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OSCILLATORY PROPERTIES OF SOLUTIONS OF n-TH ORDER DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS DEPENDING ON THE UNKNOWN FUNCTION

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ABSTRACT: In this paper differential equation of the type

$$(r_{n-1}(t)(\dots r_2(t)(r_1(t)x'(t))'\dots)')' + \delta f(t, x(\Delta_1(t, x(t))), \dots, x(\Delta_m(t, x(t)))) = 0 \quad \text{(E)}$$

is considered, where $n \geq 2$, $\delta = \pm 1$ and the deviating arguments Δ_j, $j = 1, \dots, m$ depend on the independent variable t as well as on the unknown function x.

Sufficient conditions are found under which equation (E) is almost oscillatory.

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1. INTRODUCTION

We study the n-th order differential equation

$$L_n x(t) + \delta f(t, x(\Delta_1(t, x(t))), \dots, x(\Delta_m(t, x(t)))) = 0\,, \qquad (\text{E}, \delta)$$

where the deviating arguments Δ_i, $i = 1, \dots, m$ depend on the independent variable t as well as on the unknown function x.

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Here $n \geq 2$ is an integer, $\delta = \pm 1$, $t \in J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$ and

$$L_0 x(t) = x(t)\,, \qquad L_k x(t) = r_k(t)(L_{k-1}x(t))'\,, \qquad k = 1, \ldots, n\,.$$

The domain $\mathcal{D}(L_n)$ of L_n is defined to be the set of all functions $x : [t_0, +\infty) \to \mathbb{R}$ such that $L_k x(t)$, $k = 1, \ldots, n$ exist and are continuous on the interval $[t_0, +\infty) \subseteq J$. By a *proper* solution of equation (E, δ) is meant a function $x \in \mathcal{D}(L_n)$ which satisfies (E, δ) for all sufficiently large t and $\sup\{|x(t)| : t \geq T\} > 0$ for $T \geq t_0$. We assume that equation (E, δ) do possess proper solutions. A proper solution of equation (E, δ) is called *oscillatory* if it has arbitrarily large zeros; otherwise it is called *nonoscillatory*. Equation (E, δ) is said to be oscillatory if all its proper solutions are oscillatory.

Equation (E, δ) is said to be *almost oscillatory*, if:

(i) for $\delta = 1$ and n even, equation (E, δ) is oscillatoy.

(ii) for $\delta = 1$ and n odd, every proper solution x of equation (E, δ) is either oscillatory or $|L_i x(t)| \to 0$ monotonically as $t \to +\infty$, $i = 0, 1, \ldots, n-1$.

(iii) for $\delta = -1$ and n even, every proper solution x of equation (E, δ) is oscillatory, $|L_i x(t)| \to 0$ monotonically as $t \to +\infty$, $i = 0, 1, \ldots, n-1$, or $|L_i x(t)| \to +\infty$ monotonically as $t \to +\infty$, $i = 0, 1, \ldots, n-1$;

(iv) for $\delta = -1$ and n odd, every proper solution x of equation (E, δ) is either oscillatory or $|L_i x(t)| \to +\infty$ monotonically as $t \to +\infty$, $i = 0, 1, \ldots, n-1$.

We assume that

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty \qquad \text{and} \qquad f(t, x_1, \ldots, x_m)\operatorname{sgn} x_1 \geq a(t) \prod_{j=1}^{m} |x_j|^{\lambda_j}$$

for $i = 1, \ldots, n-1$, $t \in J$ and $x_1 x_j > 0$, $j = 1, \ldots, m$, where $a \in C(J, \mathbb{R}_+)$, $\lambda_j \geq 0$, $j = 1, \ldots, m$ and $\lambda = \sum_{j=1}^{m} \lambda_j > 0$.

In this paper sufficient conditions are found under which equation (E, δ) is almost oscillatory. In the main Theorems 1, 2 and 3 the cases $\lambda > 0$, $0 < \lambda < 1$ and $\lambda = 1$ are considered, respectively. These theorems generalize and improve analogous oscillatory results obtained in Grace [2] where the deviating arguments Δ_j do not depend on $x : \Delta_j = g_j(t)$. Analogous results are given in the papers Grace [1], Grace and Lalli [3], Grace and Lalli [4], Grace and Lalli [5], Dahiya and Akinyele [6], Kitamura [8], Philos [13].

The oscillatory and asymptotic behavior of equations with deviating arguments depending on the unknown functions have been considered in Markova and Simeonov [9], Markova and Simeonov [10], Markova and Simeonov [11], Markova and Simeonov [12].

2. PRELIMINARY NOTES

Let $J = [\alpha, +\infty) \subseteq \mathbb{R}_+ = [0, +\infty)$.

Introduce the following conditions:

H1. $r_i \in C(J, (0, +\infty))$, $i = 1, \ldots, n-1$, $r_n \equiv 1$ and

$$\int^{\infty} \frac{ds}{r_i(s)} = +\infty\,, \qquad i = 1, \ldots, n-1\,. \tag{1}$$

H2. $f \in C(J \times \mathbb{R}^m, \mathbb{R})$ and there exist a function $a \in C(J, \mathbb{R}_+)$ and constants $\lambda_j \geq 0$, $j = 1, \ldots, m$ such that $\sum_{j=1}^m \lambda_j = \lambda > 0$ and

$$f(t, x_1, \ldots, x_m)\mathrm{sgn} x_1 \geq a(t) \prod_{j=1}^m |x_j|^{\lambda_j} \tag{2}$$

for $t \in J$ and $x_1 x_j > 0$, $j = 1, \ldots, m$.

H3. $\Delta_j \in C(J \times \mathbb{R}, \mathbb{R})$, $j = 1, \ldots, m$ and there exist functions $g_j \in C(J, \mathbb{R})$, $j = 1, \ldots, m$ and $T_0 \geq \alpha$ such that

$$\lim_{t \to +\infty} g_j(t) = +\infty\,, \qquad g_j(t) \leq \Delta_j(t, x)$$

for $j = 1, \ldots, m$, $t \geq T_0$ and $x \in \mathbb{R}$.

