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NONOSCILLATORY SOLUTIONS OF NEUTRAL DIFFERENTIAL EQUATIONS WITH RETARDED ARGUMENTS DEPENDING ON THE UNKNOWN FUNCTION

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ABSTRACT: The neutral differential equation

$$[x(t) + px(t-\tau)]' + \sum_{i=1}^{m} Q_i(t)x(\Delta_i(t, x(t))) = 0 \qquad (*)$$

is considered, where $p \in \mathbb{R}$, $\tau > 0$, $Q_i \in C(\mathbb{R}_+, \mathbb{R})$, $\Delta_i \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $\Delta_*(t) \leq \Delta_i(t, x) \leq t$, $i = 1, \ldots, m$ and $\lim_{t\to+\infty} \Delta_*(t) = +\infty$.

It is shown that equation (*) has a nonoscillatory solution, if $p \neq -1$ and

$$\int_0^\infty \sum_{i=1}^{m} |Q_i(t)| dt < +\infty \, .$$

AMS (MOS) Subject Classification: 34K15

1. INTRODUCTION

During the last years considerably increased the number of publications devoted to the oscillatory properties of functional-differential equations. A lot of results on this topic are presented in the monographs Erbe et al [6], Györi and Ladas [7] and Ladde et al [11].

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The mathematical modelling in various branches of the science and technics studies functional-differential equations where the transformed argument depends on the independent variable as well as on the unknown function Kolesov and Shvirta [9], Kolesov and Shvirta [10], Norkin [17]. It is of special interest the study of the oscillatory and the asymptotic properties of equations of that type. At authors knowledge, preliminary results in this direction have been obtained in the articles Angelova and Bainov [1], Angelova and Bainov [2], Angelova and Bainov [3], Bainov et al [5], Markova and Simeonov [14], Markova and Simeonov [15], Markova and Simeonov [16].

In the present paper we consider the linear neutral differential equation

$$[x(t) + px(t-\tau)]' + \sum_{i=1}^{m} Q_i(t)x(\Delta_i(t, x(t))) = 0 \tag{1}$$

with retarded arguments $\Delta_i(t, x)$ which depend on the independent variable t as well as on the unknown function x.

We will prove that equation (1) has a nonoscillatory solution, if $\tau > 0$, $p \neq -1$ and

$$\int_0^{\infty} \sum_{i=1}^{m} |Q_i(t)| dt < +\infty\,. \tag{2}$$

This result generalizes Theorem 1 from Ming-Po Chen et al [13] which states that the equation

$$[x(t) + px(t-\tau)]' + Q(t)x(t-\sigma) = 0 \tag{3}$$

has a positive solution, if

$$p \neq -1\,, \quad \tau > 0\,, \quad \sigma \geq 0 \quad \text{and} \quad \int_0^{\infty} |Q(t)| dt < +\infty\,.$$

Here the restriction $p \neq -1$ is essential, since it is possible that all solutions of equation (3) to be oscillatory, if $p = -1$ and $Q(t) > 0$, $t \in \mathbb{R}_+$, see Ladas and Y.G. Sficas [12], Yu et al [18].

We notice that the proof of Theorem 1, Ming-Po Chen et al [13], is based on the Banach Fixed Point Theorem. An analogous application of this theorem in our case is imposible. That is why the existence of a nonoscillatory solution of equation (1) is proved by successive application of a Fixed Point Theorem for a contracting operator depending on a parameter Hamilton [8] and the Schauder's Second Fixed Point Theorem, see Ladde et al [11].

2. PRELIMINARY NOTES

Consider equation (1) under the following conditions:

H1. $p \in \mathbb{R}$, $\tau \in (0, +\infty)$, $Q_i \in C(\mathbb{R}_+, \mathbb{R})$, $\Delta_i \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R})$, $i = 1, \ldots, m$.

