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MULTIPOINT SINGULAR BOUNDARY-VALUE PROBLEM FOR SYSTEM OF SEMILINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT: A singular Cauchy-Nicoletti problem for system of ordinary differential equations is considered. An approach which combines topological method of T. Ważewski and Schauder's principle is used. A theorem concerning the existence of a solution of this problem (a graph of which lies in a given domain) is proved. Moreover, estimations of its coordinates are obtained.

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1. INTRODUCTION

In the presented paper the following Cauchy-Nicoletti problem

$$y_i'(x) = \omega_i(x)y_i + f_i(x,y)\,, \quad i = 1,\dots,n, \tag{1.1}$$

$$y_p(x_p^+) = A_p, \quad y_q(x_q^\pm) = A_q, \quad y_r(x_r^-) = A_r, \tag{1.2}$$

$$p = 1,\dots,k;\; q = k+1,\dots,s;\; r = s+1,\dots,n$$

is considered, where $y = (y_1,\dots,y_n)$, $x \in I = [a,b]$ and $a = x_1 = \dots = x_k < x_{k+1} \le \dots \le x_s < x_{s+1} = \dots = x_n = b$; A_i, $i = 1,\dots,n$ are real constants. Denote $I_i = I \setminus \{x_i\}$, $i = 1,\dots,n$ and $J = \bigcap_{i=1}^{n} I_i$. We shall suppose that $\omega_i \in C(I_i,\mathbb{R})$ and $f_i \in C(\Theta_i,\mathbb{R})$, $i = 1,\dots,n$, where the

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domain $\Theta_i \subset I_i \times \mathbb{R}^n$ (satisfying relation $\Theta_i \cap \{x = x^*\} \neq \emptyset$ for $x^* \in I_i$) is more precisely specified below. Note that the continuity of the functions ω_i and f_i is not required at the point x_i, $i = 1, \dots, n$. Solution of the problem (1.1), (1.2) is defined in the following sense.

Definition 1.1. A vector-function $y(x) = (y_1(x), \dots, y_n(x)) \in C(I, \mathbb{R}^n)$ where $y_i \in C^1(I_i, \mathbb{R})$, $i = 1, \dots, n$, is said to be a solution of the problem (1.1), (1.2) if it satisfies the system (1.1) on J and, moreover, condition (1.2) holds.

Although singular boundary value problems were widely considered by using various methods (see e.g. Chechyk [2] – Diblík and Růžičková [7], Kiguradze [9] – Půža [13]), the method used here is based on a different approach, namely, simultaneously uses together the topological method of T. Ważewski and Schauder's principle. (Note that the method of T. Ważewski was applied to investigation of various asymptotic and singular problems, e.g., in Diblík [3] – Diblík and Růžičková [7], Vrdoljak [14] – Ważewski [15].) Each equation of the system (1.1) is considered separately (as a scalar equation) under supposition that nondiagonal variables are changed by functions taken from a given set of vector functions M. For every scalar equation (together with corresponding Cauchy initial condition which is subtracted from (1.2)) is shown (with the aid of Ważewski's method and qualitative properties of solutions of differential equations) that there is its solution with the same properties which are supposed for corresponding coordinate of vector functions from M. In this way an operator T is defined. For verification of conditions of Schauder's principle, Ważewski's method is used again. Stationary point of operator T defines a solution of the problem (1.1), (1.2).

2. EXISTENCE OF SOLUTIONS OF THE PROBLEM (1.1), (1.2)

We define $\Omega_i = \{(x, y_1, \dots, y_n) : x \in I_i, (x, y_1, \dots, y_n) \in \Omega\}$, where

$$\Omega = \{(x, y_1, \dots, y_n) : x \in I, \alpha_i(x) \leq y_i \leq \beta_i(x), i = 1, \dots, n\},$$

$\alpha_i, \beta_i \in C^1(I, \mathbb{R}), \alpha_i(x_i) = \beta_i(x_i) = A_i$ and $\alpha_i(x) < \beta_i(x)$ on $I_i, i = 1, \dots, n$. Let us suppose that there exists a domain $\Theta_i, i = 1, 2 \dots, n$ such that $\Theta_i \subset I_i \times \mathbb{R}^n, \Omega_i \subset \Theta_i$, cross section $S(x) = \{(x, y) \in \Theta_i\}$ is an open set for every $x \in I_i$ and $f_i \in C(\Theta_i, \mathbb{R})$. This assumption will be tacitly supposed in the sequel. Define, moreover, $\Gamma_i = \{(x, y_{i_1}, \dots, y_{i_{n-1}}) : x \in I_i, \{i_1, \dots, i_{n-1}\} = \{1, \dots, n\} \setminus \{i\},\ \alpha_s(x) \leq y_s \leq \beta_s(x), s = i_1, \dots, i_{n-1}\}$. Let us define for $i = 1, \dots, n$ auxiliary functions

$$F_i(x, y) \equiv \omega_i(x) y_i - y_i' + f_i(x, y).$$

The result of the paper is given in the following theorem.

Theorem 2.1. *Assume that:*

$$F_i(x,y)\,|_{y_i=\alpha_i(x)}\cdot F_i(x,y)\,|_{y_i=\beta_i(x)}<0 \tag{2.1}$$

if $(x,y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\in\Gamma_i, i=1,\dots,n$. *Let, moreover,*

$$|f_i(x,y)-f_i(x,z)|\le\sum_{j=1}^{n}M_{ij}(x)|y_j-z_j| \tag{2.2}$$

for any $(x,y_1,\dots,y_n),(x,z_1,\dots,z_n)\in\Omega_i$, *where* $M_{ij}(x)$ *are continuous on* I_i *functions,* $i,j=1,\dots,n$,

$$|\omega_i(x)|>\sum_{j=1}^{n}M_{ij}(x),\ x\in I_i,\quad i=1,\dots,n, \tag{2.3}$$

$$\omega_i(x)F_i(x,y)\,|_{y_i=\beta_i(x)}>0,\quad i=1,\dots,n \tag{2.4}$$

if $(x,y_1,\dots,y_{i-1},y_{i+1},\dots,y_n)\in\Gamma_i$.

Then there exists at least one solution $y(x)=(y_1(x),\dots,y_n(x))$ *of the problem* (1.1), (1.2) *such that for* $x\in I_i$: $\alpha_i(x)<y_i(x)<\beta_i(x)$, $i=1,\dots,n$.

