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STEADY-STATE THERMAL VISCOUS INCOMPRESSIBLE FLOWS WITH CONVECTIVE-RADIATIVE EFFECTS AND A NONLOCAL COULOMB FRICTION LAW

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ABSTRACT: We deal with a coupled system of elliptic motion and energy equations motivated by the thermal flow of a class of non-Newtonian fluids. A nonlocal Coulomb friction condition on a part of the liquid-solid boundary is taken into account. On this part of the boundary it is also considered a convective-radiative heat transfer related to the frictional work. The existence of a weak solution constitutes the main result of the present work which proof is based on a fixed point argument for multivalued mappings. The nonlinear boundary conditions as well as the energy dependent viscosities and the thermal conductivity are the crucial contribution on the interdependence on the fluid velocity vector, the stress tensor and the internal energy. The mathematical framework of the paper includes the classical monotone theory on elliptic equations, the duality theory of convex analysis in order to describe the Lagrange multipliers; and the L^1-theory on partial differential equations due to the existence of the Joule effect.

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1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The slip boundary conditions have been of interest of several authors along the last decades. Among them, we refer to Beirão da Veiga [4], Jäger and

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Mikelić [16] and the references therein. On the other hand, the literature on theoretical contributions to the heat conducting viscous incompressible flows has been concerned on the Navier-Stokes-Fourier problems under Dirichlet conditions with respect to the fluid velocity vector (see, for instance, Duvaut and Lions [11], Amann [1], Antontsev et al [2], Rodrigues [20] and the references therein).

In the present work, we show an existence result of weak solution to the steady-state heat conducting viscous incompressible flow problem under general constitutive laws for the heat flux, the Cauchy stress tensor, the slip and the convective-radiative boundary conditions. The Navier-Stokes-Fourier problem is included with a nonlocal Coulomb friction law and the convective-radiative boundary exponent (l) upper bounded by $(n-1)/(n-2)$. Indeed the radiation behaviour ($l = 4$) is only considered for the generalized Fourier law with the exponents $q > (5n-1)/(n+3)$, in particular in the three dimensional case $q > 7/3$ (cf. Remark 2.4).

Let Ω be a bounded open subset of $\mathbb{R}^n$ ($n \geq 2$) with a C^1-boundary $\partial\Omega$ which consists of the union of the closure of two open disjoint complementar subsets, i.e., $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}$. The heat conducting viscous fluid under study in this paper has the following constitutive laws for the heat flux $\mathbf{q} = (q_i)$ and the stress tensor $\sigma = (\sigma_{ij})$ (see Rajagopal [19], Cioranescu [7])

$$\begin{aligned} \mathbf{q} &= -\chi(\cdot, e)\mathbf{a}(\nabla e), \\ \tau &= \pi I + \sigma \in \partial\mathcal{F}(e, D\mathbf{u}). \end{aligned} \tag{1}$$

Here χ denotes the thermal diffusity, e represents the specific internal energy, π denotes the pressure, I is the identity matrix and τ is the viscous part of the stress tensor that belongs to the subdifferential of a functional $\mathcal{F}$ at the point given by $D\mathbf{u} = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$, for a fixed internal energy e. When $\mathbf{a}$ is the identity function, the relation

$$e = \int_0^\theta c_p(z)dz,$$

where c_p denotes the specific heat capacity, yields the Fourier law

$$\mathbf{q} = -\chi(\cdot, e(\theta))c_p(\theta)\nabla\theta,$$

with the thermal conductivity $k = \chi c_p$.

The internal energy e and the fluid velocity $\mathbf{u}$ satisfy the energy, motion and incompressibility equations in the following form

$$\mathbf{u}\cdot\nabla e - \nabla\cdot(\chi(\cdot, e)\mathbf{a}(\nabla e)) = \tau : D\mathbf{u} \qquad \text{in } \Omega; \tag{2}$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\tau = -\nabla\pi + \mathbf{f} \qquad \text{in } \Omega; \tag{3}$$

$$\nabla\cdot\mathbf{u} = \sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{in } \Omega, \tag{4}$$

where the density is assumed constant and equal to one, for simplicity. The functionals $\mathcal{F}$, $\mathbf{f}$ and $\mathbf{a}$ are, respectively, a given scalar field and given vector fields satisfying (10)-(14), (17), and (19)-(21).