H2. There exist $T \in \mathbb{R}_+$ and a function $\Delta_* \in C(\mathbb{R}_+, \mathbb{R})$ such that

$$\lim_{t\to+\infty} \Delta_*(t) = +\infty \quad \text{and} \quad \Delta_*(t) \le \Delta_i(t,x) \le t$$
$$\text{for} \quad t \ge T\,, \quad x \in \mathbb{R}\,, \quad i = 1,\dots,m\,.$$

Let $T_0 \in \mathbb{R}_+$ and $T_{-1} = \min_{1\le i\le m} \inf\{\Delta_i(t,x) : t \ge T_0, x \in \mathbb{R}\}$.

Definition 1. The function $x(t)$ is said to be a *solution* of equation (1) in the interval $[T_0, +\infty)$ if $x \in C([T_{-1}, +\infty), \mathbb{R})$, $x(t)+px(t-\tau) \in C^1([T_0, +\infty), \mathbb{R})$ and $x(t)$ satisfies (1) for $t \ge T_0$.

As usual, a solution of equation (1) is called *nonoscillatory*, if it is either eventually positive or eventually negative.

We need the following two assertions.

Lemma 1. (see Hamilton [8], Corollary 5.2.2) *Let X and Y be a metric spaces with X complete and let $P : X \times Y \to X$ be a continuous map with*

$$d(P(x_1, y), P(x_2, y)) \le \varrho d(x_1, x_2)$$

for some $\varrho \in [0, 1)$. Then for every $y \in Y$ there exists a unique $x \in X$ with $P(x, y) = x$. If we let $x = S(y)$ then the map $S : Y \to X$ is continuous.

Lemma 2. (see Bainov and Simeonov [4], Lemma 11.1) *Let $q \in [0, 1)$, $r_n \ge 0$, $n \in \mathbb{N}$ and $\lim_{n\to+\infty} r_n = 0$. Then*

$$\lim_{n\to+\infty} \sum_{k=1}^{n} q^{n-k} r_k = 0\,.$$

3. MAIN RESULTS

In proving the main Theorem 3 we will use the following assertions concerning the asymptotic behavior of bounded solutions of equation (1) as $t \to +\infty$.

Theorem 1. *Let conditions H1, H2 and (2) hold and the numbers p, a, b, c and T_0 satisfy the inequalities*

$$|p| < 1\,, \quad a \le \frac{c}{1+p} \le b\,, \quad T_0 \ge T\,.$$

Let to each function $y \in D = C(\mathbb{R}, [a, b])$ there corresponds a function $x \in D$ such that

$$x(t) = \begin{cases} c - px(t-\tau) + \displaystyle\int_t^\infty \sum_{i=1}^{m} Q_i(s) x(\Delta_i(s, y(s)))ds\,, & t \ge T_0\,, \\ x(T_0), & t < T_0\,. \end{cases} \tag{4}$$

Then

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+p} \quad \textit{uniformly on} \quad y \in D\,. \tag{5}$$

Proof. From (4) it follows that

$$x(t) - \frac{c}{1+p} \;=\; -p[x(t-\tau) - \frac{c}{1+p}] + \int_t^\infty \sum_{i=1}^m Q_i(s)x(\Delta_i(s, y(s)))ds\,, \quad t \geq T_0\,. \tag{6}$$

Set $T_n = T_0 + n\tau$,

$$r_n = \max\{|a|, |b|\} \int_{T_{n-1}}^\infty \sum_{i=1}^m |Q_i(s)|ds\,,$$

$$u_n = \sup\left\{ \left|x(t) - \frac{c}{1+p}\right| : T_{n-1} \leq t \leq T_n \right\}.$$

