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CONVERSE ROGERS-HÖLDER'S INEQUALITY ON TIME SCALES

Chen-Huang Hong[1], and Cheh-Chih Yeh[2]

[1]General Education Center
National Taipei University of Technology
Taipei, 106, Taiwan, Republic of China
hongch@ntut.edu.tw

[2]Department of Information Management
Lunghwa University of Science and Technology
Kueishan Taoyuan, 33306, Taiwan, Republic of China
ccyeh@mail.lhu.edu.tw

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ABSTRACT: Converse Rogers-Hölder's inequalities are established on time scales by using elementary method.

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1. INTRODUCTION

In 1964, Diaz et al [2] and Marshall and Olkin [6] established independently the following converse Rogers-Hölder's inequality, see also Liu [4], Mitrinovič [7], Mond and Ječarić [8], Wang [9].

Theorem A. *Let f and g be nonnegative functions on $[a, b]$. If there exist two positive constants m and M such that*

$$m \leq \frac{f^p(x)}{g^q(x)} \leq M$$

on $[a, b]$, where $p, q \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$c^*(p, q, l)\Big(\int_a^b f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int_a^b g^q(x)dx\Big)^{\frac{1}{q}} \leq \int_a^b f(x)g(x)dx,$$

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where

$$\begin{cases} l = \frac{M}{m}; \\ c^*(p,q,l) = p^{\frac{1}{p}} q^{\frac{1}{q}} l^{\frac{1}{pq}} (l^{\frac{1}{q}} - 1)^{\frac{1}{q}} (l^{\frac{1}{p}} - 1)^{\frac{1}{p}} (l-1)^{-1}. \end{cases}$$

The purpose of this note is to extend Theorem A to time scale version by using Rogers-Hölder's inequality (Lemma B below).

2. PRELIMINARIES AND LEMMAS

We first briefly introduce the time scales calculus.

By a times scale $\mathbb{T}$ we mean any closed subset of the set $\mathbb{R}$ of all real numbers with order and topological structure in a canonical way. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

Definition. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale, then two mappings

$$\sigma,\ \rho : \mathbb{T} \to \mathbb{R}$$

satisfying

$$\sigma(t) = \inf\{s \in \mathbb{T} | s > t\},\ \rho(t) = \sup\{s \in \mathbb{T} | s > t\}$$

are called the jump operators.

A point $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, is said to be *right-scattered* if $\sigma(t) > t$, *left-scattered* if $\rho(t) < t$, *right-dense* if $\sigma(t) = t$, *right-dense* if $\rho(t) = t$.

Throughout this paper, we suppose that:

(a) $\mathbb{R} = (-\infty, \infty)$;

(b) $\mathbb{T}$ is a time scale;

(c) an interval means the intersection of a real interval with the given time scale.

Definition. A mapping $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if the following two conditions hold:

(a) f is continuous at each right-dense point or maximal point of $\mathbb{T}$;

(b) $\lim\limits_{s \to t^-} g(s) = g(t^-)$ exists for each left-dense point $t \in \mathbb{T}$.

The set of all rd-continuous functions from $\mathbb{T}$ to $\mathbb{R}$ is denoted by $C_{rd}[\mathbb{T}, \mathbb{R}]$.

Let

$$\mathbb{T}^k := \begin{cases} \mathbb{T} - \{m\},\ \text{if } \mathbb{T} \text{ has a left-scattered maximal point } m, \\ \mathbb{T},\ \text{otherwise.} \end{cases}$$

Definition. Assume that $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, then we define $f^{\Delta}(t)$ to be the number (if it exists) with property that if for any given $\epsilon > 0$, there exists a neighborhood U of t such that

$$\Big| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \Big| \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$. In this case, f is said to be *delta*-differentiable at t.

Definition. A function $g : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $g^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. In this case, we define the integral of f by

$$\int_s^t f(u)\Delta u = g(t) - g(s)$$

for all $s, t \in T$, and we say that f is integrable on $\mathbb{T}$.

It follows from Theorem 1.74 of Bohner and Peterson [1] that every rd-continuous function has an antiderivative.

For further concerning the time scale, we refer to Bohner and Peterson [1], Lakshmikantham et al [3].

