

## ABSOLUTE EXTREMA OF INVARIANT OPTIMAL CONTROL PROBLEMS

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**ABSTRACT:** Optimal control problems are usually addressed with the help of the famous Pontryagin Maximum Principle (PMP) which gives a generalization of the classical Euler-Lagrange and Weierstrass necessary optimality conditions of the calculus of variations. Success in applying the PMP permits to obtain candidates for a local minimum. In 1967 a direct method, which permits to obtain global minimizers directly, without using necessary conditions, was introduced by Leitmann. Leitmann's approach is connected, as showed by Carlson in 2002, with "Carathéodory's royal road of the calculus of variations". Here we propose a related but different direct approach to problems of the calculus of variations and optimal control, which permit to obtain global minima directly, without recourse to needle variations and necessary conditions. Our method is inspired by the classical Noether's Theorem and its recent extensions to optimal control. We make use of the variational symmetries of the problem, considering parameter-invariance transformations and substituting the original problem by a parameter-family of optimal control problems. Parameters are then fixed in order to make the problem trivial, in some sense. Finally, by applying the inverse of the chosen invariance-transformation, we get the global minimizer for the original problem. The proposed method is illustrated, by solving concrete problems, and compared with Leitmann's approach.

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## 1. INTRODUCTION

The main goal in optimal control is to find a global (or local) minimizer. One of the most important tools is given by the famous Pontryagin Maximum Principle (PMP) which is a first order necessary optimality condition Pontryagin et al [13]. The PMP provides a generalization of the classical Euler-Lagrange and Weierstrass necessary conditions of the calculus of variations and permits to obtain candidates for a local minimum. Further analysis is then needed, to effectively find the extremum.

In 1967 a different approach, based on a coordinate transformation, was introduced by Leitmann [7], allowing the direct global extremization of certain functional integrals of the calculus of variations, without the use of variational methods or field techniques Leitmann [9]. The method is also valid for multiple integrals of the calculus of variations Carlson and Leitmann [3] and is proved Carlson [2] to be connected with “Carathéodory’s royal road of the calculus of variations” Carathéodory [1]. Here we provide a new look to Leitmann’s approach.

We propose a different direct approach to certain problems of the calculus of variations and optimal control, which permit to obtain global minima directly, without recourse to needle variations and necessary conditions. Differently from Leitmann, our method is based on the variational symmetries of the problem: a notion introduced by Emmy Noether in the classical context of the calculus variations Noether [12] and then extended to the more general context of optimal control Torres [14], Torres [16]. Our method proceeds in three steps: (i) we consider parameter-invariance transformations of the problem, generalizing the original problem to an equivalent one; (ii) parameters are then fixed in order to make the generalized problem trivial in some sense; (iii) finally, the desired global minimizer is obtained by applying the inverse of the chosen invariance-transformation and imposing the fulfilment of the boundary conditions.

The paper is organized as follows. In Section 2 we formulate the optimal control problem, providing all the necessary background. In Section 3 we recall Leitmann’s approach and apply it to a simple problem of the calculus of variations. The same problem is then solved in Section 4, for comparison and motivational purposes, by our direct optimization method. After summarizing the main ideas and steps of the proposed method, we end Section 4 by considering the minimum fuel rendezvous of a constant-power rocket. Finally, some conclusions are presented in Section 5.

## 2. PRELIMINARIES

Without loss of generality, we consider the problem of optimal control in Lagrange form: to minimize an integral functional

$$I[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) dt \quad (1)$$

subject to a control system

$$\dot{x}(t) = \varphi(t, x(t), u(t)) \quad \text{a.e. on } [a, b], \quad (2)$$

together with appropriate boundary conditions  $x(a) = \alpha$ ,  $x(b) = \beta$ . The Lagrangian  $L(\cdot, \cdot, \cdot)$  is a real function, assumed to be continuously differentiable in  $[a, b] \times \mathbb{R}^n \times \mathbb{R}^m$ ;  $t \in \mathbb{R}$  is the independent variable;  $x : [a, b] \rightarrow \mathbb{R}^n$  the vector of state variables;  $u : [a, b] \rightarrow \Omega \subset \mathbb{R}^m$  the vector of controls, assumed to be a piecewise continuous function; and  $\varphi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  the velocity vector, assumed to be a continuously differentiable vector function. In the particular case  $\varphi(t, x, u) = u$  one gets the fundamental problem of the calculus of variations.