Although the field of heat transfer is generally subdivided into conduction, radiation, and convection, in most practical situations heat is transferred by these modes simultaneously Kreith [15]. The total rate of heat flow $\mathbf{q}$ to the surface Γ is

$$\mathbf{q}\cdot\mathbf{n} = h_c(\theta-\theta_s) + \sigma\bar{\gamma}(\theta)(\theta^4-\theta_s^4),$$

where h_c denotes the convective heat transfer coefficient, θ_s a known surface temperature, σ the Stefan-Boltzmann constant, and $\bar{\gamma}=\bar{\gamma}(\theta)$ is a function on the temperature describing the total emittance, absorptance, reflectance and/or transmittance of an arbitrary black or nonblack body.

We assume Dirichlet conditions on Γ_D:

$$\mathbf{u}=\mathbf{0} \qquad \text{and} \qquad e=0; \tag{5}$$

and on the remaining part Γ of the boundary

$$\chi(\cdot,e)\mathbf{a}(\nabla e)\cdot\mathbf{n}+\gamma(\cdot,e) = -\tau_T\cdot\mathbf{u}_T \text{ on } \Gamma; \tag{6}$$

$$\mathbf{u}_N = 0 \text{ and } -\tau_T \in \partial[\varphi(\cdot,e)\Phi(\sigma_N)|\mathbf{u}_T|]. \tag{7}$$

The condition (6) means that the frictional work is related to the radiation emitted, absorbed, reflected and /or transmitted on the given surface. The condition (7) corresponds to a nonlocal Coulomb friction law in the subdifferential form Hadrian and Panagiotopoulos [13] when a energy dependant friction yield φ is taken into account, that is, the boundary operator Φ is a nonnegative regularizer (see Consiglieri [9] and the references therein).

The outline of this work is as follows. Next section we introduce the functional setting and the hypotheses on data used in the following and we state the main result. The proof of the main theorem is based on auxiliary results developed in the Section 3. Section 4 is devoted to the final proof by applying the Tychonov-Kakutani-Glicksberg Fixed Point Theorem for a multivalued mapping.

2. MAIN RESULT

Let us fix the functional Banach spaces in the framework of Lebesgue and Sobolev spaces. For $p,q>1$,

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{u}\in\mathbf{C}^\infty(\Omega):\ \nabla\cdot\mathbf{u}=0 \text{ in } \Omega\},\\
H_p &= \overline{\mathcal{V}}^{\|\cdot\|_{p,\Omega}} = \{\mathbf{u}\in\mathbf{L}^p(\Omega):\ \nabla\cdot\mathbf{u}=0 \text{ in } \Omega,\quad \mathbf{u}\cdot\mathbf{n}=0 \text{ on } \partial\Omega\},\\
V_p &= \{\mathbf{u}\in\mathbf{W}^{1,p}(\Omega):\ \nabla\cdot\mathbf{u}=0 \text{ in } \Omega,\ \mathbf{u}=\mathbf{0} \text{ on } \Gamma_D,\ u_N=0 \text{ on } \Gamma\},\\
V_T &= \{\mathbf{u}|_\Gamma:\ \mathbf{u}\in V_p\},
\end{aligned}$$

$$L^p_{\rm sym} = \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^p(\Omega)\},$$

$$W_q = \{e \in W^{1,q}(\Omega) : \ e = 0 \text{ on } \Gamma_0\},$$

endowed with their canonical norms, assuming always that meas $(\Gamma_D) > 0$ such that the Poincaré inequality holds in Ω. In notation concerning norms, we will not distinguish between scalar and vector fields.

