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MONOTONE TECHNIQUE FOR FIRST ORDER DISCONTINUOUS FUNCTIONAL DIFFERENTIAL INCLUSIONS

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ABSTRACT: In this paper, an existence theorem for first order ordinary functional differential inclusions is proved without assuming the continuity of the multi-functions on the right hand side. Sufficient conditions for the existence of a minimal solution and a maximal solution are also proved.

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1. INTRODUCTION

Let $\mathbb{R}$ be the real line and let $\mathcal{P}_p(\mathbb{R})$ denote the class of all non-empty subsets of $\mathbb{R}$ with property p. Thus, $\mathcal{P}_{cl}(\mathbb{R})$ and $\mathcal{P}_{cp}(\mathbb{R})$ denote, respectively, the classes of all closed and compact subsets of $\mathbb{R}$. Let $I_0 = [-\delta, 0]$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$ for some $\delta > 0$ and $T > 0$, and let $J = I_0 \cup I = [-\delta, T]$. Let $\mathcal{C}$ be the Banach space of continuous real-valued functions on I_0 with the supremum norm $\|\cdot\|_{\mathcal{C}}$ defined by

$$\|x\|_{\mathcal{C}} = \sup_{t \in I_0} |x(t)|.$$

We let $C(J, \mathbb{R})$ denote the space of continuous real-valued functions on J and $AC(I, \mathbb{R})$ denote the space of absolutely continuous real-valued functions on

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I. For any continuous function x on J and for any $t \in I$, let $x_t : I_0 \to \mathcal{C}$ be the function defined by

$$x_t(\theta) = x(t+\theta), \quad -\delta \le \theta \le 0.$$

Consider the first order functional differential inclusion (in short FDI)

$$\left.\begin{aligned} x'(t) \in F\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \quad \text{a.e.} \quad t \in I, \\ x(t) = \phi(t), \quad t \in I_0, \end{aligned}\right\} \tag{1}$$

where, $k : I \times I \times \mathcal{C} \to \mathbb{R}$ and $F : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$.

By a *solution* of the FDI (1) we mean a function $x \in C(J,\mathbb{R}) \cap AC(I,\mathbb{R})$ such that $x'(t) = v(t)$ for some $v \in L^1(I,\mathbb{R})$ satisfying

$$v(t) \in F\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \quad \text{a.e.} \quad t \in I$$

and $x(t) = \phi(t)$ for $t \in I_0$.

The problem of the existence of solutions of the FDI (1) has already been studied in the literature under different continuity conditions on F. An existence theorem for FDI (1) for lower semi-continuous multi-functions F is proved in Fryszkowski [6]. When F has closed convex values and is lower semi-continuous, the existence results for FDI (1) reduce to existence results for ordinary differential equations

$$\left.\begin{aligned} x'(t) = f\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \quad \text{a.e.} \quad t \in I, \\ x(t) = \phi(t), \quad t \in I_0, \end{aligned}\right\} \tag{2}$$

where $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and

$$f\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \in F\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \quad \text{a.e.} \quad t \in I.$$

In this case, the solution can be obtained under a Carathéodory condition on F by using Schauder's Fixed Point Theorem.

The case of a discontinuous multi-function F has been treated in many works (see, for example, Dhage [2], Dhage [3] and the references contained therein). The monotonicity condition used in the above papers is of a very strong nature, and the existence of solutions is proved using a Lattice Fixed Point Theorem for multi-valued mappings in complete lattices. It is known that every Banach space is not a complete lattice. These facts motivate us to pursue the study in the present paper. Here, we prove existence results for the FDI (1) under a monotonicity condition that is weaker than those used in Dhage et al [5].

2. AUXILIARY RESULTS

Let X be a Banach space and let $A, B \in \mathcal{P}_p(X)$. We set

$$A + B = \{a + b : a \in A \text{ and } b \in B\}$$

and

$$\lambda A = \{\lambda a : \lambda \in \mathbb{R} \text{ and } a \in A\}.$$

Also, we let

$$\|A\| = \{\|a\| : a \in A\}$$

and

$$\|A\|_{\mathcal{P}} = \sup\{\|a\| : a \in A\}.$$

Let the Banach space X be equipped with the order relation $\leq$ and define an order relation in $\mathcal{P}_p(X)$ as follows.

