Communications in Applied Analysis **10** (2006), no. 4, 551–562

A POROSITY RESULT FOR VARIATIONAL PROBLEMS ARISING IN CRYSTALLOGRAPHY

Alexander J. Zaslavski
Department of Mathematics
Technion – Israel Institute of Technology
Haifa, 32000, Israel
ajzasl@tx.technion.ac.il

Communicated by S. Reich

ABSTRACT: In this paper we study the structure of minimizers of variational problems which describe step-terraces on surfaces of crystals.

AMS (MOS) Subject Classification: 49J99, 74E15

1. INTRODUCTION

In this paper we study the structure of minimizers of variational problems considered in Hannon et al [6], Hannon te al [7], Jeng and Williams [8], Mizel and Zaslavski [9] which describe step-terraces on surfaces of crystals. It is well-known in surface physics that when a crystalline substance is maintained at a temperature T above its *roughening temperature* T_R then the surface stored energy integrand, usually referred to as *surface tension*, is a smooth function β of the azimuthal angle of orientation θ. Furthermore, β obeys the following:

$$\beta(-\theta) = \beta(\pi - \theta) = \beta(\theta),\ 0 < \beta(\pi/2) \leq \beta(\theta) \leq \beta(0)$$

(cf. for example Dacorogna and Pfister [2], Fonseca [5], Jeng and Williams [8]). The classical model studied in Mizel and Zaslavski [9] and in the prsent paper is given by

$$J(y) = \int_0^S \beta(\theta) ds\,,$$

where s is arclength and y is a function defined on a fixed interval $[0, L]$ whose graph is the locus under consideration:

$$y \in W^{1,1}(0, L),\ \theta = \arctan y' \in [-\pi/2, \pi/2],$$

Received August 28, 2006

while β is a positive π-periodic function which belongs to a space of functions described below. Minimization of J subject to appropriate boundary data is a parametric variational problem. It is closely related to the variational problem defining the Wulff crystal shape as that shape for a domain of prescribed area such that the boundary integral with respect to arclength involving the integrand in J (referred to as the surface tension) attains its minimum value Dacorogna and Pfister [2], Fonseca [5].

For each function $f : X \to R$ set $\inf(f) = \inf\{f(x) : \ x \in X\}$.

Denote by $\mathcal{M}$ the set of all functions $\beta \in C^2(R)$ which satisfy the following assumption:

(A)

$$\beta(t) \geq 0 \text{ for all } t \in R, \tag{1.1}$$

$$\beta(\pi/2) \leq \beta(t) \leq \beta(0) \text{ for all } t \in R, \tag{1.2}$$

$$\beta(t) = \beta(-t) \text{ for all } t \in R, \tag{1.3}$$

$$\beta(t + \pi) = \beta(t) \text{ for all } t \in R, \tag{1.4}$$

$$\beta(0) + \beta''(0) \leq 0. \tag{1.5}$$

For each $\beta_1, \beta_2 \in \mathcal{M}$ set

$$\rho(\beta_1, \beta_2) = \sup\{|\beta_1^{(i)}(t) - \beta_2^{(i)}(t)| : \ t \in R, \ i = 0, 1, 2\}. \tag{1.6}$$

It is not difficult to see that the metric space $(\mathcal{M}, \rho)$ is complete.

Denote by $\mathcal{M}_r$ the set of all $\beta \in \mathcal{M}$ such that

$$\beta(t) > 0 \text{ for all } t \in R, \tag{1.7}$$

$$\beta(0) + \beta''(0) < 0. \tag{1.8}$$

It was shown in Mizel and Zaslavski [9], Proposition 1.1 that $\mathcal{M}_r$ is an open everywhere dense subset of $(\mathcal{M}, \rho)$. Let $\beta \in \mathcal{M}$. Define

$$G_\beta(z) = \beta(\arctan(z))(1 + z^2)^{1/2}, \ z \in R. \tag{1.9}$$

Clearly G_β is a continuous function and if $\beta \in \mathcal{M}_r$, then

$$G_\beta(z) \to \infty \text{ as } z \to \pm\infty. \tag{1.10}$$

Remark 1.1. Note that if $\beta \in \mathcal{M}_r$, then $\inf(G_\beta) < \beta(0)$ (see Hannon te al [7]).

We can rewrite the variational functional J in the form

$$J(y) = \int_0^L G_\beta(y')dx.$$

It was shown in Hannon te al [7] that $y \in W^{1,1}(0, L)$ is a minimizer of J if and only if

$$|y'| \in \{z \in R : \ G_\beta(z) = \inf(G_\beta)\} \text{ a.e..}$$

In Mizel and Zaslavski [9] we showed that for a generic function β the set

$$\{z \in R : \ G_\beta(z) = \inf(G_\beta)\} = \{z_\beta, -z_\beta\},$$

where z_β is a unique positive number depending only on β.