3. MAIN RESULTS

To prove our main result, we need the following Rogers-Hölder's inequality.

Lemma B. *Let $a, b \in [0, \infty)$ and $p, q \in \mathbb{R} - \{0, 1\}$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then:*

(a) $ab > \frac{a^p}{p} + \frac{b^q}{q}$ if $p < 1$ and $p \neq 0$;

(b) $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ if $p > 1$, with equality if and only if $a^p = b^q$.

We now state and prove our main result as follows:

Theorem 1. *Let $h, f, g \in C_{rd}([a, b], [0, \infty))$ and there exist two positive constants m and M such that*

$$m \leq \frac{f^p(x)}{g^q(x)} \leq M,$$

where $p, q \in \mathbb{R} - \{0, 1\}$ with $\frac{1}{p} + \frac{1}{q} = 1$. If

$$c(p, q, l) = |p|^{-\frac{1}{p}} |q|^{-\frac{1}{q}} l^{-\frac{1}{pq}} (l^{\frac{1}{q}} - 1)^{-\frac{1}{q}} (l^{\frac{1}{p}} - 1)^{-\frac{1}{p}} (l - 1),$$

where $l = \frac{M}{m}$, then the following two statements hold:

(A) If $p < 1$ and $p \neq 0$, then

$$\Big(\int_a^b h(x) f^p(x) \Delta x \Big)^{\frac{1}{p}} \Big(\int_a^b h(x) g^q(x) \Delta x \Big)^{\frac{1}{q}} \geq c(p, q, l) \int_a^b h(x) f(x) g(x) \Delta x \qquad (R_1)$$

provided either $\int_a^b h(x)f^p(x)\Delta x > 0$ *or* $\int_a^b h(x)g^q(x)\Delta x > 0.$

(B) If $p > 1$, *then*

$$\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}} \leq c(p,q,l)\int_a^b h(x)f(x)g(x)\Delta x, \tag{R_2}$$

Proof of (A). Without loss of generality, we may assume that

$$\int_a^b h(x)f(x)g(x)\Delta x < \infty.$$

It follows from (a) of Lemma B that for each $y > 0$,

$$\begin{aligned}
&\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}\\
&= \Big(\int_a^b y^{\frac{1}{q}}h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b y^{-\frac{1}{p}}h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}}\\
&\geq \frac{1}{p}\int_a^b y^{\frac{1}{q}}h(x)f^p(x)\Delta x + \frac{1}{q}\int_a^b y^{-\frac{1}{p}}h(x)g^q(x)\Delta x\\
&= \int_a^b h(x)f(x)g(x)\Big[\frac{1}{p}\Big(\frac{yf^p(x)}{g^q(x)}\Big)^{\frac{1}{q}} + \frac{1}{q}\Big(\frac{yf^p(x)}{g^q(x)}\Big)^{-\frac{1}{p}}\Big]\Delta x.
\end{aligned}$$

Let

$$k(x) = \frac{1}{p}x^{\frac{1}{q}} + \frac{1}{q}x^{-\frac{1}{p}} \tag{1}$$

on $(0,\infty)$. Then

$$k'(x) = \frac{x^{-\frac{1}{p}}}{pq}(1 - \frac{1}{x}).$$

Obviously, $k(x)$ is increasing on $(0,1)$ and $k(x)$ is decreasing on $(1,\infty)$. Hence $k(x) \leq k(1) = 1$ for each $x > 0$. Thus for each $y > 0$,

$$k\Big(\frac{yf^p(x)}{g^q(x)}\Big) \geq \min\big\{k(my), k(My)\big\}.$$

It follows from $\frac{1}{p} + \frac{1}{q} = 1$ that:

(i) $q < 0$ if $0 < p < 1$;

(ii) $q > 0$ if $p < 0$.

By $M > m > 0$, we see that:

case (i) implies $M^{-\frac{1}{p}} - m^{-\frac{1}{p}} < 0$ and $m^{\frac{1}{q}} - M^{\frac{1}{q}} > 0$;

case (ii) implies $M^{-\frac{1}{p}} - m^{-\frac{1}{p}} > 0$ and $m^{\frac{1}{q}} - M^{\frac{1}{q}} < 0$.