Definition. We say that $(\mathbf{u}, \tau, e)$ is a *weak solution* to the problem (1)-(7) if $(\mathbf{u}, \tau, e) \in V_p \times L^{p'}_{\rm sym} \times W_r$ satisfies (1) and

$$\int_\Omega D\mathbf{u} : \mathbf{u} \otimes \mathbf{v} dx + \int_\Omega \{\mathcal{F}(e, D\mathbf{v}) - \mathcal{F}(e, D\mathbf{u})\} dx$$

$$+ \int_\Gamma \varphi(e)\Phi(\sigma_N)\{|\mathbf{v}_T| - |\mathbf{u}_T|\} ds \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) dx, \quad \forall \mathbf{v} \in V_p; \tag{8}$$

$$\int_\Omega \mathbf{u} \cdot \nabla e \phi dx + \int_\Omega \chi(e) \mathbf{a}(\nabla e) \cdot \nabla \phi dx + \int_\Gamma \gamma(e) \phi ds$$

$$= \int_\Omega \tau : D\mathbf{u} \phi dx + \int_\Gamma \varphi(e) \Phi(\sigma_N) |\mathbf{u}_T| \phi ds, \qquad \forall \phi \in W_{r/(r-q+1)}. \tag{9}$$

We assume that

$$\mathcal{F}(e, \kappa) = \mu(\cdot, e) F_1(|\kappa|) + \eta(\cdot, e) F_2(|\kappa|), \qquad \kappa \in \mathbb{M}_{n\times n}$$

where the viscosities $\mu, \eta : \Omega \times \mathbb{R} \to \mathbb{R}^+_0$ are Carathéodory functions, $F_1, F_2 : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ are convex functions such that $F_1(0) = F_2(0) = 0$ and

$$\exists \mu_\#, \mu^\# > 0 : \qquad \mu_\# \leq \mu(\cdot, e) \leq \mu^\#; \tag{10}$$

$$\exists \eta_\#, \eta^\# > 0 : \qquad \eta_\# \leq \eta(\cdot, e) \leq \eta^\#; \tag{11}$$

$$\exists p > 1, \ \alpha_\# > 0 : \qquad F_1(d) \geq \alpha_\# d^p; \tag{12}$$

$$\exists \alpha^\# > 0 : \qquad F_1(d) \leq \alpha^\#(d^p + 1); \tag{13}$$

$$\exists 1 \leq p_2 \leq p, \ \beta > 0 : \qquad 0 \leq F_2(d) \leq \beta(d^{p_2} + 1), \tag{14}$$

almost everywhere in Ω, for every $e \in \mathbb{R}$ and for all $d \geq 0$. Furthermore suppose that F_1 is strictly convex, $\varphi : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\Phi : L^{p'}(\Gamma) \to L^{p'}(\Gamma)$ is weakly continuous

$$\exists \varphi^\# > 0 : \qquad 0 \leq \varphi(s, e) \leq \varphi^\#; \tag{15}$$

$$\exists 0 < \iota < 1/\varphi^\# : \qquad 0 \leq \Phi(d) \leq \iota |d|; \tag{16}$$

$$\mathbf{f} \in \mathbf{L}^{p'}(\Omega), \tag{17}$$

almost everywhere $s \in \Gamma$, for every $e, d \in \mathbb{R}$.

We also assume that $\chi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $\gamma : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, $\gamma(\cdot, 0) = 0$, such that, respectively,

$$\exists \chi_\#, \chi^\# > 0 : \qquad \chi_\# \leq \chi(x, e) \leq \chi^\#; \tag{18}$$

$$\exists q > 1,\ \upsilon^\# > 0 : \qquad |\mathbf{a}(\kappa)| \leq \upsilon^\#(|\kappa|^{q-1} + 1); \tag{19}$$

$$\exists \upsilon_\# > 0 : \qquad \mathbf{a}(\kappa) \cdot \kappa \geq \upsilon_\# |\kappa|^q; \tag{20}$$

$$\exists \upsilon > 0 : \qquad (\mathbf{a}(\kappa) - \mathbf{a}(\zeta)) \cdot (\kappa - \zeta) \geq \upsilon |\kappa - \zeta|^q, \text{ if } q \geq 2;$$

$$(\mathbf{a}(\kappa) - \mathbf{a}(\zeta)) \cdot (\kappa - \zeta) \geq \upsilon |\kappa - \zeta|^2 (1 + |\kappa| + |\zeta|)^{q-2} \text{ if } q < 2; \tag{21}$$

$$\exists l \geq 1,\ \gamma^\# > 0 : \qquad |\gamma(s, e)| \leq \gamma^\#(|e|^l + 1); \tag{22}$$

$$(\gamma(s, e) - \gamma(s, \xi))\text{sign}(e - \xi) \geq 0, \tag{23}$$

almost everywhere $x \in \Omega$ and $s \in \Gamma$, for every $e, \xi \in \mathbb{R}$ and for all $\kappa, \zeta \in \mathbb{R}^n$.

