Communications in Applied Analysis **10** (2006), no. 4, 563–571

ONE CLASS OF SEPARABLE SYNCHRONIZATION PROBLEMS

Karel Zimmermann
Faculty of Mathematics and Physics
Charles University Prague
Prague, 11800, Czech Republic
Karel.Zimmermann@mff.cuni.cz

Communicated by —???—

AMS (MOS) Subject Classification: 90B26, 90B35

1. INTRODUCTION

Operations research problems occurring e.g. in technology or economics contain interdependent processes with a given duration. The interdependency means that some processes or activities must be finished before other processes or activities can begin. Such situations were studied in the recent literature (Cuninghame-Green [2], Olsder and Woude van Der [4]). If we want to determine release times and/or deadlines of the interdependent processes satisfying certain additional conditions, it is necessary to synchronize the release times and deadlines in such a way that the interdependency is taken into account. In these situations we have to find out whether the set of feasible release times and deadlines satisfying given technological conditions and taking into account the interdependency restrictions is nonempty and if the answer is positive find one element of such set.

In the present paper we study systems of equations and inequalities, which make possible to solve a special class of such synchronization problems, in which so called max-separable or min-separable functions can be made use of to describe the interdependency restrictions. A function $f : R^n \to R^1$ is called max-separable if $f(x_1, ..., x_n) = \max_{1 \le j \le n} f_j(x_j)$, and in a similar way the concept of a min-separable function is defined. The synchronization problems mentioned above are described by a finite systems of equations and inequalities with special max- or min-separable functions on both sides

Received September 3, 2006

of the equations. The set of solutions of such systems is in general a nonconvex set in R^n. Originally such systems with variables only on one side of such equations and inequalities were studied (e.g. Cuninghame-Green [2], Vorobjov [3], Zimmermann [5]). Recently problems with variables on both sides of equations and inequalities became the subject of research in some papers as an appropriate tool for solving various types of activity synchronization problems (Butkovic and Zimmermann [1], Olsder and Woude van Der [4]). In Butkovic and Zimmermann [1] systems of equations with special max-separable functions, so called $(\max, +)$-linear functions, in which $f_j(x_j) = c_j + x_j$, where c_j are real numbers are considered. In Olsder and Woude van Der [4] systems of inequalities with $(\max, +)$-linear functions on one side and $(\min, +)$-linear functions (i.e. functions of the form $f(x) = \min_{1\leq j\leq n}(c_j + x_j)$) on the other side of the inequalities are investigated. In the present paper unlike to Olsder and Woude van Der [4] systems of equations with $(\max, +)$-linear functions on one side and $(\min, +)$-linear functions on the other side of the equations are studied. A finite algorithm with complexity $O(m^2n^2)$ for solving such systems is proposed.

2. A MOTIVATING EXAMPLE

Let us have n railway stations S_j, $j = 1, \dots, n$, from which passengers are delivered to m railway stations C_i, $i = 1, \dots, m$. Let us assume that traveling times from S_j to C_i are equal to a_{ij}, so that if x_j denotes a departure time from S_j, the arrival time to C_i will be equal to $a_{ij} + x_j$. Under these assumptions the last train coming from stations $S_1, S_2, \dots, S_n$ will come to C_i at a time $a_i(x) \equiv \max_{1\leq\ j\leq n}(a_{ij} + x_j)$. Let us assume that all passengers coming from S_j, $j = 1, \dots, n$ must have the possibility o change at C_i for trains going to stations $T_1, \dots, T_p$. Let travelling times from C_i to T_k be equal to b_{ik} for $i = 1, \dots, m$, for $k = 1, \dots, p$ and let $y_1, \dots, y_p$ be arrival times to stations $T_1, \dots, T_p$. Then the earliest time for departure from station C_i is equal to $b_i(y) = \min_{1\leq k\leq p}(y_k - b_{ik})$. To minimize the waiting times at stations C_i we require to synchronize (if possible) the departure and arrival times in such a way that $a_i(x) = b_i(y)$ for all $i = 1, \dots, m$. We assume further that the departure and arrival times cannot be chosen arbitrarily, but within prescribed time intervals, i.e. that $\underline{x} \leq x \leq \overline{x}$, $\underline{y} \leq y \leq \overline{y}$ for given $\underline{x}$, $\overline{x}$, $\underline{y}$, $\overline{y}$. Therefore we have to find out whether such synchonized departure and arrival times exist and if the answer is positive, find a feasible (x, y). Let us note that if there is no connection between some stations C_i and S_j or T_k or if it is not necessary to wait for the passengers from some station S_j at some station C_i, we can include formally such situations in the model by setting the corresponding travelling times a_{ij} equal to $-\infty$, and the corresponding $b_{ik} = \infty$ (for practical calculations, it is evidently possible to choose the corresponding coefficients as sufficiently low negative or large positive finite numbers respectively). For similar reasons, we can assume w.l.o.g. that $n = p$.