To formulate our results we use the following notations:

$$I_0 = 1\,,$$

$$I_j(t, s, p_j, \ldots, p_1) = \int_s^t \frac{1}{p_j(u)} I_{j-1}(u, s, p_{j-1}, \ldots, p_1)du\,, \qquad j = 1, 2, \ldots\,,$$

where $p_j \in C(J, (0, +\infty))$, $j = 1, 2, \ldots$.

It is easy to verify that for $j = 1, 2, \ldots, n-1$

$$I_j(t, s; p_j, \ldots, p_1) = (-1)^j I_j(s, t; p_1, \ldots, p_j)\,, \tag{3}$$

$$I_j(t, s; p_j, \ldots, p_1) = \int_s^t \frac{1}{p_1(u)} I_{j-1}(t, u; p_j, \ldots, p_2)du\,, \tag{4}$$

$$\frac{\partial I_j}{\partial t}(t, s; p_j, \ldots, p_1) = \frac{1}{p_j(t)} I_{j-1}(t, s; p_{j-1}, \ldots, p_1)\,, \tag{5}$$

$$\frac{\partial I_j}{\partial s}(t, s; p_j, \ldots, p_1) = -\frac{1}{p_1(s)} I_{j-1}(t, s; p_j, \ldots, p_2)\,. \tag{6}$$

We will need the following lemmas.

Lemma 1. *Let $p_j \in C(J, (0, +\infty))$, $j = 1, 2, \ldots$ and $\alpha \leq T < u$. Then:*

(i) $I_j(u, t; p_1, \ldots, p_j) \leq I_1(u, t; p_j) I_{j-1}(u, t; p_1, \ldots, p_{j-1})$, $T \leq t \leq u$*;*

(ii) *The function* $\dfrac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}$ *is nonincreasing in* $t \in [T,u)$*;*

(iii) $\dfrac{I_j(u,T;p_1,\dots,p_j)}{I_1(u,T;p_j)} \geq \dfrac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}$, $T \leq t < u$*;*

(iv) $I_j(t,T;p_1,\dots,p_j) \geq \dfrac{I_j(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j)$, $T \leq t < u$*;*

(v) $\displaystyle\int_T^t \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds \geq \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j),$
$T \leq t < u.$

Proof. Using that $I_{j-1}(u,s;p_1,\dots,p_{j-1})$ is nonincreasing in $s \in [t,u]$ we obtain (i):

$$\begin{aligned} I_j(u,t;p_1,\dots,p_j) &= \int_t^u \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds \\ &\leq I_{j-1}(u,t;p_1,\dots,p_{j-1}) \int_t^u \frac{ds}{p_j(s)} = I_{j-1}(u,t;p_1,\dots,p_{j-1}) I_1(u,t;p_j)\,. \end{aligned}$$

Using (6) and (i) we obtain the inequality

$$\frac{\partial}{\partial t}\left(\frac{I_j(u,t;p_1,\dots,p_j)}{I_1(u,t;p_j)}\right) \leq 0\,, \qquad T \leq t < u\,,$$

which proves (ii) and (iii).

It follows from (iii) that

$$I_1(u,t;p_j) I_j(u,T;p_1,\dots,p_j) \geq I_1(u,T;p_j) I_j(u,t;p_1,\dots,p_j)\,, \qquad T \leq t \leq u\,.$$

Using this we obtain successively

$$\begin{aligned} &[I_1(u,T;p_j) - I_1(t,T;p_j)] I_j(u,T;p_1,\dots,p_j) \\ &\qquad \geq I_1(u,T;p_j) I_j(u,t;p_1,\dots,p_j)\,, \qquad T \leq t \leq u\,, \end{aligned}$$

$$\begin{aligned} &I_1(u,T;p_j)[I_j(u,T;p_1,\dots,p_j) - I_j(u,t;p_1,\dots,p_j)] \\ &\qquad \geq I_1(t,T;p_j) I_j(u,T;p_1,\dots,p_j)\,, \qquad T \leq t \leq u\,, \end{aligned}$$

$$I_j(t,T;p_1,\dots,p_j) \geq \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j)\,, \qquad T \leq t < u\,,$$

$$\begin{aligned} &\int_T^t \frac{1}{p_j(s)} I_{j-1}(u,s;p_1,\dots,p_{j-1})ds \geq \int_T^t \frac{1}{p_j(s)} I_{j-1}(t,s;p_1,\dots,p_{j-1})ds \\ &= I_j(t,T;p_1,\dots,p_j) \geq \frac{I_1(t,T;p_j)}{I_1(u,T;p_j)} I_j(u,T;p_1,\dots,p_j)\,, \qquad T \leq t < u\,, \end{aligned}$$

which proves (iv) and (v). $\square$

Lemma 2. (see Grace [2]) *If $x \in \mathcal{D}(L_n)$, then for $t, s \in J$ and $0 \le i < \nu \le n$:*

$$\text{(i) } L_i x(t) = \sum_{j=i}^{\nu-1} I_{j-i}(t, s; r_{i+1}, \ldots, r_j) L_j x(s) + \int_s^t I_{\nu-i-1}(t, u; r_{i+1}, \ldots, r_{\nu-1}) \frac{L_\nu x(u)}{r_\nu(u)} du;$$

$$\text{(ii) } L_i x(t) = \sum_{j=i}^{\nu-1} (-1)^{j-i} I_{j-i}(s, t; r_j, \ldots, r_{i+1}) L_j x(s) + (-1)^{\nu-i} \int_t^s I_{\nu-i-1}(u, t; r_{\nu-1}, \ldots, r_{i+1}) \frac{L_\nu x(u)}{r_\nu(u)} du.$$

Lemma 3. *Suppose condition* (1) *holds and the functions $L_n x$ and $x \in \mathcal{D}(L_n)$ are of constant sign and not identically zero for $t \ge t_* \ge \alpha$. Then:*

(i) *There exist a $t_k \ge t_*$ and an integer k, $0 \le k \le n$ with $n + k$ even for $x(t) L_n x(t)$ nonnegative or $n+k$ odd for $x(t) L_n x(t)$ nonpositive and such that for every $t \ge t_k$*

$$x(t) L_i x(t) > 0\,, \qquad i = 0, 1, \ldots, k\,,$$
$$(-1)^{k-i} x(t) L_i x(t) > 0\,, \qquad i = k, k+1, \ldots, n\,.$$

(ii) *The following inequality is valid*

$$\frac{I_\nu(t, t_0; r_{k-\nu+1}, \ldots, r_k)}{|L_{k-\nu} x(t)|} \ge \frac{I_{\nu+1}(t, t_0; r_{k-\nu}, \ldots, r_k)}{|L_{k-\nu-1} x(t)|} \tag{7}$$

for $t \ge t_0 \ge t_k$ and $\nu = 0, 1, \ldots, k-1$.