3. PRELIMINARIES

To obtain our results, we will apply topological method of T. Ważewski (e.g. Hartman [8], Ważewski [15]). Therefore we give a short summary of it. Let us consider the system of ordinary differential equations

$$y'=g(x,y), \tag{3.1}$$

where $y\in\mathbb{R}^n$. Below it will be assumed that the right-hand sides of the system (3.1) are continuous functions defined on an open (x,y)-set $\Omega^*\subset\mathbb{R}\times\mathbb{R}^n$.

Definition 3.1. (see Hartman [8]) An open subset Ω^0 of the set Ω^* is called an (n,p)-subset of Ω^* with respect to the system (3.1) if following conditions are satisfied:

(1) There exist continuously differentiable functions $n_i:\Omega\to\mathbb{R}, i=1,\dots,l$ and $p_j:\Omega\to\mathbb{R},\ j=1,\dots,m,\ l+m>0$ such that

$$\Omega^0=\{(x,y)\in\Omega^*:n_i(x,y)<0,p_j(x,y)<0\ \text{ for all }\ i,j\}.$$

(2) $\dot n_\alpha(x,y)<0$ holds for the derivatives of the functions $n_\alpha(x,y),\alpha=1,\dots,l$ along trajectories of (3.1) on the set

$$N_\alpha=\{(x,y)\in\Omega^*,n_\alpha(x,y)=0,n_i(x,y)\le 0,p_j(x,y)\le 0$$
$$\text{for all } i,j \text{ and } \alpha,i\ne\alpha\}.$$

(3) $\dot{p}_\beta(x,y) > 0$ holds for the derivatives of the functions $p_\beta(x,y), \beta = 1,\dots,m$ along trajectories of (3.1) on the set

$$P_\beta = \{(x,y) \in \Omega^*, p_\beta(x,y) = 0, n_i(x,y) \le 0, p_j(x,y) \le 0 \quad \text{for all } i,j \text{ and } \beta, j \ne \beta\}.$$

As usual, if $\omega \subset \mathbb{R} \times \mathbb{R}^n$, then $\operatorname{int}\omega, \partial\omega$ and $\overline{\omega}$ denote the interior, the boundary and the closure of ω, respectively.

Definition 3.2. (Hartman [8]) The point $(x_0, y_0) \in \Omega^* \cap \partial\Omega^0$ is called an egress point of Ω^0 with respect to the system (3.1) if, for every solution of the problem $y(x_0) = y_0$, there is an $\varepsilon > 0$ such that $(x, y(x)) \in \Omega^0$ for $x_0 - \varepsilon \le x < x_0$. An egress point (x_0, y_0) of Ω^0 is called a strict egress point of Ω^0 if $(x, y(x)) \notin \overline{\Omega}^0$ on interval $x_0 < x \le x_0 + \varepsilon_1$ for a small $\varepsilon_1 > 0$. The set of all points of egress (strict egress) is denoted by Ω^0_e (Ω^0_{se}).

Definition 3.3. (Hartman [8]) The point $(x_0, y_0) \in \Omega^* \cap \partial\Omega^0$ is called an ingress point of Ω^0 with respect to the system (3.1) if for every solution of the problem $y(x_0) = y_0$ there is an $\varepsilon > 0$ such that $(x, y(x)) \in \Omega^0$ for $x_0 < x \le x_0 + \varepsilon$. An ingress point (x_0, y_0) of Ω^0 is called a strict ingress point of Ω^0 if $(x, y(x)) \notin \overline{\Omega}^0$ on interval $x_0 - \varepsilon_2 \le x < x_0$ for a small $\varepsilon_2 > 0$. The set of all points of ingress (strict ingress) is denoted by Ω^0_i (Ω^0_{si}).

Lemma 3.4. (Hartman [8]) *Let Ω^0 be an (n,p)-subset of Ω^* with respect to the system* (3.1)*. Then*

$$\Omega^0_{se} = \Omega^0_e = \bigcup_{\beta=1}^{m} P_\beta \setminus \bigcup_{\alpha=1}^{l} N_\alpha.$$

Theorem 3.5. (Hartman [8]) *Let Ω^0 be some (n,p)-subset of Ω^* with respect to the system* (3.1)*. Let S be a nonempty compact subset of $\Omega^0 \cup \Omega^0_e$ such that the set $S \cap \Omega^0_e$ is not a retract of S but is a retract of Ω^0_e. Then there is at least one point $(x_0, y_0) \in S \cap \Omega^0$ such that the graph of the solution $y(x)$ of the Cauchy problem $y(x_0) = y_0$ lies in Ω^0 on its right-hand maximal interval of existence.*

4. PARTIAL SINGULAR PROBLEMS

In this part we will be interested in existence of solutions of some auxiliary singular problems for one scalar equation. We consider three cases below with respect to location of singular point (at the left end of an interval, at the right end, or within an interval).

4.1. SINGULAR POINT COINCIDES WITH THE LEFT END OF INTERVAL

Consider the initial problem

$$y' = A(x)y + B(x, y), \tag{4.1}$$

$$y(u^+) = K \tag{4.2}$$

on an interval $(u, v]$ with $u < v$. By a solution of the problem (4.1), (4.2) on interval $(u, v]$ we mean the function $y \in C([u, v]) \cap C^1((u, v])$ which satisfies equation (4.1) on $(u, v]$ and the condition (4.2).

Let us involve functions $\lambda(x), \mu(x)$ which are continuously differentiable on $(u, v]$, $\lambda(u^+) = \mu(u^+) = K$ and $\lambda(x) < \mu(x)$ on $(u, v]$. Denote

$$\Theta^+ = \{(x, y) : x \in (u, v], \lambda(x) < y < \mu(x)\}.$$

Let us suppose that there exists a domain $\tilde{\Theta}$, such that $\tilde{\Theta} \subset (u, v] \times \mathbb{R}, \Theta^+ \subset \tilde{\Theta}$ and cross section $S(x) = \{(x, y) \in \tilde{\Theta}\}$ is an open set for every $x \in (u, v]$. Define an auxiliary function

$$H(x, y) = A(x)y - y' + B(x, y).$$

Lemma 4.1. *Suppose $A \in C((u, v], \mathbb{R}), B \in C(\tilde{\Theta}, \mathbb{R})$ satisfies the local Lipschitz condition with respect to the variable y in Θ^+ and, moreover,*

$$H(x, \lambda(x)) < 0 < H(x, \mu(x)) \quad \text{if} \quad x \in (u, v]. \tag{4.3}$$