Remark 2.1. The convective term $\int_\Omega D\mathbf{u} : \mathbf{w} \otimes \mathbf{v} dx$ in (8) has meanful for $\mathbf{w} \in H_s, \mathbf{u}, \mathbf{v} \in V_p$ if $s \geq pn/(np + p - 2n)$ and $p < n$, or $s \geq p'$ and $p \geq n$. The antisymmetry property is valid, and the compact imbedding $V_p \hookrightarrow\hookrightarrow H_s$ occurs when $p > 3n/(n + 2)$ (see Ladyzenskaya [17] or Lions [18]).

Remark 2.2. For $1 < r < n(q - 1)/(n - 1)$ and $q > 2 - 1/n$, we have $r/(r-q+1) > n$ and then it is valid the Sobolev imbedding $W^{r/(r-q+1)}(\Omega) \hookrightarrow L^\infty(\overline{\Omega})$. The convective term in (9) has meanful for $\mathbf{w} \in H_s,\ e \in W_r$ and $\phi \in W_{r/(r-q+1)}$ if $s \geq r'$. From Remark 2.1, the requirement

$$\max(\frac{pn}{p(n+1) - 2n}, r') \leq s < \frac{pn}{n - p}$$

leads to the restrictions

$$q > \frac{n(2p - 1)}{p(n + 1) - n} \text{ and } \frac{3n}{n + 2} < p < n. \tag{24}$$

If $p \geq n$, the restriction on s is such that $s \geq \max(r', p')$, where the exponents p and q have no dependence on each other:

$$p \geq n \text{ and } q > 2 - 1/n. \tag{25}$$

Then if $n = 3$ and $p = 2$, then $q > 9/5$.

Remark 2.3. The convective term in (9) is well defined for $\mathbf{w} \in H_s,\ e, \phi \in W_q$. Indeed:

- if $q \geq n$, from Remark 2.1, we can choose an arbitrary s such that $s \geq \max(q', p')$ if $p \geq n$, and $s \geq \max(\frac{pn}{p(n+1)-2n}, q')$ if $p < n$ and taking into account that $q \geq \max(\frac{pn}{p(n+1)-n}, n) = n$ $(\forall p > 1)$.
- if $q < n$, from Remark 2.1 we can choose an arbitrary s such that $s \geq \max(\frac{qn}{q(n+1)-2n}, p')$ if $p \geq n$, where the exponents p and q have no dependence on each other. If $n > p > 3n/(n+2)$, the requirement

$$\max(\frac{pn}{p(n+1)-2n}, \frac{qn}{q(n+1)-2n}) \leq s < \frac{np}{n-p}$$

is valid for q satisfying (24) since $n > q > \frac{2np}{p(n+2)-n}$ $(n = 2, 3, 4)$. Indeed we have

$$\frac{n(2p-1)}{p(n+1)-n} \geq \frac{2np}{p(n+2)-n} \quad \text{if} \quad p \geq n/2 \geq 1.$$

Notice that if $n > 4$, $p \geq n/2 > 3n/(n+2)$.

Remark 2.4. If $1 \leq l < (q-1)(n-1)/(n-q)$ for $q < n$ or $1 \leq l < +\infty$ for $q \geq n$, then there exists $nl/(l+n-1) < r < (q-1)n/(n-1)$, and $1 \leq nl/(l+n-1) < r$ means that $W_r \hookrightarrow\hookrightarrow L^l(\Gamma)$.

Theorem 1. *Under the assumptions (10)-(25), if $1 \leq l < (q-1)(n-1)/(n-q)$ for $q < n$ or $1 \leq l < +\infty$ for $q \geq n$, there exists a weak solution to the problem (1)-(7), for all $nl/(l+n-1) < r < (q-1)n/(n-1)$.*

3. AUXILIARY RESULTS

In this section we will adapt some known results in order to apply to our problem. The known result (see Cioranescu [6]) for stationary fluid velocity with prescribed coefficients is stated in the following proposition.