Proof. This lemma generalizes the well-known lemma of Kiguradze [7] and can be proved similarly. We omit the proof of part (i) and prove part (ii) only.

Suppose without loss of generality that $x(t) > 0$, $t \ge t_*$. Since $L_k x(t)$ is nonincreasing for $t \ge t_k$ and $L_{k-1} x(t_0) > 0$, then

$$L_{k-1} x(t) = L_{k-1} x(t_0) + \int_{t_0}^t (L_{k-1} x(s))' ds = \int_{t_0}^t \frac{L_k x(s)}{r_k(s)} ds \ge \int_{t_0}^t \frac{ds}{r_k(s)} L_k x(t) = I_1(t, t_0; r_k) L_k x(t)\,, \qquad t \ge t_0\,,$$

which proves (7) for $\nu = 0$.

Suppose that (7) is true for ν, $0 \le \nu \le j < k-1$. Then

$$\frac{I_j(t,t_0;r_{k-j+1},\dots,r_k)}{L_{k-j}x(t)} \ge \frac{I_{j+1}(t,t_0;r_{k-j},\dots,r_k)}{L_{k-j-1}x(t)}\,, \qquad t \ge t_0\,. \tag{8}$$

We prove that (7) is true for $\nu = j+1$, that is,

$$f(t) = \frac{I_{j+1}(t,t_0;r_{k-j},\dots,r_k)}{L_{k-j-1}x(t)} \ge \frac{I_{j+2}(t,t_0;r_{k-j-1},\dots,r_k)}{L_{k-j-2}x(t)} = g(t) \tag{9}$$

for $t \ge t_0$.

We have that $f(t_0) = g(t_0)$. We prove that (9) holds for each $t \ge t_0$. Assume the opposite, that $f(t_1) < g(t_1)$ for some $t_1 > t_0$. Then there exists $T \in [t_0, t_1)$ such that

$$f(T) = g(T) \quad \text{and} \quad f(t) < g(t)\,, \qquad T < t \le t_1\,. \tag{10}$$

Since

$$f'(t) = \frac{I_j(t,t_0;r_{k-j+1},\dots,r_k)L_{k-j-1}x(t) - I_{j+1}(t,t_0;r_{k-j},\dots,r_k)L_{k-j}x(t)}{r_{k-j}(t)(L_{k-j-1}x(t))^2}\,,$$

$$g'(t) = \frac{I_{j+1}(t,t_0;r_{k-j},\dots,r_k)L_{k-j-2}x(t) - I_{j+2}(t,t_0;r_{k-j-1},\dots,r_k)L_{k-j-1}x(t)}{r_{k-j-1}(t)(L_{k-j-2}x(t))^2}\,,$$

then it follows from (8) and (10)

$$f'(t) \ge 0 > g'(t)\,, \qquad T < t \le t_1\,.$$

This implies $f(t) > g(t)$, $T < t \le t_1$ which contradicts (10). □

Remark 1. From inequality (7) with $\nu = 0, 1, \dots, k-1$ it follows

$$|L_{k-1}x(t)| \ge I_1(t,t_0;r_k)|L_k x(t)|\,, \qquad t \ge t_0 \ge t_k\,, \tag{11}$$

$$|x(t)| \ge I_k(t,t_0;r_1,\dots,r_k)|L_k x(t)|\,, \qquad t \ge t_0 \ge t_k\,, \tag{12}$$

$$|x(t)| \ge \frac{I_k(t,t_0;r_1,\dots,r_k)}{I_1(t,t_0;r_k)}|L_{k-1}x(t)|\,, \quad t > t_0 \ge t_k\,. \tag{13}$$

For every $T \geq \alpha$ and $t \geq s \geq T$ we set

$$\sigma_j(t) = \min\{t, g_j(t)\}\,, \qquad j = 1, \dots, m\,,$$

$$R_j(t,T) = I_1(t,T;r_j) = \int_T^t \frac{ds}{r_j(s)}\,, \qquad j = 1, 2, \dots, n-1\,,$$

$$\alpha_j(t,s) = I_j(t,s;r_1,\dots,r_j)\,, \qquad j = 1, 2, \dots, n-1\,,$$

$$\beta_j(t,s) = I_{n-j-1}(t,s;r_{n-1},\dots,r_{j+1})\,, \qquad j = 0, 1, \dots, n-1\,,$$

$$\gamma_i(\sigma_j(t),T) = \int_T^{\sigma_j(t)} \alpha_{i-2}(g_j(t),s)\frac{R_i(s,T)}{r_{i-1}(s)}ds\,, \qquad i = 2, \dots, n-1\,, \quad j = 1, \dots, m$$

and

$$\gamma_1(\sigma_j(t),T) = R_1(\sigma_j(t),T)\,, \qquad j = 1, \dots, m\,.$$

3. MAIN RESULTS

3.1. CASE $\lambda > 1$

Theorem 1. *Let conditions H1-H3 hold and $\lambda > 1$. Then a sufficient condition for equation (E,δ) to be almost oscillatory is that:*

(i) *when $\delta = 1$ and n is even, for every $k \in \{1, 3, \dots, n-1\}$ and all sufficiently large T*

$$\int_T^\infty \frac{1}{r_k(t)} \int_t^\infty \beta_k(u,t)a(u) \prod_{j=1}^m \left(\frac{\gamma_k(\sigma_j(u),T)}{R_k(u,T)}\right)^{\lambda_j} dudt = +\infty\,; \qquad (14;k)$$

(ii) *when $\delta = 1$ and n is odd, condition* (14; *k*)*, $k \in \{2, 4, \dots, n-1\}$ hold and for all sufficiently large T*

$$\int^\infty \beta_0(s,T)a(s)ds = +\infty\,; \qquad (15)$$

(iii) *when $\delta = -1$ and n is even, conditions* (14; *k*)*, $k \in \{2, 4, \dots, n-2\}$ and* (15) *hold and for all sufficiently large T*

$$\int^\infty a(s)\prod_{j=1}^m (\alpha_{n-1}(g_j(s),T))^{\lambda_j} ds = +\infty\,; \qquad (16)$$

(iv) *when $\delta = -1$ and n is odd, conditions* (14; *k*)*, $k \in \{1, 3, \dots, n-2\}$ and* (16) *hold.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E,δ). Without loss of generality we assume that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists $t_1 \geq t_0$ such that $x(\Delta_j(t,x(t))) > 0$, $x(\sigma_j(t)) > 0$ for $t \geq t_1$, $j = 1, \dots, m$.