More precisely, denote by $\mathcal{F}$ the set of all $\beta \in \mathcal{M}_r$ which satisfy the following condition:

(C) There is $z_\beta \in R$ such that

$$G_\beta(z) > G_\beta(z_\beta) \text{ for all } z \in R \setminus \{z_\beta, -z_\beta\}. \tag{1.11}$$

In Mizel and Zaslavski [9], Theorem 1.1 we showed that $\mathcal{F}$ is a countable intersection of open everywhere dense subsets of $(\mathcal{M}, \rho)$. In that paper, instead of considering condition (C) for a single function $\beta \in \mathcal{M}$, we investigated it for the whole space $\mathcal{M}$ and showed that this condition held for most of the functions in $\mathcal{M}$. This approach has also been successfully applied in many areas of Analysis. In the present paper we show that the complement of the set $\mathcal{F}$ is not only of the first category, but also σ-porous in the space $(\mathcal{M}, \rho)$. Before we continue we recall the concept of porosity Benyamini and Lindenstrauss [1], De Blasi and Myjak [3], De Blasi et al [4], Zaslavski [10].

Let (Y, d) be a complete metric space. For $x \in Y$, $r > 0$ set

$$B_d(x, r) = \{y \in Y : \ d(x, y) \le r\}.$$

A set $E \subset Y$ is a porous subset of (Y, d) if there exist $\alpha \in (0, 1)$, $r_0 > 0$ such that for each $x \in Y$, each $r \in (0, r_0]$ there exists $y \in Y$ such that

$$B_d(y, \alpha r) \subset B_d(x, r) \setminus E.$$

A subset of (Y, d) is called a σ-porous subset of (Y, d) if it is a countable union of porous subsets of (Y, d).

Other notions of porosity have been used in the literature Benyamini and Lindenstrauss [1]. We use the rather strong notion which appears in De Blasi and Myjak [3], De Blasi et al [4], Zaslavski [10].

Since porous sets are nowhere dense, all σ-porous sets are of the first category. If Y is a finite-dimensional Euclidean space, then σ-porous sets are of Lebesgue measure 0. In fact, the class of σ-porous sets in such a space is much smaller than the class of sets which have measure 0 and are of the first category.

To point out the difference between porous and nowhere dense sets note that if $E \subset Y$ is nowhere dense, $y \in Y$ and $r > 0$, then there is a point $z \in Y$ and a number $s > 0$ such that $B_d(z, s) \subset B_d(y, r) \setminus E$. If, however, E is also porous, then for small enough r we can choose $s = \alpha r$, where $\alpha \in (0, 1)$ is a constant which depends only on E.

The following theorem is our main result.

Theorem 1.1. *$\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, \rho)$.*

Theorem 1.1 will be proved in Section 4. In Section 2 we will show that $\mathcal{M} \setminus \mathcal{M}_r$ is a porous subset of $(\mathcal{M}, \rho)$. Section 3 contains auxuliary results.

For each $\psi \in C^2(R)$ set

$$||\psi||_{C^2} = \sup\{|\psi^{(i)}(t)| : \ t \in R, \ i = 0, 1, 2\}. \tag{1.12}$$

2. THE SET $\mathcal{M} \setminus \mathcal{M}_r$ IS POROUS

Proposition 2.1. *$\mathcal{M} \setminus \mathcal{M}_r$ is a porous subset of $(\mathcal{M}, \rho)$.*

Proof. Consider the function

$$\tilde{\beta}(t) = \cos(2t) + 3/2, \ t \in R.$$

Clearly $\tilde{\beta} \in \mathcal{M}_r$,

$$\tilde{\beta}(t) \geq 1/2 \text{ for all } t \in R, \tag{2.1}$$

$$\tilde{\beta}(0) + \tilde{\beta}''(0) < -1, \tag{2.2}$$

$$||\tilde{\beta}||_{C^2} = 4. \tag{2.3}$$

Set

$$\alpha = 1/32. \tag{2.4}$$

Assume that

$$\beta \in \mathcal{M}, \ r \in (0, 1]. \tag{2.5}$$

Define

$$\beta_1(t) = \beta(t) + 8^{-1} r \tilde{\beta}(t), \quad t \in R. \tag{2.6}$$

Evidently $\beta_1 \in \mathcal{M}$. In view of (2.6) and (2.3)

$$\rho(\beta, \beta_1) = 8^{-1} r ||\tilde{\beta}||_{C^2} = 2^{-1} r. \tag{2.7}$$