3. PROBLEM FORMULATION

Let us consider the following system of equations and inequalities:

$$\max_{j \in J}(a_{ij} + x_j) = \min_{j \in J}(b_{ij} + y_j), \quad \forall i \in I, \tag{3.1}$$

$$\underline{x}_j \le x_j \le \overline{x}_j, \quad \underline{y}_j \le y_j \le \overline{y}_j\,, \quad \forall j \in J \tag{3.2}$$

where $I = \{1, \dots, m\}$, $J = \{1, \dots, n\}$, and a_{ij}, b_{ij}, $\underline{x}_j$, $\overline{x}_j$ are for all $i \in I$, $j \in J$ given real numbers. If we set $x = (x_1, \dots, x_n)$, $a_i(x) = \max_{j \in J}(a_{ij} + x_j)$, $b_i(y) = \min_{j \in J}(b_{ij} + y_j)$ for all $i \in I$, the system can be written as follows:

$$a_i(x) = b_i(y), \quad \forall i \in I, \tag{3.3}$$

$$\underline{x} \le x \le \overline{x}, \quad \underline{y} \le y \le \overline{y}\,. \tag{3.4}$$

We will solve the system by making use of the following optimization problem:

$$f(x, y) \equiv \max_{i \in I} |a_i(x) - b_i(y)| \longrightarrow \min \tag{3.5}$$

subject to

$$(\underline{x}, \underline{y}) \le (x, y) \le (\overline{x}, \overline{y}). \tag{3.6}$$

Let (x^{opt}, y^{opt}) be the optimal solution of (3.5), (3.6). If $f(x^{opt}, y^{opt}) = 0$, then x^{opt}, y^{opt} is a solution of system (3.3), (3.4). If $f(x^{opt}, y^{opt}) > 0$, then no solution of system (3.3), (3.4) exists and (x^{opt}, y^{opt}) represents the best approximate solution of (3.3), (3.4) in the sense that it minimizes the Tchebyshev distance between left-hand side vector $\mathbf{a}(x) = (a_1(x), \dots, a_m(x))$ and right-hand side vector $\mathbf{b}(y) = (b_1(y), \dots, b_m(y))$ under the constraints (3.6). In the next section a finite polynomial algorithm for solving minimization problem (3.5), (3.6) will be presented.

4. THE ALGORITHM

The algorithm for solving minimization problem (3.5), (3.6), which will be presented in this section can be characterized as a version of a feasible direction algorithm. The following notations will be introduced for the description of the algorithm:

$$I^+(x,y) = \{i \in I; a_i(x) > b_i(y)\}, \quad I^-(x,y) = I \setminus I^+(x,y), \tag{4.7}$$

$$\begin{aligned} L_i(x) &= \{j \in J; a_i(x) = a_{ij} + x_j\}, \\ R_i(y) &= \{j \in J; b_i(y) = b_{ij} + y_j\} \forall i \in I, \end{aligned} \tag{4.8}$$

$$F(x,y) = \{i \in I; f(x,y) = |a_i(x) - b_i(y)|\}\,, \tag{4.9}$$

$$L^*(x,y) = \bigcup_{i \in F(x,y,)} L_i(x), \quad R^*(x,y) = \bigcup_{i \in F(x,y)} R_i(y), \tag{4.10}$$

$$\begin{aligned} V(x) &= \{j \in J; x_j = \underline{x}_j\} \cap L^*(x,y), \\ W(y) &= \{j \in J; y_j = \overline{y}_j\} \cap R^*(x,y)\,. \end{aligned} \tag{4.11}$$