Then we have from (6)

$$u_n \leq |p|u_{n-1} + r_n\,, \quad n \in \mathbb{N}\,.$$

Therefore

$$u_n \leq |p|^n u_0 + \sum_{k=1}^n |p|^{n-k} r_k\,,$$

where $u_0 = \sup\{|x - \frac{c}{1+p}| : a \leq x \leq b\}$ does not depend on $y \in D$. Since $q = |p| \in [0, 1)$ and $\lim_{n\to+\infty} r_n = 0$, then by Lemma 2 we get

$$\lim_{n\to+\infty} u_n = 0 \quad \text{uniformly on} \quad y \in D\,,$$

which proves (5). □

Theorem 2. *Let conditions H1, H2 and (2) hold and the numbers* λ*, a, b, c and* T_0 *satisfy the inequalities*

$$|\lambda| < 1\,, \quad a \leq \frac{c}{1+\lambda} \leq b\,, \quad T_0 \geq T\,.$$

Let to each function $y \in D$ *there corresponds a function* $x \in D$ *such that*

$$x(t) = \lim_{n\to+\infty} x_n(t) \quad \textit{unformly on} \quad t \in \mathbb{R}\,, \tag{7}$$

where the functions $x_n \in D$*,* $n \in \mathbb{N}$ *are defined by the equalities*

$$x_0(t) = \frac{c}{1+\lambda}\,, \tag{8}$$

$$x_{n+1}(t) = \begin{cases} c - \lambda x_n(t+\tau) + \lambda \int_{t+\tau}^{\infty} \sum_{i=1}^{m} Q_i(s) x_n(\Delta(s, y(s)))ds\,, & t \geq T_0\,, \\ x_{n+1}(T_0)\,, & t < T_0\,. \end{cases} \tag{9}$$

Then

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+\lambda} \quad \textit{uniformly on} \quad y \in D\,. \tag{10}$$

Proof. From (9) it follows that

$$x_{n+1}(t) - \frac{c}{1+\lambda} = -\lambda[x_n(t+\tau) - \frac{c}{1+\lambda}] + \lambda \int_{t+\tau}^{\infty} \sum_{i=1}^{m} Q_i(s) x_n(\Delta_i(s, y(s)))ds\,, \quad t \geq T_0\,. \tag{11}$$

Let $\varepsilon > 0$ be given. Then there exists $T_\varepsilon \geq T_0$ independent of $y \in D$ such that

$$|\lambda| \max\{|a|, |b|\} \int_{T_\varepsilon}^{\infty} \sum_{i=1}^{m} |Q_i(s)| ds < \varepsilon(1 - |\lambda|)\,. \tag{12}$$

By induction we prove that for each $n \in \mathbb{N}$ and $y \in D$

$$\left|x_n(t) - \frac{c}{1+\lambda}\right| \leq \varepsilon\,, \quad t \geq T_\varepsilon\,. \tag{13}$$

From (11), (8) and (12) we obtain

$$\left|x_1(t) - \frac{c}{1+\lambda}\right| \leq |\lambda| \max\{|a|, |b|\} \int_{t+\tau}^{\infty} \sum_{i=1}^{m} |Q_i(s)| ds \leq \varepsilon(1 - |\lambda|) < \varepsilon$$

for $t \geq T_\varepsilon$, which proves (13) for $n = 1$ and $y \in D$.

Assume that (13) holds for $n = m \in \mathbb{N}$ and $y \in D$. Then from (11) we have

$$\left|x_{m+1}(t) - \frac{c}{1+\lambda}\right| \leq |\lambda|\varepsilon + \varepsilon(1 - |\lambda|) = \varepsilon\,, \quad t \geq T_\varepsilon\,,$$

which proves (13) for each $m \in \mathbb{N}$ and $y \in D$.