By Lemma 3(i) there exist $t_k \ge t_1$ and $k \in \{0, 1, \dots, n\}$ with $n+k$ odd if $\delta = 1$ and $n+k$ even if $\delta = -1$ such that

$$\begin{aligned} L_i x(t) > 0\,, \qquad & i = 0, \dots, k\,, \qquad t \ge t_k\,, \\ (-1)^{i-k} L_i x(t) > 0\,, \qquad & i = k, k+1, \dots, n\,, \qquad t \ge t_k\,. \end{aligned} \tag{17}$$

From Lemma 2(ii) we have

$$\begin{aligned} L_k x(t) = & \sum_{j=k}^{n-1} (-1)^{j-k} I_{j-k}(s, t; r_j, \dots, r_{k+1}) L_j x(s) \\ & + (-1)^{n-k} \int_t^s I_{n-k-1}(u, t; r_{n-1}, \dots, r_{k+1}) L_n x(u) du\,, \qquad t \le s\,. \end{aligned}$$

Using (2), (17) and letting $s \to +\infty$ we obtain

$$L_k x(t) \ge \int_t^\infty \beta_k(u, t) a(u) \prod_{j=1}^m |x(\Delta_j(u, x(u)))|^{\lambda_j} du\,, \qquad t \ge t_k\,. \tag{18}$$

We remark that if $k \ge 1$, then $x(t)$ is increasing for $t \ge t_k$ and $x(\Delta_j(t, x(t))) \ge x(g_j(t)) \ge x(\sigma_j(t))$ for $t \ge t_2$, where t_2 is such that $\sigma_j(t) \ge t_k$, $t \ge t_2$, $j = 1, \dots, m$. In this case

$$L_k x(t) \ge \int_t^\infty \beta_k(u, t) a(u) \prod_{j=1}^m (x(g_j(u)))^{\lambda_j} du\,, \qquad t \ge t_2\,. \tag{19}$$

Case 1. $k \ge 2$. From Lemma 2(i) we obtain

$$\begin{aligned} x(t) & = \sum_{j=0}^{k-2} I_j(t, t_k; r_1, \dots, r_j) L_j x(t_k) \\ & \quad + \int_{t_k}^t I_{k-2}(t, u; r_1, \dots, r_{k-2}) \frac{L_{k-1} x(u)}{r_{k-1}(u)} du \\ & \ge \int_{t_k}^t \alpha_{k-2}(t, u) \frac{L_{k-1} x(u)}{r_{k-1}(u)} du\,, \qquad t \ge t_k\,. \end{aligned}$$

Then

$$x(g_j(t)) \ge \int_{t_k}^{\sigma_j(t)} \alpha_{k-2}(g_j(t), u) \frac{L_{k-1} x(u)}{r_{k-1}(u)} du\,, \qquad j = 1, \dots, m\,, \quad t \ge t_2\,. \tag{20}$$

Since (11) implies that $\frac{L_{k-1} x(u)}{R_k(u, t_k)}$ is a nonincreasing function for $u > t_k$, then it follows from (20) that

$$x(g_j(t)) \ge \frac{L_{k-1} x(t)}{R_k(t, t_k)} \int_{t_k}^{\sigma_j(t)} \alpha_{k-2}(g_j(t), u) \frac{R_k(u, t_k)}{r_{k-1}(u)} du\,, \qquad t \ge t_2\,,$$

that is,

$$x(g_j(t)) \geq \gamma_k(\sigma_j(t), t_k) \frac{L_{k-1}x(t)}{R_k(t, t_k)}, \qquad j = 1, \ldots, m, \quad t \geq t_2. \tag{21}$$

From (19) and (21) we conclude that

$$\frac{(L_{k-1}x(t))'}{(L_{k-1}x(t))^{\lambda}} \geq \frac{1}{r_k(t)} \int_t^{\infty} \beta_k(u,t)a(u) \prod_{j=1}^{m} \left(\frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du, \qquad t \geq t_2.$$

Integrating this inequality from t_2 to $\tau \geq t_2$ we obtain

$$\int_{t_2}^{\tau} \frac{1}{r_k(t)} \int_t^{\infty} \beta_k(u,t)a(u) \prod_{j=1}^{m} \left(\frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du dt \leq \int_{L_{k-1}x(t_2)}^{L_{k-1}x(\tau)} w^{-\lambda} dw < +\infty$$

which contradicts (14; k).

Case 2. $k = 1$. Then $\delta = 1$ and n is even or $\delta = -1$ and n is odd. Since $x(t)$ is increasing and $x(g_j(t)) \geq x(\sigma_j(t))$ for $t \geq t_2$ it follows from (19) that

$$x'(t) \geq \frac{1}{r_1(t)} \int_t^{\infty} \beta_1(u,t)a(u) \prod_{j=1}^{m} (x(\sigma_j(u)))^{\lambda_j} du, \qquad t \geq t_2. \tag{22}$$

Since $\frac{x(t)}{R_1(t,t_k)}$ is nonincreasing for $t > t_k$ and $\sigma_j(u) \leq u$ we have

$$x(\sigma_j(u)) \geq \frac{R_1(\sigma_j(u), t_k)}{R_1(u, t_k)} x(t), \qquad u \geq t \geq t_2$$

and then we conclude from (22) that

$$\frac{x'(t)}{(x(t))^{\lambda}} \geq \frac{1}{r_1(t)} \int_t^{\infty} \beta_1(u,t)a(u) \prod_{j=1}^{m} \left(\frac{\gamma_1(\sigma_j(u), t_k)}{R_1(u, t_k)} \right)^{\lambda_j} du, \qquad t \geq t_2.$$

This implies the inequality

$$\int_{t_2}^{\infty} \frac{1}{r_1(t)} \int_t^{\infty} \beta_1(u,t)a(u) \prod_{j=1}^{m} \left(\frac{\gamma_1(\sigma_j(u), t_k)}{R_1(u, t_k)} \right)^{\lambda_j} du dt < +\infty$$

which contradicts condition (14; 1).