Let $A, B \in \mathcal{P}_p(X)$. Then by $A \overset{i}{\leq} B$ we mean *for every $a \in A$ there exists a $b \in B$ such that $a \leq b$*. Again, $A \overset{d}{\leq} B$ *means that for each $b \in B$ there exists an $a \in A$ such that $a \leq b$*. Furthermore, we have $A \overset{id}{\leq} B$ if and only if $A \overset{i}{\leq} B$ and $A \overset{d}{\leq} B$. Finally, $A \leq B$ implies that $a \leq b$ for all $a \in A$ and $b \in B$. Note that if $A \leq A$, then it follows that A is a singleton set (see Dhage [2], Dhage [3], Dhage [4] and the references therein).

Definition 2.1. A mapping $Q : X \to \mathcal{P}_p(X)$ is called right monotone increasing (respectively, left monotone increasing) if $Qx \overset{i}{\leq} Qy$ ($Qx \overset{d}{\leq} Qy$) for all $x, y \in X$ for which $x \leq y$. Similarly, Q is called monotone increasing if it is left as well as right monotone increasing on X. Finally, Q is strictly monotone increasing if $Qx \leq Qy$ for all $x, y \in X$ with $x < y$.

We need the following fixed point theorems in the sequel.

Theorem 2.1. (see Dhage [2]) *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a right monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$, has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then Q has a fixed point.*

Theorem 2.2. (see Dhage [4]) *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a strictly monotone increasing multi-valued mapping. If every sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$, has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then Q has a least fixed point x_* and a greatest fixed point x^* in $[a, b]$. Moreover,*

$$x_* = \min\{y \in [a, b] \mid Qy \leq y\} \quad \textit{and} \quad x_* = \max\{y \in [a, b] \mid y \leq Qy\}.$$

3. EXISTENCE RESULTS

We equip the space $C(J,\mathbb{R})$ with the norm $\|\cdot\|$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|$$

and the order relation $\leq$ determined by the cone

$$K = \{x \in C(J,\mathbb{R}) \mid x \geq 0 \text{ for all } t \in J\}.$$

Clearly, K is a normal cone in $C(J,\mathbb{R})$. It is easy to see that

$$x \leq y \quad \text{if and only if} \quad x(t) \leq y(t) \quad \text{for all} \quad t \in J.$$

We need the following definitions in the sequel.

Definition 3.1. A multi-valued map $F : I \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, F(t)) = \inf\{\|y - x\| : x \in F(t)\}$ is measurable.

Definition 3.2. A multi-function $F(t, x, y)$ is called right monotone increasing in x almost everywhere for $t \in I$ if for every $y \in \mathbb{R}$, we have $F(t, x_1, y) \overset{i}{\leq} F(t, x_2, y)$ a.e. $t \in I$ for all $x_1,\ x_2 \in \mathbb{R}$ with $x_1 \leq x_2$. Similarly, the multi-function $F(t, x, y)$ is called right monotone increasing in y almost everywhere for $t \in I$, if for every $x \in \mathbb{R}$, we have $F(t, x, y_1) \overset{i}{\leq} F(t, x, y_2)$ a.e. $t \in I$ for all $y_1,\ y_2 \in \mathbb{R}$ with $y_1 \leq y_2$.

Definition 3.3. A multi-function $F(t, x, y)$ is called strict monotone increasing in x almost everywhere for $t \in I$ if for every $y \in \mathbb{R}$, we have $F(t, x_1, y) \leq F(t, x_2, y)$ a.e. $t \in I$ for all $x_1,\ x_2 \in \mathbb{R}$ with $x_1 \leq x_2$ and $x_1 \neq x_2$. Similarly, the multi-function $F(t, x, y)$ is called right monotone increasing in y almost everywhere for $t \in I$, if for every $x \in \mathbb{R}$, we have $F(t, x, y_1) \leq F(t, x, y_2)$ a.e. $t \in I$ for all $y_1,\ y_2 \in \mathbb{R}$ with $y_1 \leq y_2$ and $y_1 \neq y_2$.