Then (10) follows from (13) and (7). □

Remark 1. The function $x(t)$ which is defined in Theorem 2 by equality (7) satisfies the following equation

$$x(t) = \begin{cases} c - \lambda x(t+\tau) + \lambda \displaystyle\int_{t+\tau}^{\infty} \sum_{i=1}^{m} Q_i(s)x(\Delta_i(s,y(s)))ds\,, & t \ge T_0\,, \\ x(T_0)\,, & t < T_0\,. \end{cases} \tag{14}$$

Theorem 3. *Let conditions H1, H2 and (2) hold and $p \neq -1$. Then equation (1) has a nonoscillatory solution.*

Proof. Let B be a set of all functions $x(t) \in C(\mathbb{R},\mathbb{R})$ which are bounded in $\mathbb{R}$ and $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}\}$ be the norm of $x \in B$. Then B is a Banach space.

The proof of this theorem will be devided into five claims depending on the five different ranges of the parameter p.

Claim 1. $p \in (-1, 0]$.

Choose $T_0 \ge T$ sufficiently large such that

$$\int_{T_0}^{\infty} \sum_{i=1}^{m} |Q_i(s)|ds \le \frac{1+p}{4} \equiv r\,. \tag{15}$$

Define the set D and the operator $P : D \times D \to B$ as follows:

$$D = \{x \in B : a \le x(t) \le b, t \in \mathbb{R}\}\,, \tag{16}$$

$$P(x,y)(t) = \begin{cases} c - px(t-\tau) + \displaystyle\int_{t}^{\infty} \sum_{i=1}^{m} Q_i(s)x(\Delta_i(s,y(s)))ds\,, & t \ge T_0\,, \\ P(x,y)(T_0)\,, & t < T_0\,, \end{cases} \tag{17}$$

where $a = \frac{2(1+p)}{3}$, $b = \frac{4}{3}$, $c = 1+p$.

Clearly D is a nonempty bounded closed and convex subset of B and P is continuous. For every $x, y \in D$ and $t \ge T_0$ we have that

$$P(x,y)(t) \le 1 + p - \frac{4}{3}p + \frac{4}{3}\frac{1+p}{4} = \frac{4}{3} = b\,,$$

$$P(x,y)(t) \ge 1 + p - \frac{4}{3}\frac{1+p}{4} = \frac{2(1+p)}{3} = a\,.$$

Hence $P(x,y) \in D$ for $x,y \in D$. For $x,y,z \in D$ and $t \ge T_0$ we get

$$\begin{aligned}|P(x,y)(t) - P(z,y)(t)| &\le -p|x(t-\tau) - z(t-\tau)| \\ &\quad + \int_t^\infty \sum_{i=1}^m |Q_i(s)||x(\Delta_i(s,y(s))) - z(\Delta_i(s,y(s)))|ds \\ &\le \big(-p + \frac{1+p}{4}\big)\,\|x-z\| = \frac{1-3p}{4}\,\|x-z\|\,.\end{aligned}$$

This implies that

$$\|P(x,y) - P(z,y)\| \le \varrho\,\|x-z\| \quad \text{for} \quad x,y,z \in D\,, \tag{18}$$

where $\varrho = \frac{1-3p}{4} \in [\frac{1}{4}, 1)$. Hence by Lemma 1 for every $y \in D$ there exists a unique $x \in D$ such that $P(x,y) = x$. If we set $x = S(y)$ then the map $S : D \to D$ is continuous.

Obviously, the set of fixed points $\mathcal{F} = \{x : x = S(y), y \in D\}$ is uniformly bounded. In order to apply the Schauder's Fixed Point Theorem it remains to prove that the set $\mathcal{F}$ is equicontinuous on $\mathbb{R}$.

Since $x = P(x,y)$ and $|p| < 1$, then $x = x(t)$ satisfies equation (4) and by Theorem 1 it satisfies also relation (5).