Algorithm

$\boxed{1}$ $(\tilde{x},\tilde{y}) := (\overline{x},\underline{y})$;

$\boxed{2}$ If $F(\tilde{x},\tilde{y}) \cap I^-(\tilde{x},\tilde{y}) \neq \emptyset$, go to $\boxed{11}$;

$\boxed{3}$ If $V(\tilde{x}) \neq \emptyset$, go to $\boxed{7}$;

$\boxed{4}$ Set for $t \geq 0$: $x_j(t) := \tilde{x}_j - t$ if $j \in L^*(\tilde{x},\tilde{y})$, $x_j(t) := \tilde{x}_j$ otherwise, $y_j(t) = \tilde{y}_j$ $\forall j \in J$;

$\boxed{5}$ Increase t until a value τ, at which one of the following cases occurs for the first time:

(1) $I(x(t),y(t)) \neq I^+(\tilde{x},\tilde{y})$,

(2) $F(x(t),y(t)) \neq F(\tilde{x},\tilde{y})$,

(3) $L_i(x(t)) \neq L_i(\tilde{x})$ for some $i \in I$,

(4)$V(x(t) \neq \emptyset$.

$\boxed{6}$ Set $(\tilde{x},\tilde{y}) := (x(\tau),y(\tau))$, go to $\boxed{2}$;

$\boxed{7}$ If either $W(\tilde{y}) \neq \emptyset$ or $F(\tilde{x},\tilde{y}) \cap I^-(\tilde{x},\tilde{y}) \neq \emptyset$, go to $\boxed{11}$;

$\boxed{8}$ Set for $t \geq 0$: $y_j(t) = \tilde{y}_j + t$ if $j \in R^*(\tilde{x},\tilde{y})$, $y_j(t) = \tilde{y}_j$ otherwise, $x_j(t) = \tilde{x}_j$ $\forall j \in J$

$\boxed{9}$ Increase t until a value $\tau^{'}$, at which one of the following cases occurs for the first time:

(1) $I(x(t), y(t)) \neq I^{+}(\tilde{x}, \tilde{y})$,

(2) $F(x(t), y(t)) \neq F(\tilde{x}, \tilde{y})$,

(3) $R_i(y(t)) \neq R_i(\tilde{y})$ for some $i \in I$,

(4)$W(y(t)) \neq \emptyset$.

$\boxed{10}$ Set $(\tilde{x}, \tilde{y}) := (x(\tau^{'}), y(\tau^{'}))$, go to $\boxed{2}$;

$\boxed{11}$ $(\tilde{x}, \tilde{y}) = (x^{opt}, y^{opt})$, STOP.

We can summarize the algorithm as follows. The algorithm consists of two iteration procedures. Each iteration of the first procedure consists of steps $\boxed{2}$-$\boxed{6}$, each iteration of the second procedure consists of steps $\boxed{7}$-$\boxed{10}$. The optimal solution of problem (3.5), (3.6) is obtained in step $\boxed{11}$. The first iteration procedure finds after a finite number of iterations (the number of iterations in each of the two procedures will be specified later on) a point $\tilde{x} = x^{opt}$ such that $f(x, \underline{y}) \geq f(x^{opt}, \underline{y})$ for any x satisfying (3.6). The second iteration procedure begins in step $\boxed{7}$ with $(\tilde{x}, \tilde{y}) = (x^{opt}, \underline{y})$, increases successively $\tilde{y}_j$, $j \in R^*(\tilde{x}, \tilde{y})$, and after a finite number of iterations finds a point $(\tilde{x}, \tilde{y}) = (x^{opt}, y^{opt})$, for which $f(x^{opt}, y) \geq f(x^{opt}, y^{opt})$ $\forall y$ satisfying (3.6) holds.