It follows from (2.6), (2.5), (1.1) and (2.1) that for all $t \in R$

$$\beta_1(t) \geq 8^{-1} r \tilde{\beta}(t) \geq 16^{-1} r. \tag{2.8}$$

By (2.6), (2.5), (1.5) and (2.2)

$$\beta_1(0) + \beta_1''(0) = \beta(0) + \beta''(0) + 8^{-1} r (\tilde{\beta}(0) + \tilde{\beta}''(0))$$
$$\leq 8^{-1} r (\tilde{\beta}(0) + \tilde{\beta}''(0)) < -8^{-1} r. \tag{2.9}$$

Assume that

$$\phi \in \mathcal{M}, \ \rho(\phi, \beta_1) \leq \alpha r = r/32. \tag{2.10}$$

Relations (2.10) and (2.7) imply that

$$\rho(\phi, \beta) \leq \rho(\phi, \beta_1) + \rho(\beta_1, \beta) \leq r/32 + r/2 < r. \tag{2.11}$$

In view of (2.10), (1.6) and (2.8) for all $t \in R$

$$\phi(t) \geq \beta_1(t) - \rho(\phi, \beta_1) \geq \beta_1(t) - r/32 \geq r/16 - 32^{-1} r = 32^{-1} r. \tag{2.12}$$

By (1.6), (2.10) and (2.9)

$$\phi(0) + \phi''(0) \leq \beta_1(0) + \beta_1''(0) + 2\rho(\phi, \beta_1)$$
$$\leq -8^{-1} r + 2\rho(\phi, \beta_1) \leq -8^{-1} r + r/16 = -r/16. \tag{2.13}$$

It follows from (2.10), (2.12) and (2.13) that $\phi \in \mathcal{M}_r$. Combined with (2.11) this inclusion implies that

$$\{\phi \in \mathcal{M} : \ \rho(\phi, \beta_1) \leq \alpha r\} \subset \{\phi \in \mathcal{M} : \ \rho(\phi, \beta) \leq r\} \cap \mathcal{M}_r.$$

Proposition 2.1 is proved. □

3. AUXILIARY RESULTS

Let n be a natural number. Set

$$\Omega_n = \{z \in R : \ 1/n \leq |z| \leq n\}. \tag{3.1}$$

Denote by $\mathcal{F}_n$ the set of all $\beta \in \mathcal{M}$ which satisfy the following condition:

(C1) There is $z_{\beta n} \in \Omega_n$ such that

$$G_\beta(z) > G_\beta(z_{\beta n})$$

for all

$$z \in \Omega_n \setminus \{z_{\beta n}, -z_{\beta n}\}.$$

Proposition 3.1. $\mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n) \subset \mathcal{F}$.

Proof. Assume that

$$f \in \mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n).$$

Since $\beta \in \mathcal{M}_r$ we have

$$\lim_{|z| \to \infty} G_\beta(z) = \infty, \ \inf(G_\beta) < \beta(0) = G_\beta(0). \tag{3.2}$$

(see (1.10) and Remark 1.1).

By (3.2) there are a natural number k and $\delta > 0$ such that

$$G_\beta(z) \geq G_\beta(0) + 4 \text{ for all } z \in R \text{ satisfying } |z| \geq k, \tag{3.3}$$

$$G_\beta(z) > \inf(G_\beta) + \delta \text{ for all } z \in [-1/k, 1/k]. \tag{3.4}$$

Since $\beta \in \mathcal{F}_k$ it follows from condition (C1) that there is $z_k \in \Omega_k$ such that

$$G_\beta(z) > G_\beta(z_k) \text{ for all } z \in \Omega_k \setminus \{z_k, -z_k\}. \tag{3.5}$$

Since the function G_β is continuous it follows from (3.2) that G has a point of minimum. Let $z \in R$ satisfy

$$G_\beta(z) = \inf(G_\beta). \tag{3.6}$$

In view of (3.3), (3.4) and (3.1)

$$1/k \leq |z| \leq k \text{ and } z \in \Omega_k.$$

Combined with (3.6) and (3.5) this implies that $z \in \{z_k, -z_k\}$. Therefore $\beta \in \mathcal{F}$. Proposition 3.1 is proved. □

Let n, i be natural numbers. Denote by $\mathcal{F}_{ni}$ the set of all $\beta \in \mathcal{M}$ which satisfy the following condition:

(C2) There are $\delta > 0$, $z_* \in \Omega_n$ such that for each $z \in \Omega_n$ satisfying

$$G_\beta(z) \leq \inf\{G_\beta(x) : \ x \in \Omega_n\} + \delta\,,$$

the following inequality is valid:

$$\min\{|z - z_*|, |z + z_*|\} \leq 1/i.$$

Proposition 3.2. *Let n be a natural number. Then $\cap_{i=1}^{\infty} \mathcal{F}_{ni} \subset \mathcal{F}_n$.*

Proof. Let $\beta \in \cap_{i=1}^{\infty} \mathcal{F}_{ni}$. By condition (C2) for each natural number i there exist

$$z_i \in \Omega_n, \ \delta_i > 0 \tag{3.7}$$

such that the following property holds:

(C3) If $z \in \Omega_n$ satisfies

$$G_\beta(z) \le \inf\{G_\beta(y) : \ y \in \Omega_n\} + \delta_i,$$

then

$$\min\{|z - z_i|, |z + z_i|\} \le 1/i.$$

We may assume without loss of generality that

$$z_i \ge 0 \text{ for all natural numbers } i. \tag{3.8}$$

Let

$$z \in \Omega_n, \ G_\beta(z) = \inf\{G_\beta(x) : \ x \in \Omega_n\}. \tag{3.9}$$

By (3.9) and property (C3) for each integer $i \ge 1$

$$\min\{|z - z_i|, |z + z_i|\} \le 1/i. \tag{3.10}$$

If $z = 0$, then by (3.10) $\lim_{i\to\infty} z_i = 0$. If $z > 0$, then (3.8) and (3.10) imply that $z = \lim_{i\to\infty} z_i$. If $z < 0$, then in view of (3.8) and (3.10) $z = -\lim_{i\to\infty} z_i$. We conclude that there exists $\lim_{i\to\infty} z_i$ and if a number z satisfies (3.9), then $z \in \{\lim_{i\to\infty} z_i, -\lim_{i\to\infty} z_i\}$. This implies that $\beta \in \mathcal{F}_n$. Proposition 3.2 is proved. □

4. PROOF OF THEOREM 1.1

We preface the proof of Theorem 1.1 by the following auxiliary result.

Proposition 4.1. *Let n, i be integers. Then the set $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$.*

Proof. There exists $\psi \in C^\infty(R)$ such that

$$0 \le \psi(t) \le 1 \text{ for all } t \in R, \ \psi(t) = 0 \text{ if } |t| \ge 1,$$
$$\psi(t) = 1 \text{ if } |t| \le 1/2. \tag{4.1}$$

Set

$$\psi_1(t) = (1 - t^2)\psi(t), \ t \in R. \tag{4.2}$$

Clearly $\psi_1 \in C^\infty(R)$,

$$0 \le \psi_1(t) \le 1 \text{ for all } t \in R, \ \psi_1(t) = 0 \text{ if } |t| \ge 1,$$
$$\psi_1(t) = (1 - t^2) \text{ if } |t| \le 1/2, \ \psi_1(t) < 1 \text{ for each } t \in R \setminus \{0\}. \tag{4.3}$$

Choose a positive number c_0 such that

$$c_0 > \max\{8n^2 i, \ 2(\arctan(1/n))^{-1}, \ 2(\arctan(n+1) - \arctan(n))^{-1}\}. \tag{4.4}$$

Set

$$\Delta = \inf\{\cos(2t) - \cos(\pi) : \ t \in [0, \arctan(n+1)]\}. \tag{4.5}$$

Evidently

$$\Delta > 0. \tag{4.6}$$

Choose a positive number α such that

$$\alpha \leq 8^{-1} \min\{(4c_0^2||\psi_1||_{C^2})^{-1},\ 64^{-1}\Delta\}n^{-1}. \tag{4.7}$$

Assume that

$$\beta \in \mathcal{M},\ r \in (0, 1]. \tag{4.8}$$

Consider the function

$$\tilde{\beta}(t) = \cos(2t) + 3/2,\ t \in R. \tag{4.9}$$

It is easy to see that $\tilde{\beta} \in \mathcal{M}_r$,

$$\tilde{\beta}(t) \geq 1/2 \text{ for all } t \in R, \tag{4.10}$$

$$\tilde{\beta}(0) + \tilde{\beta}''(0) < -1. \tag{4.11}$$

Define

$$\beta_1(t) = \beta(t) + (32)^{-1}r\tilde{\beta}(t),\ t \in R. \tag{4.12}$$

Clearly $\beta_1 \in \mathcal{M}$,

$$\rho(\beta, \beta_1) = (32^{-1})r||\tilde{\beta}||_{C^2} = 8^{-1}r. \tag{4.13}$$