Case 3. $k = 0$. Then $\delta = 1$ and n is odd or $\delta = -1$ and n is even. It follows from (18) with $k = 0$

$$x(t) \geq \int_t^{\infty} \beta_0(u,t)a(u) \prod_{j=1}^{m} (x(\Delta_j(u, x(u))))^{\lambda_j} du, \qquad t \geq t_k. \tag{23}$$

Since $x(t)$ is decreasing and positive for $t \geq t_k$, there exists $\lim_{t\to+\infty} x(t) = c \geq 0$. If $c > 0$, then there exists $t_2 \geq t_k$ such that $2c \geq x(\Delta_j(u, x(u))) \geq c$, $t \geq t_2$, $j = 1, \dots, m$. Then (23) implies the inequality

$$2c \geq x(t_2) \geq \int_{t_2}^{\infty} \beta_0(u, t_2) a(u) du \,.\, c^\lambda$$

which contradicts (15). Hence $c = 0$.

Case 4. $k = n$. Then $\delta = -1$ and n is either even or odd. From (17) we have

$$L_i x(t) > 0\,, \qquad i = 0, 1, \dots, n\,, \qquad t \geq t_k\,.$$

Furthermore, by l'Hôpital's rule,

$$\lim_{t\to+\infty} \frac{x(t)}{\alpha_{n-1}(t, t_k)} = \lim_{t\to+\infty} L_{n-1} x(t) > 0\,.$$

Since $g_j(t) \to +\infty$ as $t \to +\infty$ ($j = 1, \dots, m$), there exist a constant $C > 0$ and $t_2 \geq t_k$ such that

$$x(g_j(t)) \geq C\alpha_{n-1}(g_j(t), t_k)\,, \qquad j = 1, \dots, m\,, \quad t \geq t_2\,. \tag{24}$$

Integrating equation (E, -1) from t_2 to t and using (2) and (24) we obtain

$$\begin{aligned} L_{n-1}x(t) &\geq L_{n-1}x(t_2) + \int_{t_2}^{t} f(s, x(\Delta_1(s, x(s))), \dots, x(\Delta_m(s, x(s))))ds \\ &\geq \int_{t_2}^{t} a(s) \prod_{j=1}^{m} (x(g_j(s)))^{\lambda_j} ds \geq C^\lambda \int_{t_2}^{t} a(s) \prod_{j=1}^{m} (\alpha_{n-1}(g_j(s), t_k))^{\lambda_j} ds\,. \end{aligned}$$

Therefore (16) implies

$$\lim_{t\to+\infty} L_{n-1} x(t) = +\infty$$

and hence $\lim_{t\to+\infty} L_i x(t) = +\infty$, $i = 0, 1, \dots, n-1$ monotonically. □

Consider the equation

$$x^{(n)}(t) + \delta f(t, x(\Delta_1(t, x(t))), \dots, x(\Delta_m(t, x(t)))) = 0 \tag{E$_0, \delta$}$$

which is a particular case of equation (E, δ) with $r_1 = r_2 = \cdots = r_n = 1$.

Corollary 1. *Let conditions H2, H3 hold and* $\lambda > 1$. *Then a sufficient condition for equation* (E$_0$, δ) *to be almost oscillatory is that:*

(i) *when* $\delta = 1$ *and* n *is even, for every* $k \in \{1, 3, \dots, n-1\}$

$$\int^{\infty} u^{n-k-\lambda} a(u) \prod_{j=1}^{m} (\sigma_j(u))^{k\lambda_j} du = +\infty\,; \tag{25; k}$$

(ii) *when* $\delta = 1$ *and* n *is odd, conditions (25; k),* $k \in \{2, 4, \ldots, n-1\}$ *hold and*

$$\int^{\infty} s^{n-1} a(s) ds = +\infty\,; \tag{26}$$

(iii) *when* $\delta = -1$ *and* n *is even, conditions* (25; k), $k \in \{2, 4, \ldots, n-2\}$ *and* (26) *hold and*

$$\int^{\infty} a(s) \prod_{j=1}^{m} (g_j(s))^{(n-1)\lambda_j} ds = +\infty\,; \tag{27}$$

(iv) *when* $\delta = -1$ *and* n *is odd, conditions* (25; k), $k \in \{1, 3, \ldots, n-2\}$ *and* (27) *hold.*

3.2. CASE $0 < \lambda < 1$

Theorem 2. *Let conditions H1-H3 hold and* $0 < \lambda < 1$. *Then a sufficient condition for equation* (E, δ) *to be almost oscillatory is that:*

(i) *when* $\delta = 1$ *and* n *is even*

$$\int^{\infty} a(u) \prod_{j=1}^{m} (\gamma_{n-1}(\sigma_j(u), T))^{\lambda_j} du = +\infty \tag{28}$$

and

$$\int_T^{\infty} \frac{(R_k(t,T))^{\lambda}}{r_{k+1}(t)} \int_t^{\infty} \beta_{k+1}(u,t) a(u) \prod_{j=1}^{m} \left(\frac{\gamma_k(\sigma_j(u), T)}{R_k(u,T)} \right)^{\lambda_j} du dt = +\infty \tag{29; k}$$

for all sufficiently large T *and* $k \in \{1, 3, \ldots, n-3\}$*;*

(ii) *when* $\delta = 1$ *and* n *is odd, conditions* (29; k), $k \in \{2, 4, \ldots, n-3\}$, (15) *and* (28) *hold;*

(iii) *when* $\delta = -1$ *and* n *is even, conditions* (29; k), $k \in \{2, 4, \ldots, n-2\}$, (15) *and* (16) *hold;*

(iv) *when* $\delta = -1$ *and* n *is odd, conditions* (29; k), $k \in \{1, 3, \ldots, n-2\}$ *and* (16) *hold.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E, δ). Assume without loss of generality that $x(t) > 0$, $t \geq t_0 \geq \alpha$. Then there exists $t_1 \geq t_0$ such that $x(\Delta_j(t, x(t))) > 0$, $x(g_j(t)) > 0$, $x(\sigma_j(t)) > 0$ for $t \geq t_1$, $j = 1, \ldots, m$. As in the proof of Theorem 1, we conclude that there exist $t_k \geq t_1$ and $k \in \{0, 1, \ldots, n\}$ with $n + k$ odd if $\delta = 1$ or $n + k$ even if $\delta = -1$ such that (17) holds.