The essential concept we are going to use is that of *equivalence* between two problems of optimal control. In Carathéodory's terminology two problems of the calculus of variations are said to be equivalent when the respective Lagrangians differ by a total derivative Carathéodory [1]. The importance of this equivalence concept owes to the fact that it implies the Euler-Lagrange equations to be identical for both problems. In Torres [15] the following consequence is explored: *two Carathéodory-equivalent problems have the same conservation laws*. It turns out, has proved by Noether [12], Torres [16], that conservation laws are a consequence of the existence of invariance-transformations (variational symmetries). The method we propose here is based on the following trivial remark: the invariance-transformations define a direct relation between admissible state-control pairs, being straightforward, from the transformations which define the equivalence, to obtain a solution for any of the equivalent problems known the solution for one of them. The variational symmetries may be found with the help of a computer algebra system Gouveia and Torres [5] and, roughly speaking, a given problem (1)–(2) is solved if it admits an enough rich set of variational symmetries and there exists an equivalent formulation of the problem with a trivial solution. This will be illustrated in Section 4. Now we recall the notion of invariance (variational symmetry) of an optimal control problem with respect to a  $s$ -parameter family of transformations.

**Definition 2.1.** (cf. Torres [14], Torres [16]) Let  $h^s(\cdot, \cdot, \cdot)$  be a one-parameter family of  $C^1$  mappings satisfying:

$$\begin{aligned} h^s &: [a, b] \times \mathbb{R}^n \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \\ h^s(t, x, u) &= (t^s(t, x, u), x^s(t, x, u), u^s(t, x, u)), \\ h^0(t, x, u) &= (t, x, u), \quad \forall (t, x, u) \in [a, b] \times \mathbb{R}^n \times \Omega. \end{aligned}$$

If there exists a function  $\Phi^s(t, x, u) \in C^1([a, b], \mathbb{R}^n, \Omega; \mathbb{R})$  such that

$$\begin{aligned} L \circ h^s(t, x(t), u(t)) \frac{d}{dt} t^s(t, x(t), u(t)) \\ = L(t, x(t), u(t)) + \frac{d}{dt} \Phi^s(t, x(t), u(t)) \end{aligned} \quad (3)$$

and

$$\frac{d}{dt} x^s(t, x(t), u(t)) = \varphi \circ h^s(t, x(t), u(t)) \frac{d}{dt} t^s(t, x(t), u(t)) \quad (4)$$

for all admissible pairs  $(x(\cdot), u(\cdot))$ . Then the optimal control problem (1)–(2) is said to be *invariant* under the transformations  $h^s(t, x, u)$  up to  $\Phi^s(t, x, u)$ .

A parameter-transformation  $h^s(t, x, u)$  satisfying all the conditions of Definition 2.1 is said to be a *variational symmetry* of the optimal control problem (1)–(2). These invariance-transformations are the starting point to our direct optimization method Section 4. Next, we review, comment and illustrate Leitmann's approach.

