$$\left\| \left(\tilde{L}_N \left(\prod_{i=1}^{k} (z_i - x_i)^{\alpha_i} \right) \right)(x) \right\|_\infty \to 0,$$

as $N \to \infty$, for all $\alpha\colon |\alpha| = j$, $j = 1, \ldots, n$. The claim is now established. □

5. L_p-ESTIMATES, $p \ge 1$

From Anastassiou [5] we have:

Theorem 19. *Let K be a convex and compact subset of the real normed vector space $(V, \|\cdot\|)$. Let L be a fuzzy positive linear operator from $C_{\mathcal{F}}(K)$ into itself with the property that there exists positive linear operator $\tilde{L}$ from $C(K)$ into itself with $(Lf)_\pm^{(r)} = \tilde{L}(f_\pm^{(r)})$, respectively for all $r \in [0,1]$, $\forall f \in C_{\mathcal{F}}(K)$. Then*

$$\begin{aligned} D(L(f)(x), f(x)) &\le \|\tilde{L}(1)(x) - 1| D(f(x), \tilde{o}) \\ &\quad + \omega_1^{(\mathcal{F})}\big(f, ((\tilde{L}(\|\cdot - x\|^2))(x))^{1/2}\big) \min\big\{(\tilde{L}(1)(x) \\ &\quad + \sqrt{\tilde{L}(1)(x)}), (\tilde{L}(1)(x) + 1)\big\}, \quad \forall x \in K. \end{aligned} \tag{70}$$

Furthermore we get

$$\begin{aligned} &D^*(Lf, f) \le D^*(f, \tilde{o}) \|\tilde{L}1 - 1\|_\infty \\ &+ \min\big\{\|\tilde{L}(1) + \sqrt{\tilde{L}(1)}\|_\infty, \|\tilde{L}(1) + 1\|_\infty\big\} \omega_1^{(\mathcal{F})}(f, \|(\tilde{L}(\|\cdot - x\|^2))(x)\|_\infty^{1/2}), \end{aligned} \tag{71}$$

and $\|\cdot\|_\infty$ stands for the sup-norm over K. In particular, if $\tilde{L}(1) = 1$ then (71) reduces to

$$D^*(Lf, f) \le 2\omega_1^{(\mathcal{F})}\big(f, \|(\tilde{L}(\|\cdot - x\|^2))(x)\|_\infty^{1/2}\big). \tag{72}$$

We give

Theorem 20. *All as in the assumptions of Theorem 19, plus $(K, \mathcal{A}, \mu)$ is a Borel measure space with $\mu(K) < \infty$, $p \ge 1$. Then*

$$\left(\int_K D^p(L(f)(x), f(x)) d\mu(x) \right)^{1/p}$$

$$\leq \|\tilde{L}1 - 1\|_\infty \left(\int_K D^p(f(x), \tilde{o}) d\mu(x)\right)^{1/p}$$
$$+ \min\left\{\|\tilde{L}(1) + \sqrt{\tilde{L}(1)}\|_\infty, \|\tilde{L}(1) + 1\|_\infty\right\}$$
$$\times \omega_1^{(\mathcal{F})}\left(f, \|(\tilde{L}(\|\cdot - x\|^2))(x)\|_\infty^{1/2}\right)(\mu(K))^{1/p}. \quad (73)$$

Proof. Let $f, g \in C_{\mathcal{F}}(K)$ and let $x_n \overset{\|\cdot\|}{\longrightarrow} x_0$, $n \to \infty$, where $\{x_n\}_{n\in\mathbb{N}}$, $x_0 \in K$. We have

$$D(f(x_n), g(x_n)) \leq D(f(x_n), f(x_0)) + D(f(x_0), g(x_0)) + D(g(x_0), g(x_n))$$

and

$$D(f(x_0), g(x_0)) \leq D(f(x_0), f(x_n) + D(f(x_n), g(x_n)) + D(g(x_n), g(x_0)).$$

Letting $n \to +\infty$, from the continuity of f and g we find

$$\lim_{n\to\infty} D(f(x_n), g(x_n)) = D(f(x_0), g(x_0)).$$

Therefore the function $F(x) = D(f(x), g(x))$, $x \in K$ is a continuous real valued function. Thus $D(L(f)(x), f(x))$ is continuous, hence Borel measurable. Finally using (70) and by integrating we obtain (73). □

From now on the measure of integration will be the Lebesgue measure λ, and $\|\cdot\|_p$, $p \geq 1$ will be the L_p-norm.