Let $\varepsilon > 0$ be given and set $T_k = T_0 + k\tau$. From (5) it follows that there exists $\nu = \nu(\varepsilon) \in \mathbb{N}$ such that

$$\Big|x(t) - \frac{c}{1+p}\Big| < \frac{\varepsilon}{2} \quad \text{for} \quad t \ge T_\nu \quad \text{and} \quad y \in D\,.$$

This implies

$$|x(t) - x(s)| < \varepsilon \quad \text{for} \quad t,s \ge T_\nu \quad \text{and} \quad y \in D\,. \tag{19}$$

We choose $Q = Q(\varepsilon)$ and $\delta = \delta(\varepsilon) \in (0,\tau)$ such that

$$Q = \sup\left\{\sum_{i=1}^m |Q_i(s)| : T_{-1} \le s \le T_\nu\right\},$$

$$\delta Q b \frac{1-|p|^\nu}{1-|p|} < \frac{\varepsilon}{2}\,.$$

Let $T_0 \le t,\ s \le T_\nu$ and $0 \le t - s \le \delta$. Then $T_{n-1} \le t \le T_n$ for some $n \in \{1,\dots,\nu\}$ and $t_0 = t - n\tau \in [T_{-1}, T_0]$, $s_0 = s - n\tau < T_0$. Set $t_k = t_0 + k\tau$, $s_k = s_0 + k\tau$ and $z_k = x(t_k) - x(s_k)$. We have that

$$\begin{aligned} z_n &= x(t_n) - x(s_n) = x(t) - x(s)\,, \\ z_0 &= x(t_0) - x(s_0) = 0\,, \\ z_k &= -pz_{k-1} - r_k\,, \quad k = 1,2,\dots\,, \end{aligned}$$

where

$$r_k = \int_{s_k}^{t_k} \sum_{i=1}^{m} Q_i(u) x(\Delta_i(u, y(u))) du \,.$$

It is easy to check that

$$|z_n| \le |p|^{n-1}|r_1| + |p|^{n-2}|r_2| + \cdots + |p||r_{n-1}| + |r_n| \,.$$

Since

$$|r_k| \le \int_{s_k}^{t_k} \sum_{i=1}^{m} |Q_i(u)| b du \le Qb(t_k - s_k) = Qb(t-s) \le Qb\delta \,,$$

we conclude that the inequality

$$|x(t) - x(s)| = |z_n| \le Qb\delta \frac{1 - |p|^{\nu}}{1 - |p|} < \frac{\varepsilon}{2} \tag{20}$$

holds for $t, s \in [T_0, T_\nu]$, $|t - s| < \delta$ and $y \in D$. From (19) and (20) it follows that the set $\mathcal{F}$ is equicontinuous on $\mathbb{R}$.

Hence by the Schauder's Second Fixed Point Theorem the map S has a fixed point $x \in D$: $Sx = x$. This means that $P(x, x) = x$. It is easy to verify that $x = x(t)$ is a positive solution of equation (1) for $t \ge T_0$.

Claim 2. $p \in (0, 1)$.

The proof is the same as in Claim 1 with the difference that the numbers r, a, b, c and ϱ from (15)-(18) take the following values:

$$r = \frac{1-p}{4}\,, \quad a = 2(1-p)\,, \quad b = 4\,, \quad c = 3 + p\,, \quad \varrho = \frac{1+3p}{4} \in \left(\frac{1}{4}, 1\right).$$

Claim 3. $p \in (-\infty, -1)$.

Choose $T_0 \ge T$ sufficiently large such that

$$\int_{T_0+\tau}^{\infty} \sum_{i=1}^{m} |Q_i(s)| ds \le -\frac{1+p}{4} \equiv r \,. \tag{21}$$

Define the set D and the operator $P : D \times D \to B$ as follows:

$$D = \{x \in B : a \le x(t) \le b, t \in \mathbb{R}\}\,, \tag{22}$$

$$P(x,y)(t) = \begin{cases} c - \lambda x(t+\tau) + \lambda \displaystyle\int_{t+\tau}^{\infty} \sum_{i=1}^{m} Q_i(s) x(\Delta_i(s, y(s))) ds\,, & t \ge T_0\,, \\ P(x,y)(T_0)\,, & t < T_0\,, \end{cases} \tag{23}$$

where $\lambda = \frac{1}{p}$, $a = -\frac{p}{2}$, $b = -2p$, $c = -p - 1$.