We will describe in detail the calculations of values τ and $\tau^{'}$ in Steps $\boxed{5}$ and $\boxed{9}$ of the algorithm. For this purpose we introduce the following notations:

$$I^{(1)} = \{i \in I; L_i(\tilde{x}) \subseteq L^*(\tilde{x})\}\,, \tag{4.12}$$

$$I^{(2)} = \{i \in I; R_i(\tilde{y}) \subseteq R^*(\tilde{y})\}\,, \tag{4.13}$$

$$I^{(3)} = I \setminus (I^{(1)} \cup I^{(2)})\,, \tag{4.14}$$

$$\alpha_i(\tilde{x}) = \max_{j \in J \setminus L^*(\tilde{x})} (a_{ij} + \tilde{x}_j)\,, \quad \forall i \in I\,, \tag{4.15}$$

$$\beta_i(\tilde{y}) = \min_{j \in J \setminus R^*(\tilde{y})} (b_{ij} + \tilde{y}_j)\,, \quad \forall i \in I\,, \tag{4.16}$$

$$\gamma(\tilde{x}, \tilde{y}) = \max_{i \in (I^{(2)} \cup I^{(3)})} |a_i(x) - b_i(y)|\,, \tag{4.17}$$

$$\delta(\tilde{x}, \tilde{y}) = \min_{i \in (I^{(1)} \cup I^{(3)})} |a_i(x) - b_i(y)|\,. \tag{4.18}$$

We set $\alpha_i(\tilde{x}) = -\infty$, in case that $J = L^*(\tilde{x})$, $\beta_i(\tilde{y}) = \infty$ in case that $J = R^*(\tilde{y})$, $\gamma(\tilde{x}, \tilde{y}) = -\infty$ if $I^{(2)} \cup I^{(3)} = \emptyset$, and $\delta(\tilde{x}, \tilde{y}) = \infty$, if $I^{(1)} \cup I^{(3)} = \emptyset$.

We will investigate now the calculations of τ and $\tau^{'}$ in cases (1)-(5) of steps $\boxed{5}$, $\boxed{9}$.

Case (1) $I(x(t)) \neq I^{+}(\tilde{x}, \tilde{y})$ can occur only if either in Case (1) of step $\boxed{5}$ $i \in I^{+}(\tilde{x}, \tilde{y})) \cap I^{(1)}$ and thus $|a_i(x(t)) - b_i(y(t))| = a_i(x(t)) - b_i(y(t)) = a_i(\tilde{x}) - t - b_i(\tilde{y})$ or in Case (1) of step $\boxed{9}$ if $i \in I^{+}(\tilde{x}, \tilde{y}) \cap I^{(2)}$ so that again $|a_i(x(t))) - b_i(y(t))| = a_i(\tilde{x}) - b_i(\tilde{y}) - t$; the value of t, at which for the first time $i \notin I^{+}(x(t), y(t))$ (i.e. at which $a_i(x(t)) - b_i(x(t))$ changes the sign) must be in both cases the minimum t satisfying the equation $a_i(\tilde{x}) - b_i(\tilde{y}) - t = 0$, i.e. $t = t^{(1)} \equiv \min_{i\ I^{+} \cap I^{(1)}} (a_i(\tilde{x}) - b_i(\tilde{y}))$ in step $\boxed{5}$, and $t = t^{(1)'} \equiv \min_{i\ I^{+} \cap I^{(2)}} (a_i(\tilde{x}) - b_i(\tilde{y}))$ in step $\boxed{9}$, where I^{+} stands for $I^{+}(\tilde{x}, \tilde{y})$ to simplify the notations.

Case (2) Let $\gamma(\tilde{x}, \tilde{y})$, $\delta(\tilde{x}, \tilde{y})$ be defined as in (4.17), (4.18). Since both in step $\boxed{5}$ and in step $\boxed{9}$ for sufficiently small t the equality $f(x(t), y(t)) = f(\tilde{x}, \tilde{y}) - t$ holds, $F(x(t), y(t)) \neq F(\tilde{x}, \tilde{y})$ occurs if either $f(\tilde{x}, \tilde{y}) - t = \gamma(\tilde{x}, \tilde{y})$ in step $\boxed{5}$ or $f(\tilde{x}, \tilde{y}) - t = \delta(\tilde{x}, \tilde{y})$ in step $\boxed{9}$. The corresponding values of t are $t^{(2)} = f(\tilde{x}, \tilde{y}) - \gamma(\tilde{x}, \tilde{y})$ in step $\boxed{5}$ and $t^{(2)'} = f(\tilde{x}, \tilde{y}) - \delta(\tilde{x}, \tilde{y})$ in step $\boxed{9}$.