### 3. REMARKS ON LEITMANN'S DIRECT OPTIMIZATION METHOD

G. Leitmann has proposed in 1967 a direct optimization method for a certain class of scalar problems of the calculus of variations Leitmann [7]. Leitmann's method is based on the use of transformations that satisfy a certain functional identity and permit to obtain, in some cases, absolute extremals directly, without using variational methods. Since the pioneering work Leitmann [7] Leitmann has worked on several generalizations of his method, extending the class of problems to which the method may be applied: to problems of the calculus of variations where the trajectory is vector-valued, i.e.  $x(t) \in \mathbb{R}^n$ , and to problems with side differential conditions that arise in optimal control Leitmann [9]; by allowing constraints in the form of differential equations and by considering infinite-horizon problems Leitmann [10]. More recently, Carlson and Leitmann extended the method to free problems of the calculus of variations with multiple integrals Carlson and Leitmann [3]. In this section we synthesize Leitmann's method Carlson [2], Carlson and Leitmann [3], Leitmann [7], Leitmann [9], Leitmann

[10]. Then, we apply it to solve a simple problem of the calculus of variations which is used in Section 4 to motivate our method.

### 3.1. LEITMANN'S MAIN RESULTS

Consider the fundamental problem of the calculus of variations:

$$I[x(\cdot)] = \int_a^b L(t, x(t), \dot{x}(t)) dt \longrightarrow \min, \tag{5}$$

where  $\dot{x}(t) = \frac{dx(t)}{dt}$ ,  $[a, b]$  is a given fixed interval, the Lagrangian  $L(\cdot, \cdot, \cdot)$  is a real continuously differentiable function in  $[a, b] \times \mathbb{R} \times \mathbb{R}$ , the admissible functions  $x(\cdot)$  belong to  $PC^1$  and must satisfy the boundary conditions

$$x(a) = \alpha, \quad x(b) = \beta. \tag{6}$$

**Theorem 3.1.** *Let  $x = z(t, \tilde{x})$  be a transformation having an unique inverse  $\tilde{x} = \tilde{z}(t, x)$  for  $t \in [a, b]$ , such that there is a one-to-one correspondence*

$$x(t) \Leftrightarrow \tilde{x}(t),$$

*for all functions  $x(\cdot) : [a, b] \rightarrow \mathbb{R}$  in the class  $PC^1$  satisfying (6) and all functions  $\tilde{x}(\cdot) : [a, b] \rightarrow \mathbb{R}$  in the class  $PC^1$  satisfying*

$$\tilde{x}(a) = \tilde{z}(a, \alpha), \quad \tilde{x}(b) = \tilde{z}(b, \beta). \tag{7}$$

*If the transformation  $x = z(t, \tilde{x})$  is such that there exists a function  $G : [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$  such that the functional identity*

$$L(t, x(t), \dot{x}(t)) - L(t, \tilde{x}(t), \dot{\tilde{x}}(t)) = \frac{dG}{dt}(t, \tilde{x}(t)) \tag{8}$$

*holds, then if  $\tilde{x}^*(\cdot)$  yields the extremum of  $I[\cdot]$  with  $\tilde{x}^*(\cdot)$  satisfying (7),  $x^*(t) = z(t, \tilde{x}^*(t))$  yields the extremum of  $I[\cdot]$  for  $x^*(\cdot)$  satisfying (6).*

**Remark 3.1.** To the best of our knowledge, no one has interpreted (8) before as being Noether's invariance condition (3) in the particular case where no transformation of time is considered, i.e.  $t^s = t$ . Instead of (8), the method we propose here is based on the more rich set of identities (3)-(4).

There is a one-to-one correspondence between the minimizers of problem (5)-(6) and the minimizers of the integral functional

$$I[\tilde{x}(\cdot)] = \int_a^b L(t, \tilde{x}(t), \dot{\tilde{x}}(t)) dt$$

in the class of functions  $\tilde{x}(\cdot) \in PC^1$  satisfying the boundary conditions (7). Moreover, the transformation  $x = z(t, \tilde{x})$  and its inverse  $\tilde{x} = \tilde{z}(t, x)$  give us the desired correspondence.