For any $x \in C(J,\mathbb{R})$, let

$$S^1_F(x) = \Big\{ v \in L^1(I,\mathbb{R}) \mid v(t) \in F\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big) \text{ a.e. } t \in I \Big\},$$

where, $x_s \in \mathcal{C}$; this is our set of *selection functions.* The integral of the multi-function F is defined as

$$\int_0^t F\Big(s, x(s), \int_0^s k(s,\tau,x_\tau)\,d\tau\Big)\,ds = \bigg\{ \int_0^t v(s)\,ds : v \in S^1_F(x) \bigg\}.$$

Definition 3.4. A function $a \in C(J,\mathbb{R})$ is called a strict lower solution of the FDI (1) if for all $v \in S^1_F(a)$,

$$a'(t) \leq v(t) \text{ a.e. } t \in I,$$

$$a(t) \leq \phi(t), \text{ if } t \in I_0.$$

A strict upper solution b of FDI (1) is defined in a similar way but with the inequalities reversed.

We consider the following set of hypotheses in the sequel.

(H_0) The map $(t,s) \mapsto k(t,s,x)$ is continuous for each $x \in \mathcal{C}$ and the map $x \mapsto k(t,s,x)$ is increasing almost everywhere for t, $s \in I$.

(H_1) $F(t,x,y)$ is closed and bounded for each $t \in I$ and x, $y \in \mathbb{R}$.

(H_2) The multi-valued function $F(t,x,y)$ is measurable in $t \in I$ and right monotone increasing in x and y almost everywhere for $t \in I$.

(H_3) $S^1_F(x) \neq \emptyset$ for all $x \in C(J,\mathbb{R})$.

(H_4) The multi-valued map $x \mapsto S^1_F(x)$ is right monotone increasing in $C(J,\mathbb{R})$.

(H_5) FDI (1) has a strict lower solution a and a strict upper solution b with $a \le b$.

(H_6) The function $h : I \to \mathbb{R}$ defined by

$$h(t) = \Big\|F\Big(t, a(t), \int_0^t k(t,s,a_s)\,ds\Big)\Big\|_{\mathcal{P}} + \Big\|F\Big(t, b(t), \int_0^t k(t,s,b_s)\,ds\Big)\Big\|_{\mathcal{P}}$$

is Lebesgue integrable.

Remark 3.1. Note that if (H_2) and (H_5)–(H_6) hold, then we have

$$\Big\|F\Big(t, x(t), \int_0^t k(t,s,x_s)\,ds\Big)\Big\|_{\mathcal{P}} \le h(t) \quad \text{a.e.} \quad t \in I$$

for all $x \in [a,b]$.

Hypotheses (H_1)–(H_3) are common in the literature. Some nice sufficient conditions guaranteeing that (H_3) holds are given by Deimling [1] and Lasota and Opial [8]. A mild form of (H_5) is used in Halidias and Papageorgiou [7]. Hypotheses (H_4) and (H_5) are relatively new to the literature, but special cases of them have been appeared in the works of several authors. Note that (H_5) holds, in particular, if F is bounded on $I \times \mathbb{R} \times \mathbb{R}$ (see Dhage [2], Dhage [3] and references therein). Hypothesis (H_2) is assumed in order for (H_4) to make sense.

Theorem 3.1. *Assume that $(H_0) - (H_6)$ hold. Then the FDI (1) has a solution in $[a,b]$ defined on J.*

Proof. Let $X = C(J,\mathbb{R})$ and let $Y = C(J,\mathbb{R}) \cap AC(I,\mathbb{R}) \subset X$. Define an order interval $[a,b]$ in Y, which is well defined in view of hypothesis (H_5). Now the FDI (1) is equivalent to the integral inclusion

$$\left.\begin{aligned} x(t) \in \phi(0) + \int_0^t F\Big(s, x(s), \int_0^s k(s,\tau,x_\tau)\,d\tau\Big)\,ds,\ \text{if}\ t \in I,\\ x(t) = \phi(t),\ \text{if}\ t \in I_0, \end{aligned}\right\}$$

(see Dhage et al [5] and the references therein). Define a multi-valued operator $Q : [a,b] \to \mathcal{P}_p(X)$ by

$$Qx = \begin{cases} \left\{ u \in X : u(t) = \phi(0) + \int_0^t v(s)\,ds,\ v \in S_F^1(x) \right\}, & \text{if } t \in I, \\ \phi(t), \quad \text{if } t \in I_0, \end{cases}$$
$$= \begin{cases} (\mathcal{L} \circ S_F^1)(x), & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3}$$

where $\mathcal{L} : L^1(I,\mathbb{R}) \to C(I,\mathbb{R})$ is a continuous operator defined by

$$\mathcal{L}v(t) = \phi(0) + \int_0^t v(s)\,ds. \tag{4}$$

Clearly, the operator Q is well defined in view of hypothesis (H_3). We will show that Q satisfies all the conditions of Theorem 2.1.