It follows from (4.12), (4.8), (1.1) and (4.10) that for each $t \in R$

$$\beta_1(t) \geq (32)^{-1}r\tilde{\beta}(t) \geq (64)^{-1}r. \tag{4.14}$$

In view of (4.12), (1.5), (4.8) and (4.11)

$$\begin{aligned}\beta_1(0) + \beta_1''(0) &= \beta(0) + \beta''(0) + (32^{-1})r[\tilde{\beta}(0) + \tilde{\beta}''(0)] \\ &\leq (32)^{-1}r[\tilde{\beta}(0) + \tilde{\beta}''(0)] = -32^{-1}r.\end{aligned} \tag{4.15}$$

There is

$$\bar{z} \in \Omega_n \tag{4.16}$$

such that

$$G_{\beta_1}(\bar{z}) = \inf\{G_{\beta_1}(z) :\ z \in \Omega_n\}. \tag{4.17}$$

We may assume that

$$\bar{z} > 0. \tag{4.18}$$

Set

$$\bar{\theta} = \arctan(\bar{z}). \tag{4.19}$$

By (4.19), (4.18), (4.16) and (3.1)

$$\bar{\theta} \in [\ \arctan(1/n), \arctan(n)]. \tag{4.20}$$

Set

$$c_r = r \min\{(4c_0^2||\psi_1||_{C^2})^{-1}, 64^{-1}\Delta\} \tag{4.21}$$

and define

$$\psi_2(t) = c_r(1 - \psi_1(c_0(t - \bar{\theta}))),\ t \in R. \tag{4.22}$$

Clearly $\psi_2 \in C^\infty(R)$. Relations (4.22) and (4.3) imply that

$$0 \leq \psi_2(t) \leq c_r \text{ for all } t \in R, \tag{4.23}$$

$$\psi_2(t) = c_r \text{ if } |t-\bar\theta| \geq c_0^{-1}, \tag{4.24}$$

$$\psi_2(\bar\theta) = 0, \tag{4.25}$$

$$\psi_2(t) > 0 \text{ for each } t \in R \setminus \{\bar\theta\}. \tag{4.26}$$

It follows from (4.20) and (4.4) that

$$\begin{aligned}
&\{t \in [0,\pi/2] :\ |t-\bar\theta| \geq c_0^{-1}\} \\
&= [0,\pi/2] \cap ((-\infty, \bar\theta - c_0^{-1}] \cup [\bar\theta + c_0^{-1}, \infty)) \\
&\supset [0,\pi/2] \cap ((-\infty,\ \arctan(1/n) - c_0^{-1}] \cup [\arctan(n) + c_0^{-1}, \infty)) \\
&\supset [0,\pi/2] \cap ((-\infty, 2^{-1}\ \arctan(1/n)] \cup [\arctan(n+1), \infty)) \\
&= [0, 2^{-1}\arctan(1/n)] \cup [\arctan(n+1), \pi/2].
\end{aligned} \tag{4.27}$$

In view of (4.27) and (4.24)

$$\psi_2(t) = c_r \text{ for each } t \in [0, 2^{-1}\arctan(1/n)] \cup [\ \arctan(n+1), \pi/2]. \tag{4.28}$$

By (4.28) there exists a function $\psi_3 : R \to R$ such that

$$\psi_3(t) = \psi_2(t),\ t \in [0,\pi/2],$$

$$\psi_3(-t) = \psi_3(t+\pi) = \psi_3(t) \text{ for all } t \in R. \tag{4.29}$$

Clearly $\psi_3 \in C^\infty(R)$. In view of (4.29), (4.23), (4.25) and (4.26)

$$0 \leq \psi_3(t) \leq c_r \text{ for all } t \in R, \tag{4.30}$$

$$\psi_3(\bar\theta) = 0,$$

$$\psi_3(t) > 0 \text{ for all } t \in [0,\pi/2] \setminus \{\bar\theta\}, \tag{4.31}$$

$$\psi_3(t) > 0 \text{ for all } t \in [-\pi/2, 0] \setminus \{-\bar\theta\}.$$

Set

$$\phi(t) = \beta_1(t) + \psi_3(t),\ t \in R. \tag{4.32}$$

Evidently $\phi \in C^2(R)$. It follows from the inclusion $\beta_1 \in \mathcal{M}$, (4.32), (4.30) and (4.29) that

$$\phi(t) \geq 0 \text{ for all } t \in R,\ \phi(t) = \phi(-t) = \phi(t+\pi) \text{ for all } t \in R. \tag{4.33}$$