It is easy to show that $P(x,y) \in D$ for $x, y \in D$ and

$$\|P(x,y) - P(z,y)\| \le \varrho \, \|x - z\| \quad \text{for} \quad x, y, z \in D\,, \tag{24}$$

where $\varrho = \frac{p-3}{4p} \in (\frac{1}{4}, 1)$. Hence by Lemma 1 for every $y \in D$ there exists a unique $x \in D$ such that $P(x,y) = x$. If we set $x = S(y)$ then the map $S : D \to D$ is continuous.

Obviously, the set of fixed points $\mathcal{F} = \{x : x = S(y), y \in D\}$ is uniformly bounded. Moreover, each fixed point $x = P(x,y)$ satisfies (14) and can be obtained as a limit of the successive approximations $\{x_n\}_0^\infty$:

$$\begin{aligned} x_0(t) =& \frac{c}{1+\lambda}\,, \\ x_{n+1}(t) =& P(x_n, y)(t)\,. \end{aligned}$$

Since $\{x_n\}$ satisfy equations (8) and (9) with $\lambda = \frac{1}{p}$ and $|\lambda| < 1$ then by Theorem 2

$$\lim_{t\to+\infty} x(t) = \frac{c}{1+\lambda} = \frac{cp}{1+p} \quad \text{uniformly on} \quad y \in D\,.$$

Further on we prove as in Claim 1 that the set of fixed points $\mathcal{F}$ is equicontinuous on $\mathbb{R}$. Then by the Schauder's Second Fixed Point Theorem the map S has a fixed point $x \in D$: $Sx = x$. This means that $P(x,x) = x$, that is, the function $x = x(t)$ is a positive solution of equation (1) for $t \ge T_0 + \tau$.

Claim 4. $p \in (1, +\infty)$.

Ther proof is the same as in Claim 3 with the difference that the numbers λ, r, a, b, c and ϱ from (21)-(24) take the following values:

$$\begin{aligned} &\lambda = \frac{1}{p}\,, \quad r = \frac{p-1}{4}\,, \quad a = 2(p-1)\,, \quad b = 4p\,, \\ &c = 3p+1\,, \quad \varrho = \frac{3+p}{4p} \in \big(\frac{1}{4}, 1\big)\,. \end{aligned}$$

Claim 5. $p = 1$.

Choose $T_0 \ge T$ such that

$$\int_{T_0+\tau}^{\infty} \sum_{i=1}^{m} |Q_i(s)| ds \le \frac{1}{4}\,.$$

The set $D = \{x \in B : 2 \le x(t) \le 4, t \in \mathbb{R}\}$ is a nonempty bounded closed and convex subset of B. Define the operator $P : D \times D \to B$ by the formula

$$P(x,y)(t) = \begin{cases} 3 + \displaystyle\sum_{j=1}^{\infty} \int_{t_{2j-1}}^{t_{2j}} \sum_{i=1}^{m} Q_i(s) x(\Delta_i(s, y(s))) ds\,, & t \ge T_0\,, \\ P(x,y)(T_0)\,, & t < T_0\,, \end{cases} \tag{25}$$

where $t_k = t + k\tau$, $k \in \mathbb{N}$.

Clearly P is continuous, $P(x,y) \in D$ for $x, y \in D$ and P satisfies (24) with $\varrho = \frac{1}{4}$. Therefore by Lemma 1 for every $y \in D$ there exists a unique $x \in D$ such that $P(x,y) = x$. If we set $x = S(y)$ then the map $S : D \to D$ is continuous and the set of fixed points $\mathcal{F} = \{x : x = S(y), y \in D\}$ is uniformly bounded. We prove that $\mathcal{F}$ is equicontinuous on $\mathbb{R}$.