Step I. First, we show that Q has compact values on $[a,b]$. Observe that if $t \in I$, then the operator Q is equivalent to the composition $\mathcal{L} \circ S_F^1$ of two operators on $L^1(I,\mathbb{R})$, where $\mathcal{L} : L^1(I,\mathbb{R}) \to X$ is the continuous operator defined by (4). To show that Q has compact values, it suffices to prove that the composition operator $\mathcal{L} \circ S_F^1$ has compact values on $[a,b]$. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(x)$. Then, by the definition of S_F^1, $v_n(t) \in F(t, x(t), \int_0^t k(t,s,x_s)\,ds)$ a.e. for $t \in I$. Since $F(t, x(t), \int_0^t k(t,s,x_s)\,ds)$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F(t, x(t), \int_0^t k(t,s,x_s)\,ds)$ a.e. for $t \in I$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on I as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t,\ \tau \in I$; then

$$|\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| \le \left| \int_0^t v_n(s)\,ds - \int_0^\tau v_n(s)\,ds \right| \le \left| \int_\tau^t v_n(s)\,ds \right|.$$

Since $v_n \in L^1(I,\mathbb{R})$, the right hand side of above inequality tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an easy application of Ascoli's Theorem implies that $\{\mathcal{L}v_n\}$ has a uniformly convergent subsequence. We then have $\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F^1)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S_F^1)(x)$ is compact. Therefore, Q is a compact-valued multi-valued operator on $[a,b]$.

Step II. Next, we show that Q is right monotone increasing and maps $[a,b]$ into itself. Let $x, y \in [a,b]$ be such that $x \le y$. Since $S_F^1(x) \overset{i}{\le} S_F^1(y)$, we have that $Q(x) \overset{i}{\le} Q(y)$. From (H_4), it follows that $a \le Qa$ and $Qb \le b$.

Now Q is right monotone increasing, so we have

$$a \leq Qa \overset{i}{\leq} Qx \overset{i}{\leq} Qb \leq b$$

for all $x \in [a,b]$. Hence, Q defines a right monotone increasing multi-valued operator $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$.

Step III. Finally let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a sequence in $Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$. We will show that $\{y_n\}$ has a cluster point. This is achieved by showing that $\{y_n\}$ is an uniformly bounded and equi-continuous sequence.

First, we show that $\{y_n\}$ is uniformly bounded. From the definition of $\{y_n\}$, there is a $v_n \in S_F^1(x_n)$ such that

$$y_n(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v_n(s)\,ds, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

Therefore, if $t \in I_0$, then $|y_n(t)| \leq \|\phi\|_{\mathcal{C}}$ for all $t \in I_0$. Similarly, if $t \in I$, then

$$\begin{aligned} |y_n(t)| &\leq \|\phi\|_{\mathcal{C}} + \int_0^t |v_n(s)|\,ds \\ &\leq \|\phi\|_{\mathcal{C}} + \int_0^t \Big\| F\Big(s, x(s), \int_0^s k(s,\tau,x_\tau)\,d\tau\Big)\Big\|_{\mathcal{P}}\,ds \\ &\leq \|\phi\|_{\mathcal{C}} + \int_0^T h(s)\,ds \\ &\leq \|\phi\|_{\mathcal{C}} + \|h\|_{L^1} \end{aligned}$$

for all $t \in I$. Taking supremum over t, we have

$$\|y_n\| \leq \|\phi\|_{\mathcal{C}} + \|h\|_{L^1},$$

which shows that $\{y_n\}$ is a uniformly bounded sequence in $Q([a,b])$.