Case 1. $k \in \{1, \dots, n-2\}$. From Lemma 2(ii) we have

$$L_{k+1}x(t) = \sum_{j=k+1}^{n-1} (-1)^{j-k-1} I_{j-k-1}(s, t; r_j, \dots, r_{k+2}) L_j x(s) + (-1)^{n-k-1} \int_t^s I_{n-k-2}(u, t; r_{n-1}, \dots, r_{k+2}) L_n x(u) du$$

for $s \geq t \geq t_k$. Using (E, δ), (17), (2) and letting $s \to +\infty$ we get

$$-L_{k+1}x(t) \geq \int_t^\infty \beta_{k+1}(u, t) a(u) \prod_{j=1}^m (x(g_j(u)))^{\lambda_j} du\,, \qquad t \geq t_k\,.$$

Taking in view (21) we obtain

$$-L_{k+1}x(t) \geq (L_{k-1}x(t))^\lambda \int_t^\infty \beta_{k+1}(u, t) a(u) \prod_{j=1}^m \left(\frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du$$

for $t \geq t_2 \geq t_k$. Then using (11) we conclude

$$- \frac{(L_k x(t))'}{(L_k x(t))^\lambda} \geq \frac{(R_k(t, t_k))^\lambda}{r_{k+1}(t)} \int_t^\infty \beta_{k+1}(u, t) a(u) \prod_{j=1}^m \left(\frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du\,, \qquad t \geq t_2\,.$$

Integrating this inequality from t_2 to $\tau \geq t_2$ we obtain

$$\int_{t_2}^\tau \frac{(R_k(t, t_k))^\lambda}{r_{k+1}(t)} \int_t^\infty \beta_{k+1}(u, t) a(u) \prod_{j=1}^m \left(\frac{\gamma_k(\sigma_j(u), t_k)}{R_k(u, t_k)} \right)^{\lambda_j} du dt \leq \int_{L_k x(\tau)}^{L_k x(t_2)} w^{-\lambda} dw < +\infty$$

which contradicts (29; k).

Case 2. $k = n-1$. Then $\delta = 1$ and either n is odd or even. From (E, δ) and (21) we have

$$-L_n x(t) \geq a(t) \prod_{j=1}^m \left(\frac{\gamma_{n-1}(\sigma_j(t), t_k)}{R_{n-1}(t, t_k)} L_{n-2} x(t) \right)^{\lambda_j}, \qquad t \geq t_2$$

and applying (11) with $k = n-1$ we get

$$-\frac{L_n x(t)}{(L_{n-1}x(t))^\lambda} \geq a(t) \prod_{j=1}^m (\gamma_{n-1}(\sigma_j(t), t_k))^{\lambda_j}\,, \qquad t \geq t_2\,.$$

Integrating the above inequality from t_2 to $t \geq t_2$ we obtain

$$\int_{t_2}^{t} a(u)\prod_{j=1}^{m}(\gamma_{n-1}(\sigma_j(u),t_k))^{\lambda_j}du \leq \int_{L_{n-1}x(t)}^{L_{n-1}x(t_2)} w^{-\lambda}dw < +\infty$$

which contradicts (28).

In the cases $k = 0$ and $k = n$ the proof is the same as the proof for these cases in Theorem 1 and we omit it here. □

Corollary 2. *Let conditions H2, H3 hold and* $0 < \lambda < 1$. *Then the sufficient condition for equation* $(\mathrm{E}_0,\,\delta)$ *to be almost oscillatory is that:*

(i) *when* $\delta = 1$ *and* n *is even, for every* $k \in \{1,3,\ldots,n-1\}$

$$\int^{\infty} a(u)u^{n-k-1}\prod_{j=1}^{m}(\sigma_j(u))^{k\lambda_j}du = +\infty\,; \tag{30; k}$$

(ii) *when* $\delta = 1$ *and* n *is odd, conditions* (30; k)*,* $k \in \{2,4,\ldots,n-1\}$ *and* (26) *hold;*

(iii) *when* $\delta = -1$ *and* n *is even, conditions* (30; k)*,* $k \in \{2,4,\ldots,n-2\}$*,* (26) *and* (27) *hold;*

(iv) *when* $\delta = -1$ *and* n *is odd, conditions* (30; k)*,* $k \in \{1,3,\ldots,n-2\}$ *and* (27) *hold.*

3.3. CASE $\boldsymbol{\lambda = 1}$

Theorem 3. *Let conditions H1-H3 hold and* $\lambda = 1$. *Then a sufficient condition for equation* $(\mathrm{E},\,\delta)$ *to be almost oscillatory is that:*

(i) *when* $\delta = 1$ *and* n *is even, for every* $k \in \{1,3,\ldots,n-1\}$ *and some* $T_0 \geq \alpha$

$$\limsup_{t\to+\infty}\left\{\frac{1}{R_k(t,\alpha)}\int_{T_0}^{t}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}du\right.$$
$$\left. +R_k(t,\alpha)\int_{t}^{\infty}\frac{\beta_{k-1}(t,\alpha)}{R_k(u,\alpha)}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}du\right\} > 1\,; \tag{31; k}$$