We present:

Theorem 21. *Assume $\tilde{L}(1, x) > 0$, $\forall x \in [a, b]$. All the rest as in Theorem 7. Then*

$$\|D((Lf)(x), f(x))\|_p \leq \|\tilde{L}1 - 1\|_\infty \|D(f(x), \tilde{o})\|_p$$
$$+ \sum_{k=1}^{n} \frac{\|\tilde{L}((t-x)^k, x)\|_\infty}{k!} \|D(f^{(k)}(x), \tilde{o})\|_p$$
$$+ (b-a)^{1/p}\left[\frac{n\rho^2}{8} + \frac{\rho}{2} + \frac{1}{(n+1)}\right] \quad (74)$$
$$\times \frac{(\|\tilde{L}(1)\|_\infty)^{\frac{1}{(n+1)}}}{\rho n!} \left\|(\tilde{L}(|t-x|^{n+1}, x))\right\|_\infty^{\frac{n}{(n+1)}}$$
$$\times \omega_1^{(\mathcal{F})}\left(f^{(n)}, \rho \left\|\left(\frac{\tilde{L}(|t-x|^{n+1}, x)}{\tilde{L}(1, x)}\right)\right\|_\infty^{\frac{1}{(n+1)}}\right), \quad n \in \mathbb{N}.$$

We also have

Theorem 22. *Assume all as in Theorem 8. Then*

$$\|D((Lf)(x), f(x))\|_p \leq \|\tilde{L}1 - 1\|_\infty \|D(f(x), \tilde{o})\|_p$$
$$+ \sum_{k=1}^{n} \frac{\|(\tilde{L}((\cdot - x)^k))(x)\|_\infty}{k!} \|D(f^{(k)}(x), \tilde{o})\|_p$$

$$+\ \frac{\|(\tilde{L}(|\cdot - x|^{n+1})(x))\|_{\infty}^{\frac{n}{(n+1)}}}{n!}\left\|(\tilde{L}(1))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right\|_{\infty}$$

$$\times\ \omega_1^{(\mathcal{F})}\big(f^{(n)}, \|(\tilde{L}(|\cdot - x|^{n+1})(x))\|_{\infty}^{\frac{1}{(n+1)}}\big)(b-a)^{1/p}. \tag{75}$$

Furthermore we list our multivariate fuzzy L_p results, $p \geq 1$.

Theorem 23. *Assume all as in Theorem 15. Then*

$$\|D((Lf)(x), f(x))\|_p$$

$$\leq\ \sum_{j=1}^{n}\Bigg\{\sum_{|\alpha|=j}\Bigg[\frac{\big\|\big(\tilde{L}\big(\big(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\big)\big)(x)\big\|_{\infty}}{\prod_{i=1}^{k}\alpha_i!}\|D(f_\alpha(x),\tilde{o})\|_p\Bigg]\Bigg\}$$

$$+\ \|(\tilde{L}(\|z-x\|_{\ell_1}))(x)\|_{\infty}\left\|\frac{\phi_n(1+\|x\|_{\ell_1})}{(1+\|x\|_{\ell_1})}\right\|_{\infty} w^{(\mathcal{F})}(\lambda(Q))^{1/p}. \tag{76}$$

We continue with:

Theorem 24. *Assume all as in Theorem 16. Then*

$$\|D((Lf)(x), f(x))\|_p \leq \|\tilde{L}(1)-1\|_{\infty}\|D(f(x),\tilde{o})\|_p$$

$$+\ \sum_{j=1}^{n}\Bigg[\sum_{|\alpha|=j}\Bigg(\frac{\big\|\big(\tilde{L}\big(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\big)\big)(x)\big\|_{\infty}}{\prod_{i=1}^{k}\alpha_i!}\|D(f_\alpha(x),\tilde{o})\|\Bigg)\Bigg]$$

$$+\ (\|\tilde{L}(1)\|_{\infty})^{\frac{1}{(n+1)}}\big\|(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x))\big\|_{\infty}^{\frac{n}{(n+1)}}\left(\frac{3}{2n!}+\frac{1}{8(n-1)!(n+1)}\right)$$

$$\max_{|\alpha|=n}\omega_1^{(\mathcal{F})}\left(f_\alpha, \frac{1}{(n+1)}\left\|\left(\frac{(\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x)}{\tilde{L}(1)(x)}\right)\right\|_{\infty}^{\frac{1}{(n+1)}}\right)\lambda(Q)^{1/p}. \tag{77}$$