To show that $\{y_n\}$ is an equi-continuous sequence in $Q([a,b])$, let $t, \tau \in I_0$. Then, we have

$$|y_n(t) - y_n(\tau)| \leq |\phi(t) - \phi(\tau)| \to 0$$

as $n \to \infty$. Similarly, let $t, \tau \in I$; then,

$$\begin{aligned} |y_n(t) - y_n(\tau)| &\le \Big| \int_0^t v_n(s)\,ds - \int_0^\tau v_n(s)\,ds \Big| \\ &\le \Big| \int_\tau^t v_n(s)\,ds \Big| \\ &\le \Big| \int_\tau^t |v_n(s)|\,ds \Big| \\ &\le \Big| \int_\tau^t h(s)\,ds \Big| \\ &\le |p(t) - p(\tau)|, \end{aligned}$$

where $p(t) = \displaystyle\int_0^t h(s)\,ds$. Similarly, if $\tau \in I_0$ and $t \in I$, then

$$\begin{aligned} |y_n(t) - y_n(\tau)| &\le |y_n(t) - y_n(0)| + |y_n(0) - y_n(\tau)| \\ &\le |p(t) - p(0)| + |\phi(\tau) - \phi(0)|. \end{aligned}$$

From the above inequalities, it follows that

$$|y_n(t) - y_n(\tau)| \to 0 \quad \text{as} \quad t \to \tau.$$

This shows that $\{y_n\}$ is an equi-continuous sequence in $Q([a, b])$. Now $\{y_n\}$ is uniformly bounded and equi-continuous, so it has a cluster point by the Arzelà-Ascoli Theorem. The desired conclusion then follows by an application of Theorem 2.1. □

Next, we prove a result concerning the extremal solutions of the FDI (1) on J. We need the following definition in the sequel.

Definition 3.5. A multi-function $\beta : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is called L^1-Chandrabhan if:

(i) $t \mapsto \beta(t, x, y)$ is measurable for all x, $y \in \mathbb{R}$,

(ii) $\beta(t, x, y)$ is strictly monotone increasing in x almost everywhere for $t \in I$, and

(iii) for each $r > 0$, there exists a function $h_r \in L^1(I, \mathbb{R})$ such that

$$\|\beta(t, x, y)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x, y)\} \le h_r(t) \text{ a.e. } t \in I$$

for all x, $y \in \mathbb{R}$ with $|x| \le r$ and $|y| \le r$.

We need the following additional hypotheses.

(H_7) The multi-function F is L^1-Chandrabhan on $I \times \mathbb{R} \times \mathbb{R}$.

(H_8) There exists $\alpha \in L^1(J, \mathbb{R})$ with $\|\alpha\|_{L^1} \leq 1$ such that

$$|k(t, s, x_s)| \leq \alpha(t)\|x_s\|$$

for all $t \in I$.

Remark 3.2. Note that if the multi-function $\beta(t, x, y)$ is L^1-Chandrabhan and (H_5) and (H_8) hold, then it is measurable in t and integrably bounded on $I \times [-r, r] \times [-r, r]$, where $r = \max\{\|x\| : x \in [a, b]\}$. It follows from a selection theorem (see Deimling [1]) that S^1_β is non-empty and has closed values on $[a, b]$, i.e.,

$$S^1_F(x) = \{u \in L^1(I, \mathbb{R}) \mid u(t) \in F\Big(t, x(t), \int_0^t k(t, s, x_s)\, ds\Big) \text{ a.e. } t \in I\} \neq \emptyset$$

for all $x \in [a, b] \subset C(J, \mathbb{R})$.

Theorem 3.2. *Assume that (H_0), (H_1), (H_5), (H_7), and (H_8) hold. Then the FDI (1) has a minimal solution and a maximal solution in $[a, b]$ defined on J.*

Proof. The proof is quite similar to that of Theorem 3.1. Now, $S^1_F(x) \neq \emptyset$ for each $x \in [a, b]$ in view of Remark 3.2. Also, the multi-valued map $x \mapsto S^1_F(x)$ is strictly monotone increasing on $[a, b]$. Consequently, the multi-valued operator Q defined by (3) is strictly monotone increasing on $[a, b]$. Hence, the desired result follows by an application of Theorem 2.2. □

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