By (4.29) and (4.28)

$$\psi_3(0) = \psi_2(0) = c_r,\ \psi_3''(0) = \psi_2''(0) = 0, \tag{4.34}$$

$$\psi_3(\pi/2) = \psi_2(\pi/2) = c_r.$$

It follows from (4.32), (4.34), (4.15), (4.21) and (4.5) that

$$\begin{aligned}
\phi(0) + \phi''(0) &= \beta_1(0) + \beta_1''(0) + \psi_3(0) + \psi_3''(0) \\
&= \beta_1(0) + \beta_1''(0) + c_r \leq -32^{-1}r + c_r \leq 64^{-1}r.
\end{aligned} \tag{4.35}$$

Now we show that for all $t \in R$

$$\phi(\pi/2) \leq \phi(t) \leq \phi(0).$$

In view of (4.33) it is sufficient to show that this inequality holds for all $t \in [0,\pi/2]$.

Let $t \in [0, \pi/2]$. By (4.32), (4.30), (1.2) and (4.34)

$$\phi(t) = \beta_1(t) + \psi_3(t) \le \beta_1(t) + c_r \le \beta_1(0) + c_r = \beta_1(0) + \psi_3(0) = \phi(0). \quad (4.36)$$

Let us show that $\phi(t) \ge \phi(\pi/2)$. There are two cases:

$$|t - \bar{\theta}| \ge c_0^{-1} \quad (4.37)$$

and

$$|t - \bar{\theta}| < c_0^{-1}. \quad (4.38)$$

Assume that (4.37) holds. By (4.37), (4.29) and (4.24)

$$\psi_3(t) = \psi_2(t) = c_r.$$

Combined with (4.32), (1.2) and (4.34) this equality implies that

$$\phi(t) = \beta_1(t) + \psi_3(t) = \beta_1(t) + c_r \ge \beta_1(\pi/2) + c_r = \beta_1(\pi/2)$$
$$+\psi_3(\pi/2) = \phi(\pi/2)$$

and

$$\phi(t) \ge \phi(\pi/2).$$

Assume that (4.38) holds. It follows from (4.38), (4.20) and (4.4) that

$$2^{-1}\arctan(1/n) \le \arctan(1/n) - c_0^{-1} \le \bar{\theta} - c_0^{-1} \le t \le \bar{\theta} + c_0^{-1}$$
$$\le \arctan(n) + c_0^{-1} \le \arctan(n+1). \quad (4.39)$$

In view of (4.12), (1.2), (4.39), (4.9), (4.6) and (4.5)

$$\beta_1(t) - \beta_1(\pi/2) = \beta(t) - \beta(\pi/2) + 32^{-1}r(\tilde{\beta}(t) - \tilde{\beta}(\pi/2))$$
$$\ge 32^{-1}r(\tilde{\beta}(t) - \tilde{\beta}(\pi/2)) = 32^{-1}r(\cos(2t) - \cos(\pi)) \ge 32^{-1}r\Delta. \quad (4.40)$$

It follows from (4.32), (4.30), (4.40), (4.21) and (4.34) that

$$\phi(t) = \beta_1(t) + \psi_3(t) \ge \beta_1(t) \ge \beta_1(\pi/2) + 32^{-1}r\Delta$$
$$\ge \beta_1(\pi/2) + c_r = \beta_1(\pi/2) + \psi_3(\pi/2) = \phi(\pi/2).$$

Thus in both cases $\phi(t) \ge \phi(\pi/2)$. Together with (4.36) this implies that

$$\phi(\pi/2) \le \phi(t) \le \phi(0). \quad (4.41)$$

We proved (4.41) for all $t \in [0, \pi/2]$. In view of (4.33) inequality (4.41) is valid for all $t \in R$. Combined with (4.35) and (4.33) inequality (4.41) implies that $\phi \in \mathcal{M}$. It follows from (4.13), (1.6), (4.32) ,(4.29), (1.12), (4.22), (4.3) and (4.4) that

$$\rho(\beta, \phi) \le \rho(\beta, \beta_1) + \rho(\beta_1, \phi) \le 8^{-1}r + \rho(\beta_1, \phi)$$
$$\le 8^{-1}r + ||\psi_3||_{C^2} \le 8^{-1}r + ||\psi_2||_{C^2}$$
$$\le 8^{-1}r + c_r \max\{1, c_0||\psi_1||_{C^2}, c_0^2||\psi_1||_{C^2}\} \le 8^{-1}r + c_r c_0^2|\psi_1||_{C^2}.$$