Then each point (v, y^*), where $y^* \in [\lambda(v), \mu(v)]$ defines a solution $y = y^*(x)$ of equation (4.1) on $(u, v]$ such that (4.2) holds, $y^*(v) = y^*$ and, moreover,

$$\lambda(x) < y^*(x) < \mu(x). \tag{4.4}$$

Proof of Lemma 4.1. Let us evaluate the derivative of the function $w(x, y) \equiv (y - \lambda(x))(y - \mu(x))$ along the trajectories of the equation (4.1) if $(x, y) \in \mathcal{N}$, where

$$\mathcal{N} = \{(x, y) : x \in (u, v], w(x, y) = 0\}.$$

We get

$$\frac{dw(x, y)}{dx}$$
$$= [A(x)y + B(x, y) - \lambda'(x)](y - \mu(x)) + (y - \lambda(x))[A(x)y + B(x, y) - \mu'(x)].$$

Since $(x, y) \in \mathcal{N}$, then either $y = \mu(x)$ or $y = \lambda(x)$. In the first case we have

$$\left.\frac{dw(x, y)}{dx}\right|_{y=\mu(x)} = (\mu(x) - \lambda(x)) \cdot H(x, \mu(x))$$

and in the second one

$$\left.\frac{dw(x,y)}{dx}\right|_{y=\lambda(x)} = -H(x,\lambda(x))\cdot(\mu(x)-\lambda(x)).$$

Thus, in view of condition (4.3),

$$\left.\frac{dw(x,y)}{dx}\right|_{(x,y)\in\mathcal{N}} > 0$$

and, consequently, all points of the set $\mathcal{N} = \tilde{\Theta}\cap\partial\Theta^+$ are for $x \in (u,v)$ the points of strict egress of Θ^+ with respect to equation (4.1).

Let us consider the solution $y = y^*(x)$ of the problem $y^*(v) = y^* \in [\lambda(v),\mu(v)]$ for decreasing values of $x \in (u,v]$. Let us suppose that this solution leaves the domain Θ^+ passing through a boundary point $(x^0, y^*(x^0)) \in \mathcal{N}$, where $x^0 \in (u,v)$ and $(x,y(x)) \in \Theta^+$ for $x \in (x^0,v]$. In this case this point is a point of ingress (for increasing x) with respect to equation (4.1) and this contradicts the fact that each point of the set $\mathcal{N}$ is for $x \in (u,v)$ a point of strict egress. If, nevertheless, the point $(x^0, y^*(x^0))$ is a point of strict egress, then there is a point $\tilde{x} \in (u,v)$ such that $y^{*\prime}(\tilde{x}) = \infty$. This contradicts the well known facts of theory of ordinary differential equations. By the same reason this solution cannot leave the domain Θ^+ passing through a boundary point (v, y^{**}), where $y^{**} \in [\lambda(v),\mu(v)]$.

Only one possibility remains valid: solution $y^*(x)$ is simultaneously a solution of the problem (4.1), (4.2). The lemma is proved. □

Lemma 4.2. *Let all assumptions of Lemma* 4.1 *hold except for the condition* (4.3) *which is replaced by the condition:*

$$H(x,\mu(x)) < 0 < H(x,\lambda(x)) \quad \text{if} \quad x \in (u,v]. \tag{4.5}$$

Then there is at least one solution $y = y^*(x)$ *of the problem* (4.1), (4.2) *on* $(u,v]$ *such that inequalities* (4.4) *hold.*

Proof of Lemma 4.2. Let us define the set $\mathcal{N}$ and the function $w(x,y)$ in the same way as in the proof of Lemma 4.1.

Then the derivative of $w(x,y)$ along the trajectories of the equation (4.1) satisfies, in view of condition (4.5), the inequality

$$\left.\frac{dw(x,y)}{dx}\right|_{(x,y)\in\mathcal{N}} < 0.$$

This means that all points of the set $\mathcal{N}$ for $x \in (u,v)$ are the points of strict ingress of Θ^+ with respect to the equation (4.1).

Let us change the orientation of the x-axis into reverse. Then all points of the set $\mathcal{N}$ are for $x \in (u,v)$ the points of strict egress of Θ^+ with respect to the equation (4.1).

It is easy to see that the two-point set $\{\lambda(v-\delta), \mu(v-\delta)\}$, where δ is a small positive number, is a retract of the set $\mathcal{N}$ in view of existence of the retraction

$$r(x,y) = \left(v-\delta, \mu(v-\delta) + [\lambda(v-\delta) - \mu(v-\delta)]\frac{y-\mu(x)}{\lambda(x)-\mu(x)}\right),$$

where $(x,y) \in \mathcal{N}$. Clearly, the set $S = [\lambda(v-\delta), \mu(v-\delta)]$ is not a retract of its boundary $\delta S = \{\lambda(v-\delta), \mu(v-\delta)\}$ (see, e.g. Borsuk [1]) . All assumptions of topological principle of Ważewski are valid and, by Theorem 3.5, in its formulation we put $\Omega^0 \equiv \mathrm{int}\Theta^+, p_1(x,y) \equiv w(x,y), j = 1, n_1 \equiv x - v, l = 1$, there exists at least one solution $y = y^*(x)$ of the problem (4.1), (4.2) with graph belonging to the domain Θ^+ on $(u, v-\delta]$. By the same arguments, as in the proof of Lemma 4.1, this solution can continue on the interval $(u, v]$. The lemma is proved. □

4.2. SINGULAR POINT COINCIDES WITH THE RIGHT END OF INTERVAL

Let us consider the initial problem (4.1), (4.6), where

$$y(v^-) = K \tag{4.6}$$

on an interval $[u, v)$, with $u < v$. By a solution of the problem (4.1), (4.6) on interval $[u, v)$ we mean the function $y \in C([u,v]) \cap C^1([u,v))$ which satisfies equation (4.1) on interval $[u, v)$ and the condition (4.6).

Let us involve functions $\lambda(x), \mu(x)$ which are continuously differentiable on $[u, v)$, $\lambda(v^-) = \mu(v^-) = K$ and $\lambda(x) < \mu(x)$ on $[u, v)$. Denote

$$\Theta^- = \{(x,y) : x \in [u,v), \lambda(x) < y < \mu(x)\}.$$

Let us suppose that there exists a domain $\bar{\Theta}$, such that $\bar{\Theta} \subset [u,v) \times \mathbb{R}, \Theta^- \subset \bar{\Theta}$ and cross section $S(x) = \{(x,y) \in \bar{\Theta}\}$ is an open set for every $x \in [u, v)$.

The proofs of following Lemma 4.3 and Lemma 4.4 can be made by the similar manner as the proofs of Lemma 4.1 and Lemma 4.2. Hence, they are omitted.