Case (3) First, let us consider step $\boxed{5}$. Let us note that $L_i(x(t)) \neq L_i(\tilde{x})$ can occur in step $\boxed{5}$ only for $i \in I^{(1)}$

Since for any $i \in I^{(1)}$ we have $a_i(x(t)) = a_i(\tilde{x}) - t$, the change of $L_i(x(t))$ occurs if $a_i(\tilde{x}) - t = \alpha_i(\tilde{x})$. Therefore the first such change will occur if $t = t^{(3)} \equiv \min_{i \in I^{(1)}} (a_i(\tilde{x}) - \alpha_i(\tilde{x}))$

Let us consider now step $\boxed{9}$, in which $R_i(y(t)) \neq R_i(\tilde{y})$ can occur for some $i \in I^{(2)}$. Since for any such i we have $b_i(y(t)) = b_i(\tilde{y}) + t$, the change of $R_i(y(t))$ will occur if $b_i(y(t) = b_i(\tilde{y}) + t = \beta_i(\tilde{y})$. Therefore the first such change will occur if $t = t^{(3)'} \equiv \min_{i \in I^{(2)}} (\beta_i(\tilde{y})) - b_i(\tilde{y})$.

Case (4) First, let us consider step $\boxed{5}$, in which $V(x(t)) \neq \emptyset$ can occur if $x_j(t) = \tilde{x}_j - t = \underline{x}_j$ for some $j \in L^{*}(\tilde{x}, \tilde{y})$. This situation takes place for the first time if $t = t^{(4)} \equiv \min_{j \in L^{*}(\tilde{x}, \tilde{y})} (\tilde{x}_j - \underline{x}_j)$.

Let us consider now step $\boxed{9}$, in which $W(y(t)) \neq \emptyset$ can occur if $y_j(t) = \tilde{y}_j + t = \overline{y}_j$ for some $j \in R^{*}(\tilde{x}, \tilde{y})$.This situation takes place for the first time if $t = t^{(4)'} \equiv \min_{j \in R^{*}(\tilde{x}, \tilde{y})} (\overline{y}_j - \tilde{y}_j)$.

We see that after each iteration objective function f is stricly decreased, because both τ in the first iteration procedure and $\tau^{'}$ in the second one are positive. Further, in each iteration of the both iteration procedures either $I^{+}(\tilde{x}, \tilde{y})$ changes (i.e. $\tau = t^{(1)}$, $\tau^{'} = \tau^{(1)'}$ repectively), the corresponding index i enters the set $I^-(\tilde{x},\tilde{y})$ and remains in it until the end of the calculations (since for any i the term $a_i(\tilde{x})-b_i(\tilde{y})$ can change the sign only once), or at least one new index enters either set $L^*(\tilde{x},\tilde{y})$ (if $\tau=t^{(2)}$ or $\tau=t^{(3)}$ in step $\boxed{5}$) or set $R^*(\tilde{x},\tilde{y})$ (if $\tau^{'}=t^{(2)'}$ or $\tau^{'}=t^{(3)'}$ in step $\boxed{9}$) and will never leave it until the end of the corresponding iteration procedure. Note that if $\tau=t^{(4)}$ or $\tau=t^{(4)'}$ occurs, then the process is finished on the next iteration in step $\boxed{11}$. Since $I^+(\tilde{x},\tilde{y})$ has at most m elements and J has n elements, the maximum number of iterations with a change of $I^+(\tilde{x},\tilde{y})$ is equal to m and the maximum nuber of iterations, in which either Case (2) (i.e. $\tau=t^{(2)}$, $\tau^{'}=t^{(2)'}$ or Case (3) (i.e. $\tau=t^{(3)}$, $\tau^{'}=t^{(3)'}$) occurs is for each i equal to n. Therefore the maximum number of iterations in both iteration procedures together is equal to $m+2mn$. In each iteration of any of the two iteration procedures, we have to find the maximum of m values (Case (1)), further for each i (i.e. m-times) the maximum of n values (Case (2) and Case (3)); Case (4) needs has complexity n. Further, steps $\boxed{2}$ and $\boxed{7}$ have complexity n and steps $\boxed{4}$ and $\boxed{8}$ have the complexity n. Therefore the complexity of each iteration is O(mn) and the algorithm as a whole has complexity $O(m^2n^2)$.