From (25) it follows that $x = P(x,y)$ satisfies the equality

$$x(t) = 3 + \sum_{j=1}^{\infty} \int_{t_{2j-1}}^{t_{2j}} \sum_{i=1}^{m} Q_i(s)x(\Delta_i(s, y(s)))ds\,, \quad T \geq T_0\,. \tag{26}$$

Let $\varepsilon > 0$ be given and $T_k = T_0 + k\tau$. Choose successively $\nu = \nu(\varepsilon) \in \mathbb{N}$ and $\delta \in (0, \tau)$ such that

$$4\int_{T_\nu}^{\infty} \sum_{i=1}^{m} |Q_i(s)|ds < \frac{\varepsilon}{2}\,, \quad 4Q\nu\delta < \frac{\varepsilon}{2}\,, \tag{27}$$

where $Q = \sup\{\sum_{i=1}^{m} |Q_i(s)| : T_0 \leq s \leq T_{\nu+1}\}$.

Let $T_0 \leq s$, $t \leq T_\nu$, $0 \leq t - s \leq \delta$ and $s_k = s + k\tau$.

If $s_0 = s < T_{\nu-1}$, then there exists $n \in \{1, \ldots, \nu\}$ such that $s_n \leq T_\nu$ and $s_{n+1} > T_\nu$. From (26) we have

$$x(t) - x(s) = \sum_{j=1}^{\infty} \Bigg[-\int_{s_{2j-1}}^{t_{2j-1}} \sum_{i=1}^{m} Q_i(u)x(\Delta_i(u, y(u)))du + \int_{s_{2j}}^{t_{2j}} \sum_{i=1}^{m} Q_i(u)x(\Delta_i(u, y(u)))du \Bigg]. \tag{28}$$

Then (28) and (27) imply

$$\begin{aligned} |x(t) - x(s)| &\leq \sum_{j=1}^{n} \int_{s_j}^{t_j} \sum_{i=1}^{m} |Q_i(u)||x(\Delta_i(u, y(u)))|du \\ &\quad + \sum_{j=n+1}^{\infty} \int_{s_j}^{t_j} \sum_{i=1}^{m} |Q_i(u)||x(\Delta_i(u, y(u)))|du \\ &\leq 4Q\nu\delta + 4\int_{T_\nu}^{\infty} \sum_{i=1}^{m} |Q_i(u)|du < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,. \end{aligned}$$

If $s_0 = s \geq T_{\nu-1}$, then $s_1 = s + \tau \geq T_\nu$ and from (28) it follows that

$$|x(t) - x(s)| \leq 4\int_{T_\nu}^{\infty} \sum_{i=1}^{m} |Q_i(u)|du < \frac{\varepsilon}{2}\,.$$

For $s, t \geq T_\nu$ we have that

$$|x(t) - x(s)| \leq 2.4\int_{T_\nu}^{\infty} \sum_{i=1}^{m} |Q_i(u)|du < \varepsilon\,.$$

Consequently, the set $\mathcal{F}$ is equicontinuous on $\mathbb{R}$. By the Schauder's Second Fixed Point Theorem there exists $x \in D$ such that $P(x,x) = x$, that is,

$$x(t) = \begin{cases} 3 + \sum_{j=1}^{\infty} \int_{t_{2j-1}}^{t_{2j}} \sum_{i=1}^{m} Q_i(s)x(\Delta_i(s,x(s)))ds\,, & t \geq T_0\,, \\ x(T_0)\,, & t < T_0\,. \end{cases}$$

Therefore we have that

$$x(t) + x(t-\tau) = 6 + \int_t^{\infty} \sum_{i=1}^{m} Q_i(s)x(\Delta_i(s,x(s)))ds\,, \quad t \geq T_0 + \tau\,,$$

which means that $x(t)$ is a positive solution of equation (1) with $p = 1$. □

REFERENCES

[1] D.C. Angelova and D.D. Bainov, On some oscillating properties of the solutions of a class of functional differential equations, *Tsukuba J. Math.*, **5** (1981), no. 1, 39-46.