Lemma 4.3. *Suppose $A \in C([u,v), \mathbb{R}), B \in C(\bar{\Theta}, \mathbb{R})$ satisfies the local Lipschitz condition with respect to the variable y in Θ^- and, moreover,*

$$H(x, \lambda(x)) < 0 < H(x, \mu(x)) \quad \text{if} \quad x \in [u, v). \tag{4.7}$$

Then there is at least one solution $y = y^{**}(x)$ of the problem (4.1), (4.6) *on $(u, v]$ such that*

$$\lambda(x) < y^{**}(x) < \mu(x). \tag{4.8}$$

Lemma 4.4. *Let all assumptions of Lemma* 4.3 *hold except for the condition* (4.7) *which is replaced by the condition:*

$$H(x,\mu(x)) < 0 < H(x,\lambda(x)) \quad \text{if } \; x \in [u,v). \tag{4.9}$$

Then each point (u, y^{**}), *where* $y^{**} \in [\lambda(u), \mu(u)]$ *defines a solution* $y = y^{**}(x)$ *of equation* (4.1) *on* $[u,v)$, $y^{**}(u) = y^{**}$ *and, moreover, the inequalities* (4.8) *hold.*

4.3. SINGULAR POINT IS EQUAL TO AN INTERIOR POINT OF INTERVAL

Consider the initial problem (4.1), (4.10) with

$$y(r^{\pm}) = K \tag{4.10}$$

on an interval $[u,v]$, where $u < r < v$.

Definition 4.5. By a solution of the problem (4.1), (4.10) on interval $[u,r)\cup(r,v]$ we mean the function $y \in C([u,v]) \cap C^1([u,r)\cup(r,v])$ which satisfies equation (4.1) on $[u,r)\cup(r,v]$ and the condition (4.10).

Involve functions $\lambda(x), \mu(x)$ which are continuously differentiable on $[u,r)\cup(r,v]$, $\lambda(r^{\pm}) = \mu(r^{\pm}) = K, \lambda(x) < \mu(x)$ on $[u,r)\cup(r,v]$ and define a domain

$$\Theta^{\pm} = \{(x,y) : x \in [u,r)\cup(r,v], \lambda(x) < y < \mu(x)\}.$$

Let us suppose that there exists a domain $\hat{\Theta}$, such that $\hat{\Theta} \subset [u,r)\cup(r,v] \times \mathbb{R}, \Theta^{\pm} \subset \hat{\Theta}$ and cross section $S(x) = \{(x,y) \in \hat{\Theta}\}$ is an open set for every $x \in [u,r)\cup(r,v]$.

Proofs of the next four lemmas are a composition of the proofs of Lemmas 4.1 – 4.4 and, therefore, are omitted.

Lemma 4.6. *Suppose* $A \in C([u,r)\cup(r,v],\mathbb{R}), B \in C(\hat{\Theta},\mathbb{R})$ *satisfies the local Lipschitz condition with respect to variable* y *in* $\Theta^{\pm}$ *and, moreover,*

$$H(x,\lambda(x)) < 0 < H(x,\mu(x)) \quad \text{if } \; x \in [u,r)\cup(r,v]. \tag{4.11}$$

Then through each point (v, y^{*}), *where* $y^{*} \in [\lambda(v), \mu(v)]$ *passes a solution* $y = y^{*}(x)$ *of equation* (4.1) *on* $[u,r)\cup(r,v], y^{*}(v) = y^{*}$ *and, moreover,*

$$\lambda(x) < y^{*}(x) < \mu(x) \qquad \text{on } \; [u,r)\cup(r,v]. \tag{4.12}$$

Lemma 4.7. *Let all assumptions of Lemma* 4.6 *hold except for the condition* (4.11) *which is replaced by the condition:*

$$H(x,\mu(x)) < 0 < H(x,\lambda(x)) \quad \text{if } \; x \in [u,r)\cup(r,v].$$

Then through each point (u, y^{**}), *where* $y^{**} \in [\lambda(u), \mu(u)]$ *passes a solution* $y = y^{**}(x)$ *of equation* (4.1) *on* $[u,r)\cup(r,v], y^{**}(u) = y^{**}$ *and, moreover, the inequalities* (4.12) *hold.*

Lemma 4.8. *Let all assumptions of Lemma* 4.6 *hold except for the condition* (4.11) *which is replaced by condition:*

$$H(x,\lambda(x)) < 0 < H(x,\mu(x)) \quad \text{if} \quad x \in [u,r),$$

$$H(x,\mu(x)) < 0 < H(x,\lambda(x)) \quad \text{if} \quad x \in (r,v].$$

Then there is at least one solution $y = y(x)$ of the problem (4.1), (4.10) *such that the inequalities* (4.12) *hold.*

Lemma 4.9. *Let all assumptions of Lemma* 4.6 *hold except for the condition* (4.11) *which is replaced by condition:*

$$H(x,\mu(x)) < 0 < H(x,\lambda(x)) \quad \text{if} \quad x \in [u,r),$$

$$H(x,\lambda(x)) < 0 < H(x,\mu(x)) \quad \text{if} \quad x \in (r,v].$$

Then through each two points (v,y^*), where $y^* \in [\lambda(v),\mu(v)]$ and (u,y^{**}), where $y^{**} \in [\lambda(u),\mu(u)]$ a solution $y = \tilde{y}(x)$ of the problem (4.1), (4.10) *on $[u,r) \cup (r,v], \tilde{y}(v) = y^*, \tilde{y}(u) = y^{**}$ passes and the inequalities* (4.12) *hold.*

5. PROOF OF THEOREM 1

5.1. CONSTRUCTION OF OPERATOR *T*

Let us consider the system

$$y_i' = \omega_i(x)y_i + f_i(x,\varphi_1(x),\dots,\varphi_{i-1}(x),y_i,\varphi_{i+1}(x),\dots,\varphi_n(x)),\ i = 1,2,\dots,n \tag{5.1}$$

with $(\varphi_1(x),\dots,\varphi_n(x)) \in M$, where

$$M = \{(\varphi_1(x),\varphi_2(x),\dots,\varphi_n(x)),$$

$$x \in I, \varphi_i \in C\left(I,\mathbb{R}\right), \alpha_i(x) \le \varphi_i(x) \le \beta_i(x), i = 1,2,\dots,n\}.$$

This system, strictly speaking, consists of separated scalar equations. Therefore in the sequel we shall consider equations of the system (5.1) separately.
a) Let us consider the first equation of the system (5.1) (which corresponds to the value $i = 1$) together with corresponding initial value which follows from (1.2):

$$y_1'(x) = \omega_1(x)y_1 + f_1(x,y_1,\varphi_2(x),\dots,\varphi_n(x)), \tag{5.2}$$

$$y_1(x_1^+) = A_1. \tag{5.3}$$

Let us put $A(x) \equiv \omega_1(x),\ B(x,y) \equiv f_1(x,y,\varphi_2(x),\dots,\varphi_n(x)), \lambda(x) \equiv \alpha_1(x)$, $\mu(x) \equiv \beta_1(x), u = x_1, v = x_n$ and $K = A_1$. In view of condition (2.1) we see that either condition (4.3) or condition (4.5) holds for $H(x,y) \equiv F_1(x,y)$. From Lemmas 4.1, 4.2 the existence of a solution of the problem (5.2), (5.3) satisfying inequalities (4.4) follows.