The following theorems will prove the correctness of the algorithm.

Theorem 4.1. *Let τ be the value of t obtained in step $\boxed{5}$ of the algorithm and let $(\tilde{x},\tilde{y})$ be the corresponding current upper bound. Let x be any point such that $x\not\leq x(\tau)$, $\underline{x}\leq x\leq\tilde{x}$. Then $f(x,\tilde{y})>f(x(\tau),\tilde{y})$ holds.*

Proof. Let $\tilde{x}_r\geq x_r>x_r(\tau)$. Since $x_j(\tau)=\tilde{x}_j\forall j\in J\setminus L^*(\tilde{x},\tilde{y})$, it must be $r\in L^*(\tilde{x},\tilde{y})$ so that there exists an index $s\in I^+(\tilde{x},\tilde{y})\cap F(\tilde{x},\tilde{y})$ such that $r\in L_s(\tilde{x})$ and therefore $a_s(\tilde{x})=a_{sr}+\tilde{x}_r$. We have in this case for any $t\in[0,\tau]$:

$$a_s(x(t))=a_s(\tilde{x})-t=a_{sr}+\tilde{x}_r-t,$$

$$f(x(t),\tilde{y})=a_s(\tilde{x})-t-b_r(\tilde{y})=f(\tilde{x},\tilde{y})-t$$

Since $x_r\in(x_r(\tau),\tilde{x}_r]$, there exists $t^*\in[0,\tau)$ such that $\tilde{x}_r-t^*=x_r$. We obtain:

$$f(x,\tilde{y})\geq|a_s(x)-b_s(\tilde{y})|=a_s(x)-b_s(\tilde{y})\geq a_{sr}+x_r-b_s(\tilde{y}).$$

Further we have:

$$\begin{aligned}a_{sr}+x_r-b_s(\tilde{y})&=a_{sr}+\tilde{x}_r-t^*-b_s(\tilde{y})=a_s(\tilde{x})-b_s(\tilde{y})-t^*\\&=f(\tilde{x},\tilde{y})-t^*>f(\tilde{x},\tilde{y})-\tau>f(x(\tau),\tilde{y}),\end{aligned}$$

which completes the proof. □

By analogy the following theorem can be proved.

Theorem 4.2. *Let τ' be the value of t obtained in step $\boxed{9}$ of the algorithm and let $(\tilde{x}, \tilde{y})$ be the corresponding current lower bound. Let y be any point such that $y \not\geq y(\tau')$, $\overline{y} \geq y \geq \tilde{y}$. Then $f(\tilde{x}, y) > f(\tilde{x}, y(\tau'))$ holds.*

Theorem 4.3. *Point (x^{opt}, y^{opt}) is the optimal solution of (3.5), (3.6).*

Proof. (x^{opt}, y^{opt}) was obtained either from $\boxed{3}$ as the last iteration of the first interation procedure consisting of steps $\boxed{2}$-$\boxed{6}$, or from $\boxed{7}$ as the last iteration of the second iteration procedure consisting of steps $\boxed{7}$-$\boxed{10}$. In the former case we have in step $\boxed{3}$ $(x^{opt}, y^{opt}) = (\tilde{x}, \tilde{y})$ and either $x_j^{opt} = \underline{x}_j$ for some $j \in L^*(x^{opt}, y^{opt})$ or $(x^{opt}, y^{opt}) = (x(\tau), \underline{y})$, where τ was obtained in step $\boxed{5}$ of the preceding iteration and $I^-(x^{opt}, y^{opt}) \cap F(x^{opt}, y^{opt}) \neq \emptyset$. In the latter case we have in step $\boxed{7}$ $(x^{opt}, y^{opt}) = (\tilde{x}, \tilde{y})$ and either $y_j^{opt} = \overline{y}_j$ for some $j \in R^*(x^{opt}, y^{opt})$ or $(x^{opt}, y^{opt}) = (x^{opt}, y(\tau'))$, where τ' was obtained in step $\boxed{9}$ of the preceding iteration and $I^-(x^{opt}, y^{opt}) \cap F(x^{opt}, y^{opt}) \neq \emptyset$.