Combined with (4.21) this inequality implies that

$$\rho(\beta, \phi) \le r/8 + c_r c_0^2||\psi_1||_{C^2}$$

$$\leq 8^{-1}r + r(4c_0^2||\psi_1||_{C^2})^{-1}c_0^2||\psi_1||_{C^2} \leq r/8 + r/4 \leq r/2. \tag{4.42}$$

Relations (4.32), (4.30) and (1.9) imply that

$$\phi(t) \geq \beta_1(t) \text{ for all } t \in R$$

and

$$G_\phi(t) \geq G_{\beta_1}(t) \text{ for all } t \in R. \tag{4.43}$$

Now assume that

$$h \in \mathcal{M},\ \rho(h, \phi) \leq \alpha r, \tag{4.44}$$

$$z \in \Omega_n,\ G_h(z) \leq \inf\{G_h(y) :\ y \in \Omega_n\} + \alpha r n. \tag{4.45}$$

By (4.42), (4.44) and (4.7)

$$\rho(h, \beta) \leq \rho(h, \phi) + \rho(\phi, \beta) \leq \alpha r + r/2 \leq r/8 + r/2 \leq 3r/4. \tag{4.46}$$

It follows from (1.9), (1.6), (3.1), (4.44) that for each $y \in \Omega_n$

$$|G_\phi(y) - G_h(y)| = |\phi(\arctan(y))(1 + y^2)^{1/2} - h(\arctan(y))(1 + y^2)^{1/2}|$$

$$\leq (1 + y^2)^{1/2}\rho(h, \phi) \leq (1 + n^2)^{1/2}\rho(h, \phi) \leq 2n\alpha r. \tag{4.47}$$

In particular

$$|G_\phi(z) - G_h(z)| \leq 2n\alpha r. \tag{4.48}$$

By (4.47)

$$|\inf\{G_\phi(y) :\ y \in \Omega_n\} - \inf\{G_h(y) :\ y \in \Omega_n\}| \leq 2n\alpha r. \tag{4.49}$$

In view of (4.48), (4.45) and (4.49)

$$G_\phi(z) \leq G_h(z) + 2n\alpha r \leq \inf\{G_h(y) :\ y \in \Omega_n\} + 3\alpha r n$$

$$\leq \inf\{G_\phi(y) :\ y \in \Omega_n\} + 5\alpha r n. \tag{4.50}$$

By (4.50), (1.9), (4.19), (4.32) and (4.30)

$$G_\phi(z) \leq 5\alpha r n + G_\phi(\bar{z}) = 5\alpha r n + \phi(\arctan(\bar{z}))(1 + \bar{z}^2)^{1/2}$$

$$= 5\alpha r n + \phi(\bar{\theta})(1 + \bar{z}^2)^{1/2} = 5\alpha r n + (1 + \bar{z}^2)^{1/2}(\beta_1(\bar{\theta}) + \psi_3(\bar{\theta}))$$

$$= 5\alpha r n + (1 + \bar{z}^2)^{1/2}\beta_1(\bar{\theta})$$

$$= 5\alpha r n + (1 + \bar{z}^2)^{1/2}\beta_1(\arctan(\bar{z})) = 5\alpha r n + G_{\beta_1}(\bar{z}). \tag{4.51}$$

It follows from (1.9), (4.32), (4.29), (4.17) and (4.45) that

$$G_\phi(z) = \phi(\arctan(z))(1 + z^2)^{1/2}$$

$$= \beta_1(\arctan(z))(1 + z^2)^{1/2} + \psi_3(\arctan(z))(1 + z^2)^{1/2}$$

$$= G_{\beta_1}(z) + \psi_2(|\arctan(z)|)(1 + z^2)^{1/2}$$

$$\geq G_{\beta_1}(\bar{z}) + \psi_2(|\arctan(z)|)(1 + z^2)^{1/2}.$$

Combined with (4.51) the equality above implies that

$$G_{\beta_1}(\bar{z}) + \psi_2(|\arctan\ (z)|)(1 + z^2)^{1/2} \leq G_\phi(z)$$

$$\leq 5\alpha r n + G_{\beta_1}(\bar{z})$$

and

$$\psi_2(|\arctan(z)|)(1+z^2)^{1/2} \le 5\alpha r n.$$

This inequality implies that

$$\psi_2(|\arctan(z)|) \le 5\alpha r n. \tag{4.52}$$

If

$$||\arctan(z)| - \bar{\theta}| \ge c_0^{-1},$$

then by (4.24), (4.21) and (4.7)

$$\psi_2(|\arctan(z)|) = c_r \ge 8\alpha r n.$$

The relation above contradicts (4.52). Therefore

$$|\arctan(|z|) - \bar{\theta}| < c_0^{-1}. \tag{4.53}$$

It follows from the mean value theorem that

$$\begin{aligned} |||z| - \bar{z}| &= |\arctan(|z|) - \bar{\theta}||(\tan)'(x)| \\ &= |\arctan(|z|) - \bar{\theta}|(\cos(x))^{-2}, \end{aligned} \tag{4.54}$$

where

$$x \in [\min\{\bar{\theta}, \arctan(|z|)\}, \max\{\bar{\theta}, \arctan(|z|\}]. \tag{4.55}$$