Thus, if we take

$$y = \frac{p}{q}\frac{M^{-\frac{1}{p}} - m^{-\frac{1}{p}}}{m^{\frac{1}{q}} - M^{\frac{1}{q}}}, \tag{2}$$

then $y > 0$. A straighforward calculation shows that

$$\begin{aligned} k(my) &= |p|^{-\frac{1}{p}}|q|^{-\frac{1}{q}}\Big\{\Big(\frac{l^{\frac{1}{p}}-1}{l^{\frac{1}{q}}-1}\Big)^{\frac{1}{q}} l^{-\frac{1}{pq}} + \Big(\frac{l^{\frac{1}{q}}-1}{l^{\frac{1}{p}}-1}\Big)^{\frac{1}{p}} l^{\frac{1}{p^2}}\Big\} \\ &= |p|^{-\frac{1}{p}}|q|^{-\frac{1}{q}}\frac{l^{-\frac{1}{pq}}(l^{\frac{1}{p}}-1) + l^{\frac{1}{p^2}}(l^{\frac{1}{q}}-1)}{(l^{\frac{1}{q}}-1)^{\frac{1}{q}}(l^{\frac{1}{p}}-1)^{\frac{1}{p}}} \end{aligned}$$

and

$$l^{-\frac{1}{pq}}(l^{\frac{1}{p}}-1) + l^{\frac{1}{p^2}}(l^{\frac{1}{q}}-1) = l^{-\frac{1}{pq}}(l-1).$$

Hence

$$k(my) = c(p,q,l). \tag{3}$$

Similarly,

$$k(My) = |p|^{-\frac{1}{p}}|q|^{-\frac{1}{q}}\Big\{\Big(\frac{l^{\frac{1}{q}}-1}{l^{\frac{1}{p}}-1}\Big)^{\frac{1}{p}} l^{-\frac{1}{pq}} + \Big(\frac{l^{\frac{1}{p}}-1}{l^{\frac{1}{q}}-1}\Big)^{\frac{1}{q}} l^{\frac{1}{q^2}}\Big\} = c(p,q,l). \tag{4}$$

It follows from (3) and (4) that

$$\min\{k(my), k(My)\} = c(p,q,l).$$

This completes the proof of (R_1). □

Proof of (B). It follows from $\frac{1}{p} + \frac{1}{q} = 1$ and $p > 1$ that $q > 1$. Thus, by $M > m$,

$$M^{-\frac{1}{p}} - m^{-\frac{1}{p}} < 0 \quad \text{and} \quad m^{\frac{1}{q}} - M^{\frac{1}{q}} < 0.$$

Let $k(x)$ and y be defined as (1) and (2), respectively, then $y > 0$. As in the proof of case (A), we see that

$$k\Big(\frac{yf^p(x)}{g^q(x)}\Big) \le \max\{k(my), k(My)\} = c(p,q,l).$$

This and (b) of Lemma B imply

$$\begin{aligned} &\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}} \\ &\le \int_a^b h(x)f(x)g(x)\Big[\frac{1}{p}\Big(\frac{yf^p(x)}{g^q(x)}\Big)^{\frac{1}{q}} + \frac{1}{q}\Big(\frac{yf^p(x)}{g^q(x)}\Big)^{-\frac{1}{p}}\Big]\Delta x \\ &\le c(p,q,l)\int_a^b h(x)f(x)g(x)\Delta x. \end{aligned}$$

Thus, we complete the proof of (R_2). □

Remark 1. Let $\mathbb{T} = \mathbb{R}$. If $h,\ f,\ g \in C([a,b],[0,\infty))$ and

$$m \leq \frac{f^p(x)}{g^q(x)} \leq M \quad \text{on} \quad [a,b],$$

where $p,\ q,\ m,\ M$ and $c(p,q,l)$ are defined as in Theorem 1, then the following two statements hold:

(A^*) If $p < 0$ and $p \neq 0$, then

$$\Big(\int_a^b h(x)f^p(x)dx\Big)^{\frac{1}{p}} \quad \Big(\int_a^b h(x)g^q(x)dx\Big)^{\frac{1}{q}} \geq c(p,q,l)\int_a^b h(x)f(x)g(x)dx$$

provided either $\int_a^b h(x)f^p(x)dx > 0$ or $\int_a^b h(x)g^q(x)dx > 0$.