**Corollary 3.2.** *For the validity of Theorem 3.1, the Lagrangian  $L(\cdot, \cdot, \cdot)$ , together with the transformation  $x = z(t, \tilde{x})$ , must be such that the left-hand side of the functional identity (8) is linear with respect to  $\dot{\tilde{x}}(t)$ .*

The main difficulty in applying Leitmann's method (Theorem 3.1) resides in finding the admissible transformations  $x = z(t, \tilde{x})$ . Leitmann has restricted himself to two situations for which it is easy to find the admissible transformations: (i) Corollary 3.2 is trivially satisfied if  $L(\cdot, \cdot, \cdot)$  is linear in its third argument; (ii) it can also be readily satisfied for  $L(\cdot, \cdot, \cdot)$  quadratic in its third argument, i.e. for  $L(\cdot, \cdot, \cdot)$  of the form

$$L(t, x, p) = a(t)p^2 + b(t, x)p + c(t, x), \quad (9)$$

with  $a(t) \neq 0$  for  $t \in [a, b]$ .

**Corollary 3.3.** *For a Lagrangian of type (9) the class of admissible transformations that satisfy Corollary 3.2 is of the form  $x = z(t, \tilde{x}) = \pm \tilde{x} + f(t)$ .*

**Remark 3.2.** Using our Remark 3.1 we can benefit of a well-developed theory Gouveia and Torres [4], Gouveia and Torres [5], Gouveia et al [6] on how to find Noether's invariance transformations, without the need to restrict ourselves to Lagrangians which are linear or quadratic in the velocity.<sup>1</sup> Therefore, the method we propose is applicable to a more wide class of optimization problems.

### 3.2. AN EXAMPLE

Let us apply Leitmann's method to the following simple problem of optimal control ( $a < b$ ):

$$\begin{aligned} I[u(\cdot)] &= \int_a^b (u(t))^2 dt \longrightarrow \min, \\ \dot{x}(t) &= u(t), \\ x(a) &= \alpha, \quad x(b) = \beta. \end{aligned} \quad (10)$$

This is a problem (1)-(2) with  $\varphi = u$ , so we can write (10) as a problem (5)-(6) of the calculus of variations:

$$I[x(\cdot)] = \int_a^b (\dot{x}(t))^2 dt \longrightarrow \min, \quad x(a) = \alpha, \quad x(b) = \beta.$$

The Lagrangian  $L(t, x, p) = p^2$  is of type (9), thus, by Corollary 3.3, the class of admissible transformations of Theorem 3.1 has the form  $x = z(t, \tilde{x}) =$

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<sup>1</sup>A computer algebra package to compute variational symmetries, by Paulo D. F. Gouveia and Delfim F. M. Torres, is available from the *Maple Application Center*: [http://www.maplesoft.com/applications/app\\_center\\_view.aspx?AID=1983](http://www.maplesoft.com/applications/app_center_view.aspx?AID=1983)

$\pm\tilde{x} + f(t)$ , where  $f(t)$  is some differentiable function. We consider, without loss of generality, the transformation  $x = z(t, \tilde{x}) = \tilde{x} + f(t)$ . Then,

$$L(t, f(t) + \tilde{x}, f'(t) + \tilde{p}) - L(t, \tilde{x}, \tilde{p}') = (f'(t))^2 + 2f'(t)\tilde{p},$$

and from the functional identity (8) we get

$$\frac{\partial G}{\partial t}(t, \tilde{x}) = (f'(t))^2, \quad \frac{\partial G}{\partial \tilde{x}}(t, \tilde{x}) = 2f'(t).$$

On the other hand,

$$\frac{\partial^2 G}{\partial \tilde{x} \partial t}(t, \tilde{x}) = \frac{\partial^2 G}{\partial t \partial \tilde{x}}(t, \tilde{x}),$$

and we conclude that

$$2f''(t) = 0, \tag{11}$$

that is,

$$f(t) = c_1 + c_2 t, \tag{12}$$

with  $c_1$  and  $c_2$  constants. We now determine function  $G(t, \tilde{x})$ . Substituting (12) into  $\frac{\partial G}{\partial \tilde{x}}(t, \tilde{x}) = 2f'(t)$  it follows that  $\frac{\partial G}{\partial \tilde{x}}(t, \tilde{x}) = 2c_2$ , and integrating with respect to  $\tilde{x}$  we arrive to