(ii) *when* $\delta = 1$ *and* n *is odd, conditions* (31; k)*,* $k \in \{2,4,\ldots,n-1\}$ *hold and*

$$\int^{\infty}\beta_0(s,\alpha)a(s)ds = +\infty\,; \tag{32}$$

(iii) *when* $\delta = -1$ *and* n *is even, conditions* (31; k)*,* $k \in \{2,4,\ldots,n-2\}$ *and* (32) *hold and*

$$\int^{\infty} a(s)\prod_{j=1}^{m}(\alpha_{n-1}(g_j(s),\alpha))^{\lambda_j}ds = +\infty\,; \tag{33}$$

(iv) *when* $\delta = -1$ *and* n *is odd, conditions* (31; k), $k \in \{1, 3, \dots, n-2\}$ *and* (33) *hold.*

Proof. Let $x(t)$ be a nonoscillatory solution of equation (E, δ). Assume without loss of generality that $x(t)$ is eventually positive: $x(\Delta_j(t, x(t))) > 0$, $x(g_j(t)) > 0$, $x(\sigma_j(t)) > 0$, $t \geq t_1 \geq \alpha$, $j = 1, \dots, m$. By Lemma 3(i) there exist $t_k \geq t_1$ and $k \in \{0, 1, \dots, n\}$ with $n+k$ odd if $\delta = 1$ and $n+k$ even if $\delta = -1$ such that (17) holds. Applying Lemma 2(ii) with $i = k$ and taking into account (17), (E, δ) and (2) we get

$$L_k x(t) \geq \int_t^\infty \beta_k(u,t) a(u) \prod_{j=1}^m (x(\Delta_j(u, x(u))))^{\lambda_j} du\,, \qquad t \geq t_k\,.$$

Case 1. $k \in \{1, 2, \dots, n-1\}$. Then $x(t)$ is nondecreasing and $x(\Delta_j(u, x(u))) \geq x(\sigma_j(u))$, $u \geq t_k$. Therefore

$$L_k x(t) \geq \int_t^\infty \beta_k(u,t) a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} du\,, \qquad t \geq t_k\,.$$

Integrating this inequality from t_k to $t \geq t_k$ we have

$$L_{k-1} x(t) \geq L_{k-1} x(t_k) + \int_{t_k}^t \frac{1}{r_k(s)} \int_s^\infty \beta_k(u,s) a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} du\,, \qquad t \geq t_k\,.$$

Keeping in mind (17) and changing the order of integration we obtain

$$\begin{aligned} L_{k-1} x(t) \geq{}& \int_{t_k}^t a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} \int_{t_k}^u \frac{\beta_k(u,s)}{r_k(s)} ds du \\ &+ \int_t^\infty a(u) \prod_{j=1}^m (x(\sigma_j(u)))^{\lambda_j} \int_{t_k}^t \frac{\beta_k(u,s)}{r_k(s)} ds du\,, \qquad t \geq t_k\,. \end{aligned} \tag{34}$$

From (4) and Lemma 1(v) it follows

$$\int_{t_k}^u \frac{\beta_k(u,s)}{r_k(s)} ds = \beta_{k-1}(u, t_k)\,, \qquad t \geq u \geq t_k$$

and

$$\int_{t_k}^t \frac{\beta_k(u,s)}{r_k(s)} ds \geq \frac{R_k(t, t_k)}{R_k(u, t_k)} \beta_{k-1}(u, t_k)\,, \qquad t_k \leq t < u\,.$$

This together with (34) implies

$$L_{k-1}x(t) \geq \int_{t_k}^{t} \beta_{k-1}(u,t_k)a(u)\prod_{j=1}^{m}(x(\sigma_j(u)))^{\lambda_j}\,du + R_k(t,t_k)\int_{t_k}^{\infty}\frac{\beta_{k-1}(u,t_k)}{R_k(u,t_k)}a(u)\prod_{j=1}^{m}(x(\sigma_j(u)))^{\lambda_j}\,du\,, \qquad t \geq t_k\,. \quad (35)$$

From condition H3 there exists $t_2 \geq t_k$ such that $\sigma_j(u) \geq t_k$ for $t \geq t_2$, $j = 1, \ldots, m$. Then from (13) we have

$$x(\sigma_j(u)) \geq \frac{\alpha_k(\sigma_j(u),t_k)}{R_k(\sigma_j(u),t_k)}L_{k-1}x(\sigma_j(u))\,, \qquad u \geq t_2\,. \quad (36)$$

Since $\frac{L_{k-1}x(t)}{R_k(t,t_k)}$ is nonincreasing and $L_{k-1}x(t)$ is nondecreasing in $t \geq t_k$, then

$$L_{k-1}x(\sigma_j(u)) \geq \frac{R_k(\sigma_j(u),t_k)}{R_k(t,t_k)}L_{k-1}x(t)\,, \qquad t > u \geq t_2 \quad (37)$$

and

$$L_{k-1}x(\sigma_j(u)) \geq \frac{R_k(\sigma_j(u),t_k)}{R_k(u,t_k)}L_{k-1}x(t)\,, \qquad u > t \geq t_2\,.$$

Now, we conclude from (35)-(37) that

$$1 \geq \frac{1}{R_k(t,t_k)}\int_{T_1}^{t}\beta_{k-1}(u,t_k)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),t_k))^{\lambda_j}\,du + R_k(t,t_k)\int_{t}^{\infty}\frac{\beta_{k-1}(u,t_k)}{(R_k(u,t_k))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),t_k))^{\lambda_j}\,du \quad (38)$$

for $t \geq T_1 > \max(t_2, T_0)$.

From (31; k) it follows that there exists $\eta \in (0,1)$ such that

$$\eta \limsup_{t\to+\infty}\left\{\frac{1}{R_k(t,\alpha)}\int_{T_0}^{t}\beta_{k-1}(u,\alpha)a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du + R_k(t,\alpha)\int_{t}^{\infty}\frac{\beta_{k-1}(u,\alpha)}{(R_k(u,\alpha))^2}a(u)\prod_{j=1}^{m}(\alpha_k(\sigma_j(u),\alpha))^{\lambda_j}\,du\right\} > 1\,. \quad (39)$$