Proposition 3.1. *Let the assumptions (10)-(17) be fulfilled. For all $p > 1$, $\mathbf{w} \in H_s$, $\xi \in W^{1,1}(\Omega)$ and $\psi \in L^{p'}(\Gamma)$, there exists a unique solution $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi, \psi) \in V_p$ satisfying*

$$\int_\Omega D\mathbf{u} : \mathbf{w} \otimes \mathbf{v} dx + \int_\Omega \{\mathcal{F}(\xi, D\mathbf{v}) - \mathcal{F}(\xi, D\mathbf{u})\} dx$$

$$+ \int_\Gamma \varphi(\xi)\Phi(\psi)\{|\mathbf{v}_T| - |\mathbf{u}_T|\} ds \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) dx, \quad \mathbf{v} \in V_p. \tag{26}$$

Moreover, the following estimate holds

$$\|\mathbf{u}\|_{V_p} \leq C(\Omega, \mu_\# \alpha_\#) \|\mathbf{f}\|_{p',\Omega}^{1/(p-1)} := R_1. \tag{27}$$

The following result for the existence of Lagrange multiplyer is consequence of duality theory of convex optimization (see Ekeland and Temam [12]) and its proof can be found in Consiglieri [9].

Proposition 3.2. *For each solution* $\mathbf{u}$ *given at the Proposition 3.1, there exists a stress tensor* $\sigma \in L^{p'}_{\mathrm{sym}}$ *such that*

$$\sigma = -\pi I - \zeta_1 - \zeta_2; \tag{28}$$

$$\text{and } \sigma_T = \varsigma \text{ in } (V_T)'; \tag{29}$$

where $\pi \in L^{p'}_0(\Omega)$, *the subspace of* $L^{p'}(\Omega)$ *consisting of functions with mean value equal to* 0, $\zeta_1, \zeta_2 \in L^{p'}_{\mathrm{sym}}, \varsigma \in \mathbf{L}^{p'}(\Gamma)$ *satisfy*

$$\int_\Gamma \varphi(\xi)\Phi(\psi)|\mathbf{u}_T| ds = -\int_\Gamma \varsigma \cdot \mathbf{u}_T ds \text{ and } |\varsigma| \leq \varphi(\xi)\Phi(\psi) \text{ on } \Gamma; \tag{30}$$

$$\int_\Omega \mu(\xi) F_1(|D\mathbf{u}|) dx + \int_\Omega \mu(\xi) F_1^* \left(\left| \frac{\zeta_1}{\mu(\xi)} \right| \right) dx = -\int_\Omega \zeta_1 : D\mathbf{u} dx; \tag{31}$$

$$\int_\Omega \eta(\xi) F_2(|D\mathbf{u}|) dx + \int_\Omega \eta(\xi) F_2^* \left(\left| \frac{\zeta_2}{\eta(\xi)} \right| \right) dx = -\int_\Omega \zeta_2 : D\mathbf{u} dx; \tag{32}$$

$$(\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla \cdot (\zeta_1 + \zeta_2) = \mathbf{f} - \nabla \pi \text{ in } \Omega. \tag{33}$$

Moreover, the estimates hold

$$\|\zeta_i\|^{p'}_{p',\Omega} \leq C_i(\|D\mathbf{u}\|^p_{p,\Omega} + 1) \qquad (i = 1, 2), \tag{34}$$

$$\|\sigma\|_{p',\Omega} \leq C(\|\mathbf{w}\|_{s,\Omega} + 1), \tag{35}$$

where C_i *denotes a constant dependant on* $p, |\Omega|, \mu^\# \alpha^\#$ *or* $\eta^\# \beta$, *respectively, and* C *is dependant on* $C_1, C_2, \mu_\# \alpha_\#$ *and* $\|\mathbf{f}\|_{p',\Omega}$.