[2] D.C. Angelova and D.D. Bainov, On oscillation of solutions of forced functional differential equations of second order, *Math. Nachr.*, **122** (1985), 289-300.

[3] D.Tc. Angelova and D.D. Bainov, Oscillatory and asymptotic behaviour of the solutions of first-order functional differential equations, *IMA Journal of Applied Mathematics*, **39** (1987), 75-89.

[4] D.D. Bainov and P.S. Simeonov, *Systems with Impulse Effect: Stability, Theory and Applications*, Ellis Horwood Series in Mathematics and its Applications, 1989.

[5] D.D. Bainov, N.T. Markova, and P.S. Simeonov, Asymptotic behaviour on the nonoscillatory solutions of differential equations of second order with delay depending on the unknown function, *Journal of Computational and Applied Mathematics*, **91** (1998), 87-96.

[6] L.H. Erbe, Q. Kong, and B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Pure and Applied Mathematics, **190**, Marcel Dekker, 1995.

[7] I. Györi and G. Ladas, *Oscillation Theory of Delay Differential Equations: With Applications*, Clarendon Press, Oxford, 1991.

[8] R.S. Hamilton, The inverse function theorems of Nach and Moser, *Bulletin (New Series) of the American Mathematical Society*, **7** (1982), no. 1, 65-222.

[9] Yu.S. Kolesov and D.I. Shvirta, Mathematical modeling of the process of combustion in the camera of liquid propellant rocket motors, *Litovskii Math. Sb.*, **15**, (1979), no. 4, In Russian.

[10] Yu.S. Kolesov and D.I. Shvirta, *Self-Excited Oscillations in Systems with Delay*, Vilnjus, 1979, In Russian.

[11] G.S. Ladde, V. Lakshmikantham, and B.G. Zhang, *Oscillation Theory of Differential Equations with Deviating Arguments*, Pure and Applied Mathematics, **110**, Marcel Dekker, 1987.

[12] G. Ladas and Y.G. Sficas, Oscillations of neutral delay differential equations, *Canad. Math. Bull.*, **29** (1986), 438-445.

[13] Ming-Po Chen, J.S. Yu, and Z.C. Wang, Nonoscillatory solutions of neutral delay differential equations, *Bull. Austral. Math. Soc.*, **48** (1993), 475-483.

[14] N.T. Markova and P.S. Simeonov, On the asymptotic behaviour of the solutions of a class of differential equations of second order with delay depending on the unknown function, Invited lectures delivered at the *VII-th Int. Colloquium on Differential Equations*, August 18-23, 1996, Plovdiv, Bulgaria, Volume **1**, 89-100.

[15] N.T. Markova and P.S. Simeonov, Asymptotic and oscillatory properties of the solutions of differential equations with delay depending on the unknown function, Invited lectures delivered at the *VII-th Int. Colloquium on Differential Equations*, Volume **2**, August 18-23, 1996. Plovdiv, Bulgaria, 71-78.

[16] N.T. Markova and P.S. Simeonov, Oscillatory and asymptotic behaviour of the solutions of first order differential equations with delays depending on the unknown function, Invited lectures deliverd at the *VII-th Int. Colloquium on Differential Equations*, Volume **2**, August 18-23, 1996. Plovdiv, Bulgaria, 79-92.

[17] S.B. Norkin, On self-excitation and damping of oscillations in a system with auto-regulating lag, *Proc. of the Seminar on the Theory of Differential Equations with Deviating Argument*, **5** (1967), 148-160, In Russian.

[18] J.S. Yu, Z.C. Wang, and C. Qian, Oscillation and nonoscillation of neutral differential equations, *Bull. Austral. Math. Soc.*, **45** (1992), 195-200.