Let us consider the former case. Let for some x, $x \not\leq x^{opt}$ holds. Since $x^{opt} = x(\tau)$, it follows from Theorem 4.1 that $f(x, y) > f(x^{opt}, y^{opt})$. Let us assume further that $(x, y) \neq (x^{opt}, y^{opt})$ and $(x, y) \leq (x^{opt}, y^{opt})$. Since in this case $y^{opt} = \underline{y}$), it must be $x \neq x^{opt}$ (if $y \neq y^{opt} = \underline{y}$, then (x, y) is not a feasible solution). Note that if $x_j = x^{opt}$ for any $j \in L^*(x^{opt}, y^{opt})$, then $f(x, y) = f(x^{opt}, y^{opt})$. Therefore if $x_j^{opt} = \underline{x}_j$ for some $j \in L^*(x^{opt}, y^{opt})$ (i.e. if $V(x^{opt}) \neq \emptyset$), then $f(x, y) = f(x^{opt}, y^{opt})$ so that we can exclude this case from our further considerations and assume that $V(x^{opt}) = \emptyset$. Assume now that $x_j < x_j^{opt} \forall j \in L^*(x^{opt}, y^{opt})$ and $I^-(x^{opt}, y^{opt}) \cap F(x^{opt}, y^{opt}) \neq \emptyset$. We obtain for $i \in I^-(x^{opt}, y^{opt}) \cap F(x^{opt}, y^{opt})$:

$$f(x, y) \geq b_i(y) - a_i(x) > b_i(\underline{y}) - a_(x^{opt}) = f(x^{opt}, y^{opt}).$$

Therefore we obtained that $f(x, y) \geq f(x^{opt}, y^{opt})$ for any feasible $(x, y) \neq (x^{opt}, y^{opt})$, if (x^{opt}, y^{opt}) was obtained in the first iteration procedure consisting of steps $\boxed{2}$-$\boxed{6}$.

The latter case, when (x^{opt}, y^{opt}) is obtained from the second iteration procedure consisting of steps $\boxed{7}$-$\boxed{10}$ can be proved by analogy.

Remark 4.1. Let us note that the solution method proposed above can be easily applied also for systems with the same variables on both sides of the system. We can simply add to the equations (3.3) n further equations $\max_{j \in J}(a_{ij} + x_j) = \min_{j \in J}(b_{ij} + y_j)$ for $i = m+1, \ldots, m+n$ with $a_{m+jj} = b_{m+jj} = 0$ for $j = 1, \ldots, n$ and $a_{ik} = -\infty$, $b_{ik} = \infty$ if $i = m + j$, $k \neq j$. In a similar way we can by an appropriate choice of additional coefficients include in the systems also additional precedence requirements like $x_j \leq x_k + \alpha$, $x_j \geq x_k + \beta$, $x_j \geq y_k + \gamma$ etc., where $\alpha,\ \beta,\ \gamma$ are given constants. Note that including an additional requirement that some or all variables x_j, y_j must be integer needs also only a slight technical modification of the proposed algoritm.

ACKNOWLEDGMENTS

Under the support of GA ČR 202-03/2060982, GA ČR # 402/06/1071.

REFERENCES

[1] P. Butkovic and K. Zimmermann, A strongly polynomial algorithm for solving two-sided linear systems in max-algebra, *Discrete Applied Mathematics*, **154** (2006), 437-446.

[2] R.A. Cuninghame-Green, *Minimax Algebra*, Lecture Notes in Economics and Mathematical Systems, **166**, Springer Verlag, Berlin, 1979.

[3] N.N. Vorobjov, Extremal algebra of positive matrices, *Elektronische Datenverarbeitung und Kybernetik*, **3** (1967), 39-71, In Russian.

[4] G.J. Olsder and J. Woude van Der, On $(\min, \max, +)$-inequalities, *Contemporary Mathematics*, **377** (2005), 353-361.

[5] K. Zimmermann, Disjunctive optimization, max-separable problems and extremal algebras, *Theoretical Computer Science*, **293** (2003), 45-54.