Conversely, if $\mathbf{u} \in V_p$ *and* $\sigma \in L^{p'}_{\mathrm{sym}}$ *satisfies (28)-(33), then* $\mathbf{u}$ *is the solution to (26).*

Remark 3.1. Since $\sigma \in L^{p'}_{\mathrm{sym}}$ and $\nabla \cdot \sigma \in \mathbf{L}^{p'}(\Omega)$, there exists a uniquely determined linear continuous trace mapping $\mathcal{T}$ such that $\mathcal{T}(\sigma) = \sigma \cdot \mathbf{n}$ and the following decomposition holds (see Kikuchi and Oden [14])

$$\sigma \cdot \mathbf{n} = \sigma_T + \sigma_N \mathbf{n}.$$

Subsequently the estimate holds

$$\|\sigma_N\|_{p',\Gamma} \leq C(1 + R_1 \|\mathbf{w}\|_{s,\Omega} + R_1^p) + \varphi^\# \iota \|\psi\|_{p',\Gamma}, \tag{36}$$

with a nonnegative constant C only dependant on the data.

In the following proposition an existence of a SOLA solution in accordance to L^1-theory is stated (see Boccardo and Gallouet [5]).

Proposition 3.3. *Let the assumptions (18)-(23) be fulfilled. For each* $q > 2 - 1/n$, $\mathbf{w} \in H_s$, $\xi \in W^{1,1}(\Omega)$, $g \in L^1(\Omega)$ *and* $h \in L^1(\Gamma)$, *there exists* *a SOLA solution* $e \in W_r$*, for all* $nl/(l+n-1) < r < (q-1)n/(n-1)$*, satisfying*

$$\int_\Omega \mathbf{w}\cdot\nabla e\phi dx + \int_\Omega \chi(\xi)\mathbf{a}(\nabla e)\cdot\nabla\phi dx + \int_\Gamma \gamma(e)\phi ds$$

$$= \int_\Omega g\phi dx + \int_\Gamma h\phi ds, \quad \forall \phi \in W_{r/(r-q+1)}. \tag{37}$$

Moreover, the estimate holds, independantly on $\mathbf{w}$ *and* ξ*,*

$$\|e\|_{W_r} \le R(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma}), \tag{38}$$

with a constant R *dependant on the data constants and on* $\Lambda = \|g\|_{1,\Omega} + \|h\|_{1,\Gamma}$*, and such that* $R \to 0$ *if* $\Lambda \to 0$*.*

Proof. Let us argue as in Boccardo and Gallouet [5] or Dall'aglio [10]. Under regular data $g \in L^{q'}(\Omega)$ and $h \in L^{q'}(\Gamma)$, from classical existence results for elliptic equations (for instance, see Zeidler [21]) it remains to verify the coercivity and the boundedness of the operator A defined by

$$\langle Ae,\phi\rangle = \int_\Omega \mathbf{w}\cdot\nabla e\phi dx + \int_\Omega \chi(\xi)\mathbf{a}(\nabla e)\cdot\nabla\phi dx + \int_\Gamma \gamma(e)\phi ds, \quad \forall e,\phi \in W_q.$$

Indeed, from the antisymmetry to the convective term and the assumptions (18), (20) and (23) we have

$$\langle Ae, e\rangle \ge \chi_\# \|\nabla e\|^q_{q,\Omega}.$$

Applying Hölder inequalities we have

$$|\langle Ae,\phi\rangle| \le \|\mathbf{w}\|_{s,\Omega}\|\nabla e\|_{q,\Omega}\|\phi\|_{qn/(n-q),\Omega} + \chi^\#\|\mathbf{a}(\nabla e)\|_{q',\Omega}\|\phi\|_{W_q}$$
$$+\|\gamma(e)\|_{q(n-1)/(nq-n),\Gamma}\|\phi\|_{q(n-1)/(n-q),\Gamma}.$$

Taking into account that $lq(n-1)/(nq-n) < q(n-1)/(n-q)$ for $l < n(q-1)/(n-q)$, from (18)-(19) and (22) it follows

$$|\langle Ae,\phi\rangle| \le C(\|\mathbf{w}\|_{s,\Omega}\|\nabla e\|_{q,\Omega} + \|\nabla e\|^q_{q,\Omega} + \|e\|^l_{q(n-1)/(n-q),\Gamma} + 1)\|\phi\|_{W_q},$$

where s is in accordance to Remark 2.3. Clearly for $q > n$ the estimate (38) holds in the form

$$\|\nabla e\|_{q,\Omega} \le \left(\frac{1}{\chi_\#}(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma})\right)^{1/(q-1)}.$$