$$G(t, \tilde{x}) = \int (2c_2) d\tilde{x} = 2c_2 \tilde{x} + h(t),$$

where  $h(t)$  is still to be determined. For that, we differentiate the last expression with respect to  $t$  and compare the result with  $\frac{\partial G}{\partial t}(t, \tilde{x})$ :

$$\frac{\partial}{\partial t} (2c_2 \tilde{x} + h(t)) = h'(t),$$

and since  $\frac{\partial G}{\partial t}(t, \tilde{x}) = (c_2)^2$  we must have

$$h'(t) = (c_2)^2 \Leftrightarrow h(t) = (c_2)^2 t + c_3,$$

where  $c_3$  is an arbitrary constant. Therefore,  $G(t, \tilde{x}) = 2c_2 \tilde{x} + (c_2)^2 t + c_3$ . We have all the necessary ingredients to apply Theorem 3.1. We consider the trivial problem

$$I[\tilde{x}(\cdot)] = \int_a^b (\dot{\tilde{x}}(t))^2 dt \longrightarrow \min, \quad \tilde{x}(a) = 0, \quad \tilde{x}(b) = 0,$$

which admits the global minimizer  $\tilde{x}^*(t) \equiv 0$  (the original problem (10) is trivial when  $\alpha = \beta$ ; we are interested to solve (10) in the case  $\alpha \neq \beta$ ). To

obtain the solution of problem (10) we just need to choose  $c_1$  and  $c_2$  in (12) in such a way  $f(a) = \alpha$  and  $f(b) = \beta$ , i.e.

$$\begin{cases} f(a) = \alpha, \\ f(b) = \beta, \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 a = \alpha, \\ c_1 + c_2 b = \beta, \end{cases} \Leftrightarrow \begin{cases} c_1 = \frac{\beta a - b \alpha}{a - b}, \\ c_2 = \frac{\alpha - \beta}{a - b}. \end{cases}$$

The global minimizer for problem (10) is given by  $x^*(t) = \tilde{x}^*(t) + f(t) = 0 + c_1 + c_2 t$ :

$$x^*(t) = \frac{\beta a - b \alpha}{a - b} + \frac{\alpha - \beta}{a - b} t. \quad (13)$$

We remark that (13) satisfies (11), and that (11) is nothing more than the Euler-Lagrange equation of (10). However, the Euler-Lagrange equation only gives a candidate for local minimizer, i.e. we are not sure if the candidate is indeed a local minimizer. Leitmann's method has given much more: (13) is the global minimizer of (10). Next section gives an alternative direct optimization method, which we claim to be more broad in application.

#### 4. A NEW DIRECT OPTIMIZATION METHOD

Our direct optimization method is of simple comprehension and is applicable to a wider class of optimal control problems. We first show how it can be applied to problem (10).

##### 4.1. MOTIVATIONAL EXAMPLE

The initial step of our method is the determination of the parametric transformations under which the problem is invariant (see Definition 2.1). In respect to this, the techniques found in Gouveia and Torres [5], Torres [16] are useful.

**Proposition 4.1.** (see Torres [14], Example 1) *Problem (10) is invariant up to  $\Phi^s(t, x) = s^2 t + 2sx$ , in the sense of Definition 2.1, under the  $s$ -parameter transformations ( $s \in \mathbb{R}$ )*

$$t^s = t, \quad x^s = x + st, \quad u^s = u + s. \quad (14)$$

**Proof.** We begin by showing (3):

$$\begin{aligned} \tilde{I} &= \int_a^b (u^s(t))^2 dt = \int_a^b (u(t) + s)^2 dt = \int_a^b (u^2(t) + s^2 + 2su(t)) dt \\ &= \int_a^b u^2(t) dt + \int_a^b (s^2 + 2su(t)) dt = I + \int_a^b \frac{d}{dt} (s^2 t + 2sx(t)) dt \\ &= I + \Phi^s(b, \beta) - \Phi^s(a, \alpha). \end{aligned} \quad (15)$$