In the sequel we will take the unique solution of the problem (5.2), (5.3) $y_1(x) = y^*(x)$. This solution is defined, in the case of Lemma 4.1, by means of additional condition

$$y_1(x_n^-) = y^*(v) = y_1^* = \frac{1}{2}\left(\alpha_1(x_n) + \beta_1(x_n)\right).$$

If Lemma 4.2 holds, then let us denote the set of all solutions with indicates properties as a set Y and put $y_1(x_n^-) = y^*(v) = \min\{y(v) : y \in Y\}$. Obviously $y^*(v) > \lambda(v)$.

We define $T_L(\varphi_2, \dots, \varphi_n) = y_1^*$. From inequalities (4.4) it follows that $(y_1^*, \varphi_2, \dots, \varphi_n) \in M$. The same reasonings can be repeated for $i = 2, \dots, k$.

b) Now consider the last equation of the system (5.1) (which corresponds to the value $i = n$) together with corresponding initial value which follows from (1.2):

$$y_n' = \omega_n(x)y_n + f_n(x, \varphi_1(x), \dots, \varphi_{n-1}(x), y_n), \tag{5.4}$$

$$y_n(x_n^-) = A_n. \tag{5.5}$$

Let us put $A(x) \equiv \omega_n(x), B(x,y) \equiv f_n(x, \varphi_1(x), \dots, \varphi_{n-1}(x), y), \lambda(x) \equiv \alpha_n(x)$, $\mu(x) \equiv \beta_n(x), u = x_1, v = x_n$ and $K = A_n$. In view of condition (2.1) we see that either condition (4.7) or condition (4.9) holds for $H(x,y) \equiv F_n(x,y)$. From Lemmas 4.3, 4.4 the existence of a solution of the problem (5.4), (5.5) satisfying inequalities (4.8) follows. As above, in the part a), we will take the unique solution of the problem (5.4),(5.5) $y_n^*(x) = y^{**}(x)$.

We define $T_R(\varphi_1, \dots, \varphi_{n-1}) = y_n^*$. From (4.8) it follows: $(\varphi_1, \dots, \varphi_{n-1}, y_n^*) \in M$. The same reasonings can be repeated for $i = s+1, \dots, n-1$.

c) Let us consider the equation of the system (5.1) which corresponds to the value $i = s$ together with corresponding initial value which follows from (1.2):

$$y_s' = \omega_s(x)y_s + f_s(x, \varphi_1(x), \dots, \varphi_{s-1}(x), y_s, \varphi_{s+1}(x), \dots, \varphi_n(x)), \tag{5.6}$$

$$y_s(x^{\pm}) = A_s. \tag{5.7}$$

Let $A(x) \equiv \omega_s(x), B(x,y) \equiv f_s(x, \varphi_1(x), \dots, \varphi_{s-1}(x), y, \varphi_{s+1}(x), \dots, \varphi_n(x)$, $\lambda(x) \equiv \alpha_s(x), \mu(x) \equiv \beta_s(x)$ and $K = A_s$. Consider, at first, the problem (5.6), (5.7) on interval $[x_1, x_s)$. For this, let us put $u = x_1, v = x_s$. In view of condition (2.1) we see that either condition (4.7) or condition (4.9) holds for $H(x,y) = F_s(x,y)$ and with the aid of Lemmas 4.3, 4.4 (as in the part b)) we can define the unique solution of (5.6), (5.7) $y_s(x) = y^{**}(x)$ on interval $[x_1, x_s)$.

Now consider the problem (5.6), (5.7) on interval $(x_s, x_n]$. Put $u = x_s, v = x_n$. In view of condition (2.1) we see that either condition (4.3) or condition (4.5) holds for $H(x, y) \equiv F_s(x, y)$ and with the aid of Lemmas 4.1, 4.2 (as in part a)) we define the unique solution of (5.6), (5.7) $y_s(x) = y^*(x)$ on interval $(x_s, x_n]$.

At the end we define, by a unique manner, the solution $y_s^*(x)$ of the problem (5.6), (5.7) as

$$y_s^*(x) = \begin{cases} y^{**}(x), & x \in [x_1, x_s), \\ y^*(x), & x \in (x_s, x_n]. \end{cases}$$

We define $T_M(\varphi_1, \ldots, \varphi_{s-1}, \varphi_{s+1}, \ldots, \varphi_n) = y_s^*$. Then

$$(\varphi_1, \ldots, \varphi_{s-1}, y_s^*, \varphi_{s+1}, \ldots, \varphi_n) \in M.$$

The same reasonings can be repeated for $i = k + 1, \ldots, s - 1$.

d) Now we are able to define operator T. For $\varphi = (\varphi_1, \ldots, \varphi_n) \in M$ define $T\varphi = y^*$, where $y^* = (y_1^*, \ldots, y_n^*) \in M$. Note that y^* is defined by an unique way. Obviously, $T(M) \subset M$.

5.2. VERIFICATION OF SCHAUDER'S ASSUMPTIONS

Let us consider the Banach space Ψ of functions $\psi = (\psi_1, \psi_2, \ldots, \psi_n)$, continuous on I, with the norm

$$\|\psi\| = \max_{i=1,2,\ldots,n} \left\{ \max_I |\psi_i(x)| \right\}.$$

Obviously $M \subset \Psi$ and, as it follows from the properties of the functions $\alpha_i(x), \beta_i(x)$, $i = 1, 2, \ldots, n$, M is a closed, bounded and convex set.

It remains to prove that T is a continuous mapping such that $T(M)$ is a relatively compact subset of Ψ. With respect to relatively compactness of $T(M)$ it is sufficient to prove by Arzelà–Ascoli Theorem that $T(M)$ is uniformly bounded and equicontinuous on I.

$\boldsymbol{\alpha}$) Denote $L = \max_I \{|\alpha_i(x)|, |\beta_i(x)|, i = 1, 2, \ldots, n\}$. The *uniform boundedness* follows from inequality $\|\varphi\| < L$ which holds for every $\varphi \in M$.