Arguing as in Boccardo and Gallouet [5] or Dall'aglio [10], for $q \le n$, let m be a fixed natural number and choose $e_m = \min((|e|-m)_+, 1)\mathrm{sign}(e)$ as a test function in (37), considering g_m and h_m as regular data such that

$\|g_m\|_{q',\Omega} \le \|g\|_{1,\Omega}$ and $\|h_m\|_{q',\Gamma} \le \|h\|_{1,\Gamma}$. Since the convective term vanishes, from the assumptions (18) and (20) we obtain

$$\|\nabla e\|_{r,\Omega}^r \le \left(\frac{\|g\|_{1,\Omega} + \|h\|_{1,\Gamma}}{\chi_\# \upsilon_\#}\right)^{r/q} \|e\|_{\lambda,\Omega}^{\lambda(q-r)/q} \left(\sum_{m\ge 1} \frac{1}{m^{\lambda(q-r)/r}}\right)^{r/q},$$

where $r/(q-r) < \lambda \le rn/(n-r)$ is choosen such that the imbedding $W^r(\Omega) \hookrightarrow L^\lambda(\Omega)$ can be applied and the Dirichlet series converge. We wish to emphasize that the bound $r < n(q-1)/(n-1)$ comes from this choice. Then applying the Poincaré inequality, we conclude that

$$\|\nabla e\|_{r,\Omega} \le C(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma})^{r/(qr-\lambda(q-r))}, \qquad q \le n.$$

Hence, we can extract a subsequence denoted by e_m such that $e_m \rightharpoonup e$ in W_r, $e_m \to e$ in Ω and Γ, and $e_m \to e$ in $L^l(\Gamma)$. Therefore, the limit e is the solution obtained as limit approximation (or simply SOLA, cf. Dall'aglio [10]) to the problem (37). We skip the proof and we refer the reader to Dall'aglio [10] where this step is performed for a similar problem. □

4. PROOF OF THEOREM 1

Let us set a multivalued mapping $\mathcal{L}$ from $X := V_p \times L^{p'}(\Gamma) \times L^1(\Omega) \times L^1(\Gamma) \times W_r$ defined by

$$\mathcal{L}(\mathbf{w}, \psi, g, h, \xi) = \{(\mathbf{u}, \sigma_N, \tau : D\mathbf{u}, \varphi(\xi)\Phi(\psi)|\mathbf{u}_T|, e)\}, \quad \tau = -(\zeta_1 + \zeta_2),$$

where $\mathbf{u}$ is the unique solution given at Proposition 3.1, σ_N at Remark 3.1, ζ_1 and ζ_2 are given at Proposition 3.2 and e is the unique SOLA solution given at Proposition 3.3. In order to apply the Tychonov-Kakutani-Glicksberg Fixed Point Theorem Baiocchi and Capelo [3], p. 218-220, we consider the space X endowed with the product of weak topologies. Thus X becomes a locally convex Hausdorff topological vector space, and the ball

$$K := \{(\mathbf{w}, \psi, g, h, \xi) \in X : \ \|\mathbf{w}\|_{V_p} \le R_1, \ \|\psi\|_{p',\Gamma} \le R_2, \ \|g\|_{1,\Omega} \le R_3, \\ \|h\|_{1,\Gamma} \le R_4, \ \|\xi\|_{W_r} \le R_5\}$$

is a nonempty compact convex set in X. The operator $\mathcal{L}$ maps K into $\mathcal{P}(K)$, considering R_1 chosen as in (27) and chosing R_i $(i = 2, \cdots, 5)$ such that

$$\|\sigma_N\|_{p',\Gamma} \le CR_1^p + \varphi^\# \iota R_2 := R_2\,,$$

$$\|\tau : D\mathbf{u}\|_{1,\Omega} \le C(R_1^p + 1) := R_3\,,$$

$$\|\varphi(\xi)\Phi(\psi)|\mathbf{u}_T|\|_{1,\Gamma} \le C\varphi^\# \iota R_2 R_1 := R_4\,,$$