We finish our main results with:

Theorem 25. *Assume all as in Theorem 17. Then*

$$\|D((Lf)(x), f(x))\|_p \leq \|\tilde{L}(1)-1\|_{\infty}\|D(f(x),\tilde{o})\|_p$$

$$+\ \sum_{j=1}^{n}\Bigg\{\sum_{|\alpha|=j}\Bigg[\frac{\big\|\big(\tilde{L}\big(\prod_{i=1}^{k}(z_i-x_i)^{\alpha_i}\big)\big)(x)\big\|_{\infty}}{\prod_{i=1}^{k}\alpha_i!}\|D(f_\alpha(x),\tilde{o})\|_p\Bigg]\Bigg\}$$

$$+\ \frac{\|(\tilde{L}(\|z-x\|_{\ell_1}^{n+1})(x))\|_{\infty}^{\frac{n}{(n+1)}}}{n!}\left\|(\tilde{L}(1))^{\frac{1}{(n+1)}}+\frac{1}{(n+1)}\right\|_{\infty}$$

$$\times \max_{|\alpha|=n} \omega_1^{(\mathcal{F})}\big(f_\alpha, \|((\tilde{L}(\|z-x\|_{\ell_1}^{n+1}))(x))\|_\infty^{\frac{1}{(n+1)}}\big)\lambda(Q)^{1/p}. \quad (78)$$

6. APPLICATIONS

Let $f \in C^1_{\mathcal{F}}([0,1])$, and the real Bernstein operators

$$(B_N(g))(x) := \sum_{k=0}^{N} g\left(\frac{k}{N}\right)\binom{N}{k} x^k (1-x)^{N-k},$$

$$\forall x \in [0,1],\ \forall g \in C^1([0,1]).$$

We have $B_N 1 = 1$, $(B_N(id))(x) = x$, also $(B_N(\cdot - x))(x) = 0$, $\forall x \in [0,1]$. Furthermore we get

$$(B_N((\cdot - x)^2))(x) = \frac{x(1-x)}{N} \le \frac{1}{4N},$$

with equality at $x = \frac{1}{2}$. By (11) and (17) we obtain

$$\begin{aligned} D(B_N^{(\mathcal{F})}(f)(x), f(x)) &\le \frac{3}{2}\sqrt{\frac{x(1-x)}{N}}\,\omega_1^{(\mathcal{F})}\left(f', \sqrt{\frac{x(1-x)}{N}}\right) \\ &\le \frac{3}{4\sqrt{N}}\,\omega_1^{(\mathcal{F})}\left(f', \frac{1}{2\sqrt{N}}\right), \quad (79) \end{aligned}$$

$$\forall N \in \mathbb{N},\ \forall x \in [0,1],\ \forall f \in C^1_{\mathcal{F}}([0,1]).$$

Clearly then $\lim_{N\to+\infty} D^*(B_N^{(\mathcal{F})}(f), f) = 0$, fuzzy and uniformly, $\forall f \in C^1_{\mathcal{F}}([0,1])$, at higher speed than (13).

2) Inequality (68) for $n = 1$ becomes

$$\begin{aligned} D((Lf)(x), f(x)) &\le |\tilde{L}(1)(x) - 1| D(f(x), \tilde{o}) \\ &+ \sum_{i=1}^{k}\left(|(\tilde{L}(z_i - x_i))(x)| D\left(\frac{\partial f}{\partial x_i}(x), \tilde{o}\right)\right) \\ &+ \sqrt{((\tilde{L}(\|z-x\|_{\ell_1}^2))(x))}\left[\sqrt{(\tilde{L}(1))(x)} + \frac{1}{2}\right] \\ &\times \max_{i=\{1,\dots,k\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \sqrt{((\tilde{L}(\|z-x\|_{\ell_1}^2))(x))}\right), \quad \forall x \in Q. \quad (80) \end{aligned}$$

If $\tilde{L}(1)(x) = 1$, $(\tilde{L}(z_i - x_i))(x) = 0$, $\forall x \in Q$, all $i = 1,\dots,k$, $n = 1$, then (68) reduces to

$$D((Lf)(x), f(x)) \le \frac{3}{2}\sqrt{((\tilde{L}(\|z-x\|_{\ell_1}^2))(x))}$$

$$\times \max_{i\in\{1,\dots,k\}} \omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \sqrt{((\tilde{L}(\|z-x\|_{\ell_1}^2))(x))}\right), \ \forall x \in Q. \quad (81)$$