We remark that the minimizer of  $\tilde{I}[\cdot]$  coincide with the one of  $I[\cdot]$ :  $\Phi^s(a, \alpha)$  and  $\Phi^s(b, \beta)$  are constants and adding a constant in the functional does not

change the minimizer. It remains to prove the control invariance condition (4):

$$\frac{d}{dt}(x^s(t)) = \frac{d}{dt}(x(t) + st) = \dot{x}(t) + s = u(t) + s = u^s(t). \quad (16)$$

Equalities (15) and (16) prove that problem (10) is invariant under the one-parameter transformations (14) up to the gauge term  $\Phi^s$ .  $\square$

Using the invariance transformations (14) we generalize problem (10) to a parameter family of problems which include the original problem for  $s = 0$ : we substitute  $x(\cdot)$  and  $u(\cdot)$  in (10) respectively by  $x^s(\cdot)$  and  $u^s(\cdot)$ , obtaining

$$I^s[u^s(\cdot)] = \int_a^b (u^s(t))^2 dt \longrightarrow \min, \quad (17)$$

$$\dot{x}^s(t) = u^s(t),$$

$$x^s(a) = \alpha + sa, \quad x^s(b) = \beta + sb, \quad s \in \mathbb{R}.$$

Problem (10) is nontrivial for  $\alpha \neq \beta$ , but the crucial point is that there exists always a problem in the parameter family of problems (17), i.e. there exists always a specific value of  $s$ , which only depend on the concrete values of  $\alpha$ ,  $\beta$ ,  $a$  and  $b$ , admitting the trivial global minimizer  $u^s(t) = 0 \forall t \in [a, b]$ . The invariance properties asserted by Proposition 4.1 give the general solution to our original problem (10) from the trivial solution of this  $s$ -chosen problem.

**Proposition 4.2.** *Function (13) is a global minimizer of problem (10).*

**Proof.** It is clear that  $I^s \geq 0$  and that  $I^s = 0$  if  $u^s(t) \equiv 0$ . From the control system  $\dot{x}^s(t) = u^s(t)$ ,  $u^s(t) \equiv 0$  implies that  $x^s(a) = x^s(b)$ :

$$\alpha + sa = \beta + sb \Leftrightarrow s = \frac{\beta - \alpha}{a - b}.$$

Hence, the global minimizing trajectory of problem (17) for  $s = \frac{\beta - \alpha}{a - b}$  is given by

$$x^s(t) = \alpha + sa \Leftrightarrow x^s(t) = \frac{\beta a - \alpha b}{a - b}.$$

We solve (10) using the inverse functions of the variational symmetries (14):

$$\begin{cases} u(t) = u^s(t) - s, \\ x(t) = x^s(t) - st, \end{cases} \Leftrightarrow \begin{cases} u(t) = \frac{\alpha - \beta}{a - b}, \\ x(t) = \frac{\beta a - \alpha b}{a - b} - \frac{\beta - \alpha}{a - b}t. \end{cases}$$

We have just found the global minimizer (13) of problem (10).  $\square$

## 4.2. THE METHOD

As just illustrated, our direct optimization method permits to find global extremizers (minimizers or maximizers) of sufficiently rich invariant optimal control problems. The method consists of the following four steps:

- (1) Determine parameter invariant transformations  $t^s$ ,  $x_i^s$ , and  $u_j^s$ ,  $i = 1, \dots, n$ ,  $j = 1, \dots, m$ , under which the problem is invariant (cf. Definition 2.1). The results in Gouveia and Torres [5], Torres [16] are useful.
- (2) Applying the parameter transformations found in the previous step, write the generalized problem together with the generalized boundary conditions, i.e. substitute  $x_i(\cdot)$  and  $u_j(\cdot)$  respectively by  $x_i^s(\cdot)$  and  $u_j^s(\cdot)$ ,  $i = 1, \dots, n$  and  $j = 1, \dots, m$ .
- (3) Analyze the generalized problem and determine a specific value for the parameters for which it is easy to find a global optimal solution.
- (4) Define the inverse of the transformations  $t^s$ ,  $x^s$ , and  $u^s$ , for the particular choice of parameters  $s$  fixed on step (3), and obtain a global solution to the initial problem.