$$\|e\|_{W_r} \le R(R_3 + R_4) := R_5,$$

taking into account the estimates (36), (34), (27) and (38), respectively. The set $\mathcal{L}(\mathbf{w}, \psi, g, h, \xi)$ is convex due to convex property of the set of Lagrange multipliers and the uniqueness of the solutions $\mathbf{u}$ and e. To conclude the proof it remains to prove the closeness in $K \times K$ of the graph set:

$$G_{KK}(\mathcal{L}) := \{(y, z) \in K \times K : \ z \in \mathcal{L}(y)\}.$$

Take the sequences $(\mathbf{w}_m, \psi_m, g_m, h_m, \xi_m) \in K$ and

$$(\mathbf{u}_m, \sigma_{mN}, \tau_m : D\mathbf{u}_m, \varphi(\xi_m)\Phi(\psi_m)|\mathbf{u}_{mT}|, e_m) \in \mathcal{L}(\mathbf{w}_m, \psi_m, g_m, h_m, \xi_m)$$

satisfying

$$\begin{aligned} \mathbf{w}_m \rightharpoonup \mathbf{w}, \ \mathbf{u}_m \rightharpoonup \mathbf{u} \quad & \text{in} \quad V_p; \\ \psi_m \rightharpoonup \psi, \ \sigma_{mN} \rightharpoonup \kappa_1 \quad & \text{in} \quad L^{p'}(\Gamma); \\ g_m \rightharpoonup g, \ \tau_m : D\mathbf{u}_m \rightharpoonup \kappa_2 \quad & \text{in} \quad L^1(\Omega); \\ h_m \rightharpoonup h, \ \varphi(\xi_m)\Phi(\psi_m)|\mathbf{u}_{mT}| \rightharpoonup \kappa_3 \quad & \text{in} \quad L^1(\Gamma); \\ \xi_m \rightharpoonup \xi, \ e_m \rightharpoonup e \quad & \text{in} \quad W_r. \end{aligned}$$

From the weak convergence property, it follows that $(\mathbf{u}, \kappa_1, \kappa_2, \kappa_3, e) \in K$. Taking into account Remark 2.1, we have $\mathbf{w}_m \to \mathbf{w}$ in H_s. From the compact imbedding $V_p \hookrightarrow\hookrightarrow \mathbf{L}^p(\Gamma)$ we have $|\mathbf{u}_{mT}| \to |\mathbf{u}_T|$ in $\mathbf{L}^p(\Gamma)$. The compact imbeddings $W^r(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ and $W^r(\Omega) \hookrightarrow\hookrightarrow L^1(\Gamma)$ imply that $\xi_m \to \xi$ a.e. in Ω and a.e. on Γ. Then using the continuity property of the Niemytski operators μ, η and φ and applying classical monotonicity argument, we can pass to the limit in (26), when m tends to infinity, obtaining $\mathbf{u} = \mathbf{u}(\mathbf{w}, \psi, \xi)$ solution to (26). Hence, we conclude that $\kappa_3 = \varphi(\xi)\Phi(\psi)|\mathbf{u}_T|$.

Using Remark 3.1 and Proposition 3.2, we can extract subsequences denoted by π_m, $\tau_m = -(\zeta_{1m} + \zeta_{2m})$ and ς_m weakly convergent to π, τ and ς in $L^{p'}(\Omega)$, $L^{p'}_{\text{sym}}$ and $\mathbf{L}^{p'}(\Gamma)$, respectively, in order to prove that $\kappa_1 = \sigma_N$ and $\kappa_2 = \tau : D\mathbf{u}$ (see Consiglieri [8], Consiglieri [9], for details).

Finally, in order to recognise that the weak limit e is the SOLA solution to (37) we recall the convergence of ξ_m to ξ a.e. in Ω and a.e. on Γ. Hence the continuity of the Niemytski operator χ allows the passage to the limit on the thermal diffusity coefficient. The proof of the weak convergence of $\mathbf{a}(\nabla e_m)$ to $\mathbf{a}(\nabla e)$ in $L^{r/(q-1)}(\Omega)$ results from the convergence in measure of ∇e_m to ∇e (see Dall'aglio [10]). Thus the passage to the limit in (37) is finished.

Then we fulfil the conditions of the Tychonov-Kakutani-Glicksberg Fixed Point Theorem and therefore the required solution to the problem (8)-(9) arises.

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