Set $\mu = \eta^{1/3}$. Since $\mu \in (0,1)$ and the functions $\sigma_j(t)$, $R_k(t,t_k)$, $\alpha_k(t,t_k)$ and $\beta_{k-1}(t,t_k)$ tend to infinity as $t \to +\infty$, there exists $t_\mu \geq t_2$ such that

$$R_k(u,\alpha) \geq R_k(u,t_k) \geq \mu R_k(u,\alpha)\,, \qquad u \geq t_\mu\,,$$
$$\alpha_k(\sigma_j(u),t_k) \geq \mu\alpha_k(\sigma_j(u),\alpha)\,, \qquad j = 1,\ldots,m\,, \quad u \geq t_\mu\,, \quad (40)$$
$$\beta_{k-1}(u,t_k) \geq \mu\beta_{k-1}(u,\alpha)\,, \qquad u \geq t_\mu\,.$$

Keeping in mind (40) and (38) we obtain

$$1 \ge \eta \Bigg\{ \frac{1}{R_k(t,\alpha)} \int_{T_1}^{t} \beta_{k-1}(u,\alpha) a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u),\alpha))^{\lambda_j} du$$

$$+ R_k(t,\alpha) \int_t^{\infty} \frac{\beta_{k-1}(u,\alpha)}{(R_k(u,\alpha))^2} a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u),\alpha))^{\lambda_j} du \Bigg\}, \quad t \ge T_1 . \qquad (41)$$

Taking into account (41) and the equality

$$\lim_{t\to+\infty} \frac{1}{R_k(t,\alpha)} \int_{T_0}^{T_1} \beta_{k-1}(u,\alpha) a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u),\alpha))^{\lambda_j} du = 0 ,$$

we obtain the inequality

$$1 \ge \eta \lim_{t\to+\infty} \sup \Bigg\{ \frac{1}{R_k(t,\alpha)} \int_{T_0}^{t} \beta_{k-1}(u,\alpha) a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u),\alpha))^{\lambda_j} du$$

$$+ R_k(t,\alpha) \int_t^{\infty} \frac{\beta_{k-1}(u,\alpha)}{R_k(u,\alpha)^2} a(u) \prod_{j=1}^{m} (\alpha_k(\sigma_j(u),\alpha))^{\lambda_j} du \Bigg\} ,$$

which contradicts (39).

In the cases $k = 0$ and $k = n$ the proof is the same as in the proof of Theorem 1. □

Corollary 3. *Let conditions H2, H3 hold and $\lambda = 1$. Then a sufficient condition for equation* (E_0, δ) *to be almost oscillatory is that:*

(i) *when $\delta = 1$ and n is even, for every $k \in \{1, 3, \dots, n-1\}$ and some $T_0 \ge \alpha$*

$$\lim_{t\to+\infty} \sup \Bigg\{ \frac{1}{t} \int_{T_0}^{t} \frac{u^{n-k}}{(n-k)!k!} a(u) \prod_{j=1}^{m} (\sigma_j(u))^{k\lambda_j} du$$

$$+ t \int_t^{\infty} \frac{u^{n-k-2}}{(n-k)!k!} a(u) \prod_{j=1}^{m} (\sigma_j(u))^{k\lambda_j} du \Bigg\} > 1 ; \qquad (42; k)$$

(ii) *when $\delta = 1$ and n is odd, conditions* (42; k), $k \in \{2, 4, \dots, n-1\}$ *and* (26) *hold;*

(iii) *when $\delta = -1$ and n is even, conditions* (42; k), $k \in \{2, 4, \dots, n-2\}$, (26) *and* (27) *hold;*

(iv) *when $\delta = -1$ and n is odd, conditions* (42; k), $k \in \{1, 3, \dots, n-2\}$ *and* (27) *hold.*

REFERENCES

[1] S.R. Grace, Oscillation theorems for n-th order differential equations with deviating arguments, *J. Math. Anal. Appl.*, **101** (1984), 268-296.

[2] S.R. Grace, Oscillatory properties of functional differential equations, *J. Math. Anal. Appl.*, **160** (1991), 60-78.

[3] S.R. Grace and B.S. Lalli, Oscillation theorems for n-th order delay differential equations, *J. Math. Anal. Appl.*, **91** (1983), 352-366.

[4] S.R. Grace and B.S. Lalli, Oscillatory and asymptotic behavior of solutions of differential equations with deviating arguments, *J. Math. Anal. Appl.*, **104** (1984), 79-94.

[5] S.R. Grace and B.S. Lalli, Oscillation of even order differential equations with deviating arguments, *J. Math. Anal. Appl.*, **147** (1990), 569-579.

[6] R.S. Dahiya and O. Akinyele, Oscillation theorems of n-th order functional differential equations with forcing terms, *J. Math. Anal. Appl.*, **109** (1985), no. 2, 325-332.

[7] I.T. Kiguradze, On the oscillation of solutions of the equation $\frac{d^m u}{dt^m} + a(t)|u|^n \mathrm{sign} u = 0$, *Mat. Sb.*, **65** (1964), no. 2, 172-187, In Russian.

[8] Y. Kitamura, Oscillation of functional differential equations with general deviating argument, *Hiroshima Math. J.*, **15** (1985), 445-491.

[9] N.T. Markova and P.S. Simeonov, Asymptotic and oscillatory properties of the solutions of differential equations with delays depending on the unknown function, Invited lectures delivered at the *VII-th Int. Colloquium on Differential Equations*, Volume **2**, August 18-23, 1996. Plovdiv, Bulgaria, 71-78.

[10] N.T. Markova and P.S. Simeonov, On the asymptotic behaviour of the solutions of a class of differential equations of second order with delay depending on the unknown function, Invited lectures delivered at the *VII-th Int. Colloquium on Differential Equations*, August 18-23, 1996, Plovdiv, Bulgaria, Volume **1**, 89-100.

[11] N.T. Markova and P.S. Simeonov, Oscillation theorems for n-th order nonlinear differential equations with forcing terms and deviating arguments depending on the unknown function, *Communications in Applied Analysis*, **9** (2005),no. 3, 417-427.

[12] N.T. Markova and P.S. Simeonov, Asymptotic and oscillatory behavior of n-th order forced differential equations with deviating argument depending on the unknown function, *PanAmerican Mathematical Journal*, **16** (2006), no. 1, 1-15.

[13] Ch.G. Philos, Oscillatory and aymptotic behavior of all solutions of differential equations with deviating arguments, *Proc. Roy. Soc. Edinburgh Sect. A*, **81** (1978), 195-210.