We shall now apply our simple method to the minimum fuel rendezvous problem of a constant-power rocket.

## 4.3. AN APPLICATION

Let us consider the problem of minimizing the amount of fuel consumed by a rocket operating at constant propulsive power. This is a classical problem of optimal control, “solved” by the Pontryagin Maximum Principle in most books (see e.g. Leitmann [8], Macki and Strauss [11]). We assume the following situation: (i) a positive prescribed transfer time  $\tau = t_1 - t_0$  is given; (ii) at the end (at time  $t_1$ ) the rocket car is to be at the origin with zero-velocity; (iii) the rocket is initially on the negative axis (at a given position  $-\alpha$ ,  $\alpha > 0$ ). Thus, we have:

$$\int_{t_0}^{t_1} u^2(t) dt \longrightarrow \min, \quad t_1 - t_0 = \tau,$$

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = u(t), \end{cases} \quad (18)$$

$$x_1(t_0) = -\alpha, \quad x_1(t_1) = 0, \quad x_2(t_1) = 0,$$

where  $t$  is the time variable,  $x_1$  the position,  $x_2$  the velocity, and  $u$  is the acceleration due to the thrust. We are assuming that the thrust-acceleration is not constrained, i.e.  $|u(t)| < \infty$ , and that  $\tau > 0$  and  $\alpha > 0$  are given. The thrust-acceleration program that results in the minimum fuel consumption can also be obtained by Leitmann’s method (cf. Leitmann [9], Section 9) but the analysis is enough-complex: it is not easy to guess functions  $f_1(t)$  and

$f_2(t)$  of Corollary 3.3, associated respectively with  $x_1(t)$  and  $x_2(t)$ . Here we show that there exists a simple way to obtain a global minimizer to problem (18).

**Proposition 4.3.** *A global minimizer of problem (18) is given by*

$$\begin{aligned} x_1(t) &= -\frac{\alpha}{\tau^2}t^2 + \frac{2\alpha}{\tau}\left(\frac{t_0}{\tau} + 1\right)t - \frac{2\alpha}{\tau}\left(\frac{t_0}{2\tau} + 1\right)t_0 - \alpha, \\ x_2(t) &= -\frac{2\alpha}{\tau^2}t + \frac{2\alpha}{\tau}\left(\frac{t_0}{\tau} + 1\right), \\ u(t) &= -\frac{2\alpha}{\tau^2}. \end{aligned} \tag{19}$$

**Proof.** We follow the four-step method of Section 4.2.

(1) Problem (18) is invariant under the parameter transformations ( $s \in \mathbb{R}$ )

$$t^s = t, \quad x_1^s(t) = x_1(t) + \frac{s^2}{2}t^2, \quad x_2^s(t) = x_2(t) + s^2t, \quad u^s(t) = u(t) + s^2, \tag{20}$$

up to  $\Phi^s(t, x_2) = s^4t + 2s^2x_2$ : the functional is invariant,

$$\begin{aligned} \int_{t_0}^{t_1} (u^s(t))^2 dt &= \int_{t_0}^{t_1} (u(t) + s^2)^2 dt = \int_{t_0}^{t_1} (u^2(t) + s^4 + 2s^2u(t)) dt \\ &= \int_{t_0}^{t_1} u^2(t)dt + \int_{t_0}^{t_1} (s^4 + 2s^2u(t)) dt \\ &= \int_{t_0}^{t_1} u^2(t)dt + \int_{t_0}^{t_1} \frac{d}{dt} (s^4t + 2s^2x_2(t)) dt; \end{aligned}$$

as well as the control system,

$$\begin{cases} \frac{d}{dt} (x_1^s(t)) = \frac{d}{dt} \left( x_1(t) + \frac{s^2}{2}t^2 \right) = \dot{x}_1(t) + s^2t = x_2(t) + s^2t = x_2^s(t), \\ \frac{d}{dt} (x_2^s(t)) = \frac{d}{dt} (x_2(t) + s^2t) = \dot{x}_2(t) + s^2 = u(t) + s^2 = u^s(t). \end{cases}$$