By (4.55), (4.45), (4.19), (4.16), (3.1) and (4.18)

$$x \in [\arctan(1/n), \arctan(n)].$$

Since

$$\cos(x)^{-2} = (\tan(x))^2 + 1 \in [1 + 1/n^2, n^2 + 1]$$

it follows from (4.54), (4.53) and (4.4) that

$$||z| - \bar{z}| \le |\arctan(|z|) - \bar{\theta}|(n^2+1) \le c_0^{-1}(n^2+1) \le 1/i.$$

We have shown that $||z| - \bar{z}| \le 1/i$. Therefore $h \in \mathcal{F}_{ni}$. Together with (4.46) this inequality implies that

$$\{h \in \mathcal{M} :\ \rho(h, \phi) \le \alpha r\} \subset \{h \in \mathcal{M} :\ \rho(\beta, h) \le r\} \cap \mathcal{F}_{ni}$$

and $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$. Proposition 4.1 is proved. □

Completion of the Proof of Theorem 1.1. Let n be a natural number. By Proposition 4.1 for each natural number i, $\mathcal{M} \setminus \mathcal{F}_{ni}$ is a porous subset of $(\mathcal{M}, \rho)$. Proposition 3.2 implies that

$$\mathcal{M} \setminus \mathcal{F}_n \subset \mathcal{M} \setminus (\cap_{i=1}^{\infty} \mathcal{F}_{ni}) = \cup_{i=1}^{\infty} (\mathcal{M} \setminus \mathcal{F}_{ni}).$$

Thus $\mathcal{M} \setminus \mathcal{F}_n$ is a σ-porous subset of $(\mathcal{M}, \rho)$ for any natiral number n.

By Propositions 3.1 and 2.1

$$\mathcal{M} \setminus \mathcal{F} \subset \mathcal{M} \setminus (\mathcal{M}_r \cap (\cap_{n=1}^{\infty} \mathcal{F}_n)) = (\mathcal{M} \setminus \mathcal{M}_r) \cup_{n=1}^{\infty} (\mathcal{M} \setminus \mathcal{F}_n)$$

and $\mathcal{M} \setminus \mathcal{F}$ is a σ-porous subset of $(\mathcal{M}, \rho)$. Theorem 1.1 is proved. □

REFERENCES

[1] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, Amer. Math. Soc., Providence, RI, 2000.

[2] B. Dacorogna and C.E. Pfister, Wulff Theorem and best constant in Sobolev inequality, *J. Math. Pures Appl.*, **71** (1992), 97-118.

[3] F.S. De Blasi and J. Myjak, Sur la porosité des contractions sans point fixe, *C.R. Acad. Sci. Paris*, **308** (1989), 51-54.

[4] F.S. De Blasi, J. Myjak, and P. L. Papini, Porous sets in best approximation theory, *J. London Math. Soc.*, **44** (1991), 135-142.

[5] I. Fonseca, The Wulff Theorem revisited, *Proc. R. Soc. Lond. A*, **432** (1991), 125-145.

[6] J. Hannon et al, Step faceting at the (001) surface of boron doped silicon, *Phys. Rev. Lett.*, **79** (1887), 4226-4229.

[7] J. Hannon, M. Marcus and V.J. Mizel, A variational problem modelling behavior of unorthodox silicon crystals, *ESAIM Control Optim. Calc. Var.*, **9** (2003), 145-149.

[8] H.C. Jeng and E.D. Williams, Steps on surfaces: experiment and theory, *Surface Science Reports*, **34** (1999), 175-294.

[9] V.J. Mizel and A.J. Zaslavski, Anisotropic functions: a genericity result with crystallographic implications, *ESAIM: Control, Optimization and the Calculus of Variations*, To Appear.

[10] A.J. Zaslavski, Well-posedness and porosity in optimal control without convexity assumptions,*Calculus of Variations and Partial Differential Equations*, **13** (2001), 265-293.