(B^*) If $p > 1$, then

$$\Big(\int_a^b h(x)f^p(x)dx\Big)^{\frac{1}{p}} \quad \Big(\int_a^b h(x)g^q(x)dx\Big)^{\frac{1}{q}} \leq c(p,q,l)\int_a^b h(x)f(x)g(x)dx.$$

Marshall and Olkin [6] proved (B^*) by using the probability method.

Corollary 2. *Let $f,\ g,\ p,\ q,\ l$ and $c(p,q,l)$ be defined as in Theorem 1. If there exists two positive constants m and M such that $m \leq \frac{f(x)}{g(x)} \leq M$ on $[a,b]$, then the following two statements hold:*

(a) if $p < 1$ and $p \neq 0$, then

$$\Big(\int_a^b h(x)f(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g(x)\Delta x\Big)^{\frac{1}{q}} \geq c(p,q,l)\int_a^b h(x)f^{\frac{1}{p}}(x)g^{\frac{1}{q}}(x)\Delta x,$$

provided either $\int_a^b h(x)f(x)\Delta x > 0$ or $\int_a^b h(x)g(x)\Delta x > 0$,

(b) if $p > 1$, then

$$\Big(\int_a^b h(x)f(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g(x)\Delta x\Big)^{\frac{1}{q}} \leq c(p,q,l)\int_a^b h(x)f^{\frac{1}{p}}(x)g^{\frac{1}{q}}(x)\Delta x.$$

Letting f and g be replaced by $f^{\frac{1}{pq}}$ and $f^{-\frac{1}{pq}}$ in Theorem 1, respectively, we obtain the following corollary.

Corollary 3. *Let $h,\ f,\ p,\ q,\ l$ and $c(p,q,l)$ be defined as in Theorem 1. If there exist two positive constants m and M such that $0 < m \le f(x) \le M$, then the following two statements hold:*

(a) if $p < 1$, $p \neq 0$, then

$$\Big(\int_a^b h(x) f^{\frac{1}{p}}(x)\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b h(x) f^{-\frac{1}{p}}(x)\Delta x\Big)^{\frac{1}{q}} \ge c(p,q,l)\int_a^b h(x)\Delta x,$$

provided either $\int_a^b h(x) f^{\frac{1}{p}}\Delta x > 0$ or $\int_a^b h(x) f^{-\frac{1}{p}}\Delta x > 0$;

(b) if $p > 1$, then

$$\Big(\int_a^b h(x) f^{\frac{1}{q}}(x)\Delta x\Big)^{\frac{1}{p}} \Big(\int_a^b h(x) f^{-\frac{1}{p}}(x)\Delta x\Big)^{\frac{1}{q}} < c(p,q,l)\int_a^b h(x)\Delta x.$$

Corollary 4. *Let $a_i,\ b_i, c_i \in (0,\infty)$ and there exist two positive constants m and M such that $m \le \frac{a_i^p}{b_i^q} \le M$ for each $i = 1, 2, \cdots, n, \cdots$, where $p, q \in \mathbb{R} - \{0,1\}$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following four statements hold:*
(a) if $p < 1$ and $p \neq 0$, then

$$\Big(\sum_{i=1}^{\infty} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} b_i^q c_i\Big)^{\frac{1}{q}} \ge c(p,q,l)\sum_{i=1}^{\infty} a_i b_i c_i,$$

(b) if $p < 1$ and $p \neq 0$, then

$$\Big(\sum_{i=1}^{n} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} b_i^q c_i\Big)^{\frac{1}{q}} \ge c(p,q,l)\sum_{i=1}^{n} a_i b_i c_i,$$

(c) if $p > 1$, then

$$\Big(\sum_{i=1}^{\infty} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} b_i^q c_i\Big)^{\frac{1}{q}} \le c(p,q,l)\sum_{i=1}^{\infty} a_i b_i c_i,$$

(d) if $p > 1$, then

$$\Big(\sum_{i=1}^{n} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} b_i^q c_i\Big)^{\frac{1}{q}} \le c(p,q,l)\sum_{i=1}^{n} a_i b_i c_i,$$

where l and $c(p,q,l)$ are defined as in Theorem 1.