(2) The generalized problem takes the following form:

$$\begin{aligned} I^s[u^s(\cdot)] &= \int_{t_0}^{t_1} (u^s(t))^2 dt \longrightarrow \min, \\ \begin{cases} \dot{x}_1^s(t) = x_2^s(t), \\ \dot{x}_2^s(t) = u^s(t), \end{cases} & \tag{21} \\ x_1^s(t_0) = \frac{s^2t_0^2}{2} - \alpha, \quad x_1^s(t_1) = \frac{s^2t_1^2}{2}, \quad x_2^s(t_1) = s^2t_1. \end{aligned}$$

For  $s = 0$  problem (21) reduces to (18).

(3)  $I^s \geq 0 \forall u^s(\cdot)$ , and  $I^s = 0$  if  $u^s(t) \equiv 0$ . From the control system we have for  $u^s(t) \equiv 0$  that  $x_2^s(t) = c_1$ , and  $x_1^s(t) = c_1 t + c_2$ , where  $c_1$  and  $c_2$  are constants. From the generalized boundary condition  $x_2^s(t_1) = s^2 t_1$ , it follows that  $c_1 = s^2 t_1$ . Then,  $x_2^s(t) = s^2 t_1$ ,  $x_1^s(t) = s^2 t_1 t + c_2$ . Using the boundary conditions for  $x_1^s(\cdot)$  we arrive to  $c_2 = -\frac{s^2 t_1^2}{2}$  and  $s^2 = \frac{2\alpha}{\tau^2}$ . Therefore, a global minimizer to problem (21) with  $s = \pm \frac{\sqrt{2\alpha}}{\tau}$  is given by

$$x_1^s(t) = \frac{2\alpha}{\tau^2} t_1 t - \frac{\alpha t_1^2}{\tau^2}, \quad x_2^s(t) = \frac{2\alpha}{\tau^2} t_1, \quad u^s(t) = 0, \quad t \in [t_0, t_1].$$

(4) The global solution to problem (18) is obtained using the inverses of transformations (20) for  $s^2 = \frac{2\alpha}{\tau^2}$ :

$$\begin{aligned} x_1(t) &= x_1^s(t) - \frac{s^2}{2} t^2 = \frac{2\alpha}{\tau^2} t_1 t - \frac{\alpha t_1^2}{\tau^2} - \frac{\alpha}{\tau^2} t^2, \\ x_2(t) &= x_2^s(t) - s^2 t = \frac{2\alpha}{\tau^2} t_1 - \frac{2\alpha}{\tau^2} t, \\ u(t) &= u^s(t) - s^2 = -\frac{2\alpha}{\tau^2}. \end{aligned} \tag{22}$$

It is a simple exercise to see that (19) and (22) are equivalent.  $\square$

## 5. CONCLUSIONS

In the calculus of variations, as well as in the more general setting of optimal control, the problem of minimizing an integral functional is the main issue, in general a difficult one. The standard way to attack such problems relies on necessary optimality conditions, which give candidates for a local minimum. A direct method for addressing some problems of the calculus of variations which are linear or quadratic in velocity (control) was introduced by Leitmann and further improved by Carlson, providing global minimizers directly, without using necessary conditions. Here we propose a different, simpler, and more wide applicable direct method for problems of optimal control: (i) different because instead of using transformations which keep the problem invariant in Carathéodory's sense, as in the method of Leitmann-Carlson, our method is based on transformations which keep the problems invariant in Noether's sense; (ii) simpler in finding the admissible transformations; (iii) more general because it easily covers Lagrangians which are not linear or quadratic in the control variables.

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