$\boldsymbol{\beta}$) Let us prove the *equicontinuity* of each function $\varphi \in T(M)$. On I_1 the first coordinate φ_1 of φ satisfies (as it follows from construction of T) an equation of the type

$$\varphi_1'(x) = \omega_1(x)\varphi_1(x) + f_1(x, \varphi_1(x), \nu_2(x), \ldots, \nu_n(x)), \tag{5.8}$$

where $(\varphi_1, \nu_2, \dots, \nu_n) \in M$. Since $\omega_1 \in C(I_1, \mathbb{R})$ and $f_1 \in C(\Omega_1, \mathbb{R})$, from (5.8) we get

$$|\varphi_1'(x)| < K_\delta, \quad x \in [x_1 + \delta, x_n], \quad x_1 + \delta < x_n, \quad 0 < \delta = \text{ const},$$

where the constant K_δ exists and depends on δ. Put $\delta_1 = \min(\delta/2, \varepsilon/K_{\delta/2})$, where ε is an arbitrary positive number and δ is so small that

$$\max_{[x_1, x_1+\delta]} |\beta_1(x) - A_1| < \varepsilon/2, \qquad \max_{[x_1, x_1+\delta]} |\alpha_1(x) - A_1| < \varepsilon/2.$$

Let us suppose that $|z_1 - z_2| < \delta_1, z_1, z_2 \in [x_1, x_n]$. Then either $z_1, z_2 \in [x_1, x_1 + \delta]$ or $z_1, z_2 \in [x_1 + \delta/2, x_n]$. In the first case

$$|\varphi_1(z_1) - \varphi_1(z_2)| \le |\varphi_1(z_1) - A_1| + |\varphi_1(z_2) - A_1| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and in the second one (by Lagrange's Mean Value Theorem)

$$|\varphi_1(z_1) - \varphi_1(z_2)| \le K_{\delta/2}\, |z_1 - z_2| < \varepsilon.$$

So, for each positive ε there is a $\delta_1 > 0$ such that $|\varphi_1(z_1) - \varphi_1(z_2)| < \varepsilon$ for $|z_1 - z_2| < \delta_1$ and each function of the type of $\varphi_1(x)$ is equicontinuous. By analogy we can show that the functions of the type $\varphi_j(x), j = 2, \dots, n$ are equicontinuous too. Finally, for $|z_1 - z_2| < \delta_1$, we get $\|\varphi(z_1) - \varphi(z_2)\| < \varepsilon$ and the equicontinuity of the set $T(M)$ is proved.

$\boldsymbol{\gamma}$) *Continuity* of operator T. Let us suppose that $y^0 \in M, \tilde{y}(x) \in M$ and

$$Y^0 = Ty^0, \;\; \tilde{Y} = T\tilde{y}.$$

In the sequel we prove that the operator T is continuous, i.e. that

$$\|Y^0 - \tilde{Y}\| < \varepsilon \quad \text{if} \quad \|y^0 - \tilde{y}\| < \delta \le \varepsilon. \tag{5.9}$$

The last inequality (in which ε is an arbitrary sufficiently small positive number) will be supposed at the sequel. Consider the identity

$$Y_i^{0\prime}(x) \equiv \omega_i(x) Y_i^0(x) + f_i(x, \eta_1^0(x), \eta_2^0(x), \dots, \eta_n^0(x)),$$

with $i = 1, 2, \dots, n,\ \eta_i^0(x) = Y_i^0(x),\ \eta_j^0(x) \equiv y_j^0(x), j \ne i,\ (x, \eta_1^0(x), \eta_2^0(x), \dots, \eta_n^0(x)) \in \Omega_i$ and the equation (which has a solution $\tilde{Y}_i \equiv Y_i(x)$)

$$\tilde{Y}_i' = \omega_i(x)\tilde{Y}_i + f_i(x, \tilde{\eta}_1(x), \tilde{\eta}_2(x), \dots, \tilde{\eta}_n(x)), \tag{5.10}$$

where $i = 1, 2, \dots, n,\ \tilde{\eta}_i = \tilde{Y}_i, \tilde{\eta}_j = \tilde{\eta}_j(x) \equiv \tilde{y}_j(x), j \ne i,\ (x, \tilde{\eta}_1, \tilde{\eta}_2, \dots, \tilde{\eta}_n) \in \Omega_i$. Let us define the functions

$$W_i(x, \tilde{Y}_i) = (\tilde{Y}_i - Y_i^0(x) - \varepsilon)(\tilde{Y}_i - Y_i^0(x) + \varepsilon),\ i = 1, 2, \dots, n,$$

and the sets

$$\mathcal{P}_i = \{(x, \tilde{Y}_i) \in \Omega : x \in I_i, W_i(x, \tilde{Y}_i) = 0\},\ i = 1, 2, \dots, n,$$

$$\mathcal{G}_i = \{(x, \tilde{Y}_i) \in \Omega : x \in I_i, W_i(x, \tilde{Y}_i) < 0\},\ i = 1, 2, \dots, n.$$

γ_1) Let us evaluate the derivative of $W_1(x,\tilde{Y}_1)$ along the trajectories of equation (5.10) for $i=1$ if $(x,\tilde{Y}_1)\in\mathcal{P}_1$. We get

$$\frac{dW_1(x,\tilde{Y}_1)}{dx} = [\tilde{Y}_1' - Y_1^{0\prime}(x)](\tilde{Y}_1 - Y_1^0(x)+\varepsilon) + (\tilde{Y}_1 - Y_1^0(x)-\varepsilon)[\tilde{Y}_1' - Y_1^{0\prime}(x)].$$

Since $(x,\tilde{Y}_1)\in\mathcal{P}_1$, then either $\tilde{Y}_1 = Y_1^0(x)+\varepsilon$ or $\tilde{Y}_1 = Y_1^0(x)-\varepsilon$. We have

$$\left.\frac{dW_1(x,\tilde{Y}_1)}{dx}\right|_{\tilde{Y}_1=Y_1^0(x)\pm\varepsilon} = \pm 2\varepsilon[\omega_1(x)(\pm\varepsilon) + f_1(x,Y_1^0(x)\pm\varepsilon,\tilde{y}_2(x),\dots,\tilde{y}_n(x)) - f_1(x,Y_1^0(x),y_2^0(x),\dots,y_n^0(x))].$$

According to (2.2) and (2.3)

$$|f_1(x,Y_1^0(x)\pm\varepsilon,\tilde{y}_2(x),\dots,\tilde{y}_n(x)) - f_1(x,Y_1^0(x),y_2^0(x),\dots,y_n^0(x))| \le \left(\sum_{j=1}^{n} M_{1j}(x)\right)\varepsilon < \varepsilon|\omega_1(x)|.$$