Let $g \in C([0,1])^2$, the two-dimensional Bernstein polynomials of g are defined by

$$(B_{m,\overline{n}}(g))(t_1,t_2)$$
$$:= \sum_{k=0}^{m}\sum_{\ell=0}^{\overline{n}} g\left(\frac{k}{m},\frac{\ell}{\overline{n}}\right)\binom{m}{k}\binom{\overline{n}}{\ell} t_1^k(1-t_1)^{m-k}t_2^{\ell}(1-t_2)^{\overline{n}-\ell}, \quad (82)$$

for all $t := (t_1,t_2) \in [0,1]^2$, all $(m,\overline{n}) \in \mathbb{N}^2$. It is known that $B_{m,\overline{n}}(g) \to g$ uniformly on $[0,1]^2$. Clearly $(B_{m,\overline{n}}(1))(t_1t_2) = 1$, $\forall (t_1,t_2) \in [0,1]^2$, $\forall (m,\overline{n}) \in \mathbb{N}^2$. Using Schwarz's inequality we get

$$\sqrt{((B_{m,\overline{n}}(\|\cdot - t\|_{\ell_1}^2))(t))} \le \left(\sqrt{\frac{t_1(1-t_1)}{m}} + \sqrt{\frac{t_2(1-t_2)}{\overline{n}}}\right)$$
$$\le \frac{1}{2}\left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{\overline{n}}}\right),$$
$$\forall (m,\overline{n}) \in \mathbb{N}^2, \ \forall t \in [0,1]^2. \quad (83)$$

We have easily that

$$(B_{m,\overline{n}}(z_i - t_i))(t_1,t_2) = 0, \quad i = 1,2. \quad (84)$$

Next we define the fuzzy two-dimensional Bernstein operators as follows

$$(B_{m,\overline{n}}^{(\mathcal{F})}(f))(t_1,t_2) := \sum_{k=1}^{m}{}^{*}\sum_{\ell=0}^{\overline{n}}{}^{*} f\left(\frac{k}{m},\frac{\ell}{\overline{n}}\right)$$
$$\odot \binom{m}{k}\binom{\overline{n}}{\ell} t_1^k(1-t_1)^{m-k}t_2^{\ell}(1-t_2)^{\overline{n}-\ell}, \quad (85)$$
$$\forall (t_1,t_2) \in [0,1]^2, \ \forall (m,\overline{n}) \in \mathbb{N}^2, \ \forall f \in C_{\mathcal{F}}([0,1]^2).$$

We observe as valid the following

$$(B_{m,\overline{n}}^{(\mathcal{F})}(f))_{\pm}^{(r)} = B_{m,\overline{n}}(f_{\pm}^{(r)}), \quad (86)$$

respectively, for all $r \in [0,1]$, $\forall f \in C_{\mathcal{F}}([0,1]^2)$. Finally, by (81) and (83) we derive that

$$D\big((B_{m,\overline{n}}^{(\mathcal{F})}(f))(t_1,t_2), f(t_1,t_2)\big)$$
$$\le \frac{3}{2}\left(\sqrt{\frac{t_1(1-t_1)}{m}} + \sqrt{\frac{t_2(1-t_2)}{\overline{n}}}\right)$$
$$\times \max_{i\in\{1,2\}}\left\{\omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_i}, \left(\sqrt{\frac{t_1(1-t_1)}{m}} + \sqrt{\frac{t_2(1-t_2)}{\overline{n}}}\right)\right)\right\}$$

$$\leq \frac{3}{4}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{\overline{n}}}\right)\max\left\{\omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_1},\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{\overline{n}}}\right)\right),\right. \tag{87}$$

$$\left.\omega_1^{(\mathcal{F})}\left(\frac{\partial f}{\partial x_2},\frac{1}{2}\left(\frac{1}{\sqrt{m}}+\frac{1}{\sqrt{\overline{n}}}\right)\right)\right\},$$

$$\forall (m,\overline{n})\in\mathbb{N}^2,\ \forall (t_1,t_2)\in[0,1]^2,\ \forall f\in C^1_{\mathcal{F}}([0,1]^2).$$

Clearly then $\lim_{m,\overline{n}\to\infty} D^*(B^{(\mathcal{F})}_{m,\overline{n}}(f),f) = 0$, fuzzy and uniformly, $\forall f \in C^1_{\mathcal{F}}$ $([0,1]^2)$, at a higher rate.

One can give many similar other applications of our theorems.

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