Remark 2. If $m \le \frac{a_i^p}{b_i^q} \le M$ is replaced by $m \le \frac{a_i}{b_i} \le M$ in Corollary 4 for ecah $i = 1, 2, \cdots, n$, then $(a) - (d)$ of Corollary 4 are, respectively, reduced to the following four statements:

(a^*) if $p < 1$ and $p \neq 0$, then

$$\Big(\sum_{i=1}^{\infty} a_i c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} b_i c_i\Big)^{\frac{1}{q}} \ge c(p, q, l) \sum_{i=1}^{\infty} a_i^{\frac{1}{p}} b_i^{\frac{1}{q}} c_i,$$

(b^*) if $p < 1$ and $p \neq 0$, then

$$\Big(\sum_{i=1}^{n} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} b_i^q c_i\Big)^{\frac{1}{q}} \ge c(p, q, l) \sum_{i=1}^{n} a_i^{\frac{1}{p}} b_i^{\frac{1}{q}} c_i,$$

(c^*) if $p > 1$, then

$$\Big(\sum_{i=1}^{\infty} a_i c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{\infty} b_i c_i\Big)^{\frac{1}{q}} \le c(p, q, l) \sum_{i=1}^{\infty} a_i b_i c_i,$$

(d^*) if $p > 1$, then

$$\Big(\sum_{i=1}^{n} a_i^p c_i\Big)^{\frac{1}{p}} \Big(\sum_{i=1}^{n} b_i^q c_i\Big)^{\frac{1}{q}} \ge c(p, q, l) \sum_{i=1}^{n} a_i b_i c_i.$$

Corollary 5. (Converse Minkowski's Inequality) *Let f, g, h, p, q, l and $c(p, q, l)$ be defined as in Theorem 1. Then the following two statements hold:*
(a) if $p < 1$ and $p \neq 0$, then

$$\Big(\int_a^b h(x)\big(f(x) + g(x)\big)^p \Delta x\Big)^{\frac{1}{p}}$$
$$\ge c(p, q, l)\Big[\Big(\int_a^b h(x) f^p(x) \Delta x\Big)^{\frac{1}{p}} + \Big(\int_a^b h(x) g^q(x) \Delta x\Big)^{\frac{1}{q}}\Big],$$

provided $\int_a^b h(x)\big(f(x) + g(x)\big)^p \Delta x > 0$ when $p < 0$,
(b) if $p > 1$, then

$$\Big(\int_a^b h(x)\big(f(x) + g(x)\big)^p \Delta x\Big)^{\frac{1}{p}}$$
$$\le c(p, q, l)\Big[\Big(\int_a^b h(x) f^p(x) \Delta x\Big)^{\frac{1}{p}} + \Big(\int_a^b h(x) g^q(x) \Delta x\Big)^{\frac{1}{q}}\Big].$$

Corollary 6. *Let f, g, h and $c(p, q, l)$ be defined as in Theorem 1 and satisfy*

$$m \le \frac{f^p(x)}{g^q(x)} \le M$$

on $[a,b]$ *for some positive constants* m *and* M*, where* $p,\ q \in \mathbb{R}$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ *and* $\frac{r}{p},\ \frac{r}{q} \notin \{0,1\}$*, then the following two statements hold:*

(a) if $\frac{r}{p}<1$ *and* $\frac{r}{p}\neq 0$*, then*

$$\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}} \geq c(p,q,l)^{\frac{1}{r}}\int_a^b h(x)f(x)g(x)\Delta x,$$

provided either $\int_a^b h(x)f^p\Delta x>0$ *or* $\int_a^b h(x)g^q\Delta x>0$*;*

(b) if $\frac{r}{p}>1$*, then*

$$\Big(\int_a^b h(x)f^p(x)\Delta x\Big)^{\frac{1}{p}}\Big(\int_a^b h(x)g^q(x)\Delta x\Big)^{\frac{1}{q}} \leq c(p,q,l)^{\frac{1}{r}}\int_a^b h(x)f(x)g(x)\Delta x.$$

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