Therefore

$$\left.\frac{dW_1(x,\tilde{Y}_1)}{dx}\right|_{(x,\tilde{Y}_1)\in\mathcal{P}_1} > 0 \text{ if } \omega_1(x)>0 \text{ on } I_1 \tag{5.11}$$

and

$$\left.\frac{dW_1(x,\tilde{Y}_1)}{dx}\right|_{(x,\tilde{Y}_1)\in\mathcal{P}_1} < 0 \text{ if } \omega_1(x)<0 \text{ on } I_1. \tag{5.12}$$

If (5.11) and Lemma 4.1 (in situation, described in Subsection 5.1, a)) hold simultaneously, then all points of the set $\partial\mathcal{Q}_1$, where $\mathcal{Q}_1 = \{(x,Y_1) : x\in(x_1,x_n], w(x,Y_1)<0, W_1(x,Y_1)<0\}$ and w is defined in the proof of Lemma 4.1, are for $x\in(x_1,x_n)$ the points of strict egress for $\mathcal{Q}_1$ with respect to (5.10) with $i=1$ (since this equation is at the same time an equation of the type (5.1) for $i=1$). Since $Y_1^0(x_1^+)=\tilde{Y}_1(x_1^+)$ and (in view of construction of operator T) $Y_1^0(x_n^-)=\tilde{Y}_1(x_n^-)$ then $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$. Indeed if this inequality does not hold then there is a $x^*\in I_1$ such that $|Y_1^0(x^*)-\tilde{Y}_1(x^*)|=\varepsilon$ and by (5.11) $|Y_1^0(x)-\tilde{Y}_1(x)|>\varepsilon$ on $(x^*,x_n]$. This is impossible.

If (5.12) and Lemma 4.2 (in the situation, described in Subsection 5.1, a)) hold simultaneously, then all points of the set $\partial\mathcal{Q}_1$ are, for $x\in(x_1,x_n)$, the points of strict ingress for $\mathcal{Q}_1$ with respect to (5.10) with $i=1$. If inequality $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$ does not hold, then there is a $x^*\in I_1$ such that $|Y_1^0(x^*)-\tilde{Y}_1(x^*)|=\varepsilon$ and $|Y_1^0(x)-\tilde{Y}_1(x)|>\varepsilon$ on (x_1,x^*). This is impossible. In both considered cases $|Y_1^0(x)-\tilde{Y}_1(x)|<\varepsilon$ on I_1 and, consequently, on I too.

We conclude that in the above described cases

$$|\tilde{Y}_1(x)-Y_1^0(x)|<\varepsilon \text{ on } I \quad \text{if} \quad \|y^0-\tilde{y}\|<\delta\,.$$

If (5.12) and Lemma 4.1 (in the situation, described in Subsection 5.1, a)) hold simultaneously, then for small ε: $\mathcal{N}_1\cap\mathcal{P}_1\neq\emptyset$ and there exists a point $(x^\triangle,y^\triangle)\in\mathcal{N}_1\cap\mathcal{P}_1$ which is at the same time a point of strict egress a point of strict ingress for $\mathcal{G}_1$. This is impossible in view of (2.4). Analogously we can consider equation (5.10) if $i=2,\dots,k$.

$\boldsymbol{\gamma_2}$) Let us evaluate the derivative of $W_n(x,\tilde{Y}_n)$ along the trajectories of equation (5.10) for $i=n$ if $(x,\tilde{Y}_n)\in\mathcal{P}_n$. We get

$$\frac{dW_n(x,\tilde{Y}_n)}{dx}=[\tilde{Y}_n'-Y_n^{0\prime}(x)](\tilde{Y}_n-Y_n^0(x)+\varepsilon)$$
$$+(\tilde{Y}_n-Y_n^0(x)-\varepsilon)[\tilde{Y}_n'-Y_n^{0\prime}(x)].$$

Since $(x,\tilde{Y}_n)\in\mathcal{P}_n$, then either $\tilde{Y}_n=Y_n^0(x)+\varepsilon$ or $\tilde{Y}_n=Y_n^0(x)-\varepsilon$. We have

$$\left.\frac{dW_n(x,\tilde{Y}_n)}{dx}\right|_{\tilde{Y}_n=Y_n^0\pm\varepsilon}=\pm 2\varepsilon\,[\omega_n(x)(\pm\varepsilon)$$
$$+f_n(x,\tilde{y}_1(x),\dots,\tilde{y}_{n-1}(x),Y_n^0(x)\pm\varepsilon)-f_n(x,y_1^0(x),\dots,y_{n-1}^0(x),Y_n^0(x))]\,.$$

According to (2.2) and (2.3)

$$|f_n(x,\tilde{y}_1(x),\dots,\tilde{y}_{n-1}(x),Y_n^0(x)\pm\varepsilon)-f_n(x,y_1^0(x),\dots,y_{n-1}^0(x),Y_n^0(x))|$$
$$\le\left(\sum_{j=1}^n M_{nj}(x)\right)\varepsilon<\varepsilon|\omega_n(x)|.$$

Therefore

$$\left.\frac{dW_n(x,\tilde{Y}_n)}{dx}\right|_{(x,\tilde{Y}_n)\in\mathcal{P}_n}>0 \text{ if } \omega_n>0 \text{ on } I_n \tag{5.13}$$

and

$$\left.\frac{dW_n(x,\tilde{Y}_n)}{dx}\right|_{(x,\tilde{Y}_n)\in\mathcal{P}_n}<0 \text{ if } \omega_n<0 \text{ on } I_n. \tag{5.14}$$

If (5.13) and Lemma 4.3 (in the situation described in Subsection 5.1, b)) hold simultaneously, then all points of the set $\partial\mathcal{Q}_n$, where $\mathcal{Q}_n = \{(x, Y_n) : x \in (x_1, x_n), w(x, Y_n) < 0, W_n(x, Y_n) < 0\}$ and w is defined as in the proof of Lemma 4.3, are, for $x \in (x_1, x_n)$, the points of strict egress for $\mathcal{Q}_n$ with respect to (5.10), where $i = n$ (since this equation is at the same time an equation of the type (5.1) for $i = n$). If (5.14) and Lemma 4.4 (in the situation described in Subsection 5.1, b)) hold simultaneously, then all points of the set $\partial\mathcal{Q}_n$ for $x \in (x_1, x_n)$ are points of strict ingress. In both of these cases we can prove, as in part γ_1), that $|Y_n^0(x) - \tilde{Y}_n^0(x)| < \varepsilon$ on I if $\|y^0 - \tilde{y}\| < \delta$. Cases (5.13), Lemma 4.4 and (5.14), Lemma 4.3 are impossible according to (2.4). Analogously we can proceed if $i = s+1, \dots, n-1$.

$\boldsymbol{\gamma_3}$) Let us evaluate the derivative of W_t along the trajectories of equations (5.10) for $t = k+1, \dots, s$ if $(x, \tilde{Y}_t) \in \mathcal{P}_t$. We get

$$\frac{dW_t(x, \tilde{Y}_t)}{dx} = [\tilde{Y}_t' - Y_t^{0\prime}(x)](\tilde{Y}_t - Y_t^0(x) + \varepsilon) + (\tilde{Y}_t - Y_t^0(x) - \varepsilon)[\tilde{Y}_t' - Y_t^{0\prime}(x)].$$

Since $(x, \tilde{Y}_t) \in \mathcal{P}_t$, then either $\tilde{Y}_t = Y_t^0(x) + \varepsilon$ or $\tilde{Y}_t = Y_t^0(x) - \varepsilon$. We have

$$\left.\frac{dW_t(x, \tilde{Y}_t)}{dx}\right|_{\tilde{Y}_t = Y_t^0 \pm \varepsilon} = \pm 2\varepsilon \,[\pm\varepsilon\omega_t(x) + f_t(x, \tilde{y}_1(x), \dots, \tilde{y}_{t-1}(x), Y_t^0(x) \pm \varepsilon, \tilde{y}_{t+1}(x), \dots, \tilde{y}_n(x)) - f_t(x, y_1^0(x), \dots, y_{t-1}^0(x), Y_t^0(x), y_{t+1}^0(x), \dots, y_n^0(x))]\,.$$

According to (2.2) and (2.3)

$$|f_t(x, \tilde{y}_1(x), \dots, \tilde{y}_{t-1}(x), Y_t^0(x) \pm \varepsilon, \tilde{y}_{t+1}(x), \dots, \tilde{y}_n(x)) - f_t(x, y_1^0(x), \dots, y_{t-1}^0(x), Y_t^0(x), y_{t+1}^0(x), \dots, y_t^0(x))| \le \left(\sum_{j=1}^{n} M_{tj}(x)\right)\varepsilon < |\omega_t(x)|\,\varepsilon.$$

Therefore the following four cases (5.15)-(5.18) are possible:

$$\frac{dW_t(x, \tilde{Y}_t)}{dx} > 0 \quad \text{if } \omega_t > 0, \text{ on } I_t; \tag{5.15}$$

$$\begin{cases} \dfrac{dW_t(x,\tilde{Y}_t)}{dx} > 0 \quad \text{if} \quad \omega_t > 0 \quad \text{on} \quad [x_1, x_t), \\ \dfrac{dW_t(x,\tilde{Y}_t)}{dx} < 0 \quad \text{if} \quad \omega_t < 0 \quad \text{on} \quad (x_t, x_n]; \end{cases} \tag{5.16}$$

$$\frac{dW_t(x,\tilde{Y}_t)}{dx} < 0 \quad \text{if } \omega_t < 0 \text{ on } I_t; \tag{5.17}$$

$$\begin{cases} \dfrac{dW_t(x,\tilde{Y}_t)}{dx} < 0 \quad \text{if} \quad \omega_t < 0 \quad \text{on} \quad [x_1, x_t), \\ \dfrac{dW_t(x,\tilde{Y}_t)}{dx} > 0 \quad \text{if} \quad \omega_t > 0 \quad \text{on} \quad (x_t, x_n]. \end{cases} \tag{5.18}$$

Each of the admissible cases (i.e. if Lemma 4.6 and inequality (5.15) hold, or if Lemma 4.8 and inequalities (5.16) hold, or if Lemma 4.7 and inequality (5.17) hold, or if Lemma 4.9 and inequalities (5.18) hold) can be considered as above (see parts $\gamma_1)$ and $\gamma_2)$) and, therefore, for $t = k+1, \dots, s$ hold $|Y_t^0(x) - \tilde{Y}_t(x)| < \varepsilon$ on I if $\|y^0 - \tilde{y}\| < \delta$ (the remaining cases are impossible in view of (2.4)).

Connecting all parts $\gamma_1) - \gamma_3)$ we conclude that (5.9) holds and, consequently, operator T is continuous. All conditions of Schauder's principle are valid and, therefore, operator T has a fixed point, i.e. has a solution of problem (1.1), (1.2) with indicated properties which follow from the form of the set M. The proof is complete.

6. EXAMPLE

Let us consider singular problem:

$$y_1' = \frac{y_1}{x} + \frac{y_2+1}{10}\left(\frac{y_1}{x} - 1\right)\exp\left(-\frac{1}{x}\right),$$

$$y_2' = \frac{2y_2}{x} + \frac{y_1}{10}\left(\frac{y_2}{x^2} - 1\right)\exp\left(-\frac{1}{x^2}\right),$$

$$y_3' = \frac{2y_3}{x-1} + \frac{y_4+1}{10}\left(\frac{y_3}{(x-1)^2} - 1\right)\exp\left(-\frac{1}{(x-1)^2}\right),$$

$$y_4' = \frac{2y_4}{x-2} + \frac{y_3+1}{10}\left(\frac{y_4}{(x-2)^2} - 1\right)\exp\left(-\frac{1}{(x-2)^2}\right),$$

$$y_1(0^+) = 0, \; y_2(0^+) = 0, \; y_3(1^\pm) = 0, \; y_4(2^-) = 0.$$

This problem has trivial solution. Moreover, all conditions of Theorem 1 are valid for $\alpha_1(x) = 0.2x, \beta_1(x) = 2x, \alpha_2(x) = 0.2x^2, \beta_2(x) = 2x^2, \alpha_3(x) = 0.2(x-1)^2, \beta_3(x) = 2(x-1)^2, \alpha_4(x) = 0.2(x-2)^2, \beta_4(x) = 2(x-2)^2$. Consequently, there is at least one *nontrivial* solution of this problem $y(x) =$

$(y_1(x), y_2(x), y_3(x), y_4(x))$ such that $0.2x < y_1(x) < 2x$ on $(0,2]$, $0.2x^2 < y_2(x) < 2x^2$ on $(0,2]$, $0.2(x-1)^2 < y_3(x) < 2(x-1)^2$ on $[0,1) \cup (1,2]$,

$0.2(x-2)^2 < y_4(x) < 2(x-2)^2$ on $[0,2)$.

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