MULTIVARIATE FRACTIONAL TAYLOR’S FORMULA

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Communicated by D.D. Bainov

ABSTRACT: Here is established a multivariate fractional Taylor’s formula using a suitable definition of fractional derivative. As related results we present that the order of fractional-ordinary partial differentiation is immaterial, we discuss fractional integration by parts, and we estimate the remainder of our multivariate fractional Taylor’s formula.

AMS (MOS) Subject Classification: 26A33

1. INTRODUCTION

The main motivation here is from Canavati [4], Anastassiou [1] and Anastassiou [2], where there is presented a Taylor’s univariate fractional formula by using an appropriate definition of fractional derivative introduced first in Canavati [4].

So we extend this formula to the multivariate fractional case over a compact and convex subset of $\mathbb{R}^k$, $k \geq 2$, for all fractional orders $\nu > 0$.

We give an estimate to the remainder of our multivariate fractional Taylor’s formula. We establish under mild and natural assumptions that the order of fractional-ordinary partial differentiation is immaterial. Also we present some fractional integration by parts results. The main ingredient in all here is the Riemann-Liouville integral.
2. RESULTS

Remark 2.1. We follow Anastassiou [2], p. 540, see also Canavati [4], Anastassiou [1]. Let \([a, b] \subseteq \mathbb{R}\). Let \(x, x_0 \in [a, b]\) such that \(x \geq x_0\), \(x_0\) is fixed. Let \(f \in C([a, b])\) and define

\[
(J_{\nu}^{x_0} f)(x) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} (x - t)^{\nu - 1} f(t) dt, \quad x_0 \leq x \leq b,
\]

\(\nu > 0\), the generalized Riemann-Liouville integral. We consider the subspace \(C_{x_0}^{\nu}([a, b])\) of \(C^n([a, b])\), \(n := \nu\), \(\alpha := \nu - n \quad (0 < \alpha < 1)\):

\[
C_{x_0}^{\nu}([a, b]) := \{ f \in C^n([a, b]) : J_{1-\alpha}^{x_0} f^{(n)} \in C^1([x_0, b]) \}. \tag{2.2}
\]

Hence, let \(f \in C_{x_0}^{\nu}([a, b])\), we define the generalized \(\nu\) - fractional derivative of \(f\) over \([x_0, b]\), see also Canavati [4], Anastassiou [1] as

\[
D_{x_0}^{\nu} f := (J_{1-\alpha}^{x_0} f^{(n)})'. \tag{2.3}
\]

Notice that

\[
(J_{1-\alpha}^{x_0} f^{(n)}) (x) = \frac{1}{\Gamma(1-\alpha)} \int_{x_0}^{x} (x - t)^{-\alpha} f^{(n)}(t) dt \tag{2.4}
\]

exists for \(f \in C_{x_0}^{\nu}([a, b])\).

Let \(f_{x_0}(t) := f(x_0 + t), 0 \leq t \leq b - x_0, x \geq x_0\). By change of variable we obtain

\[
(D_{x_0}^{\nu} f_{x_0})(x - x_0) = (D_{x_0}^{\nu} f)(x). \tag{2.5}
\]

When \(\nu \in \mathbb{N}\) then the fractional derivative collapses to the usual one.

We mention the fractional Taylor’s formula. See Anastassiou [2], p. 540, Canavati [4] and Anastassiou [1].

Theorem 2.1. Let \(f \in C_{x_0}^{\nu}([a, b]), \quad x_0 \in [a, b]\) fixed

(i) If \(\nu \geq 1\), then it holds

\[
f(x) = f(x_0) + f'(x_0)(x - x_0) + f''(x_0) \frac{(x - x_0)^2}{2} + \cdots + f^{(n-1)}(x_0) \frac{(x - x_0)^{n-1}}{(n-1)!} + (J_{\nu}^{x_0} D_{x_0}^{\nu} f)(x), \quad \text{all} \quad x \in [a, b] : x \geq x_0. \tag{2.6}
\]

(ii) If \(0 < \nu < 1\) we have

\[
f(x) = (J_{\nu}^{x_0} D_{x_0}^{\nu} f)(x), \quad \text{all} \quad x \in [a, b] : x \geq x_0. \tag{2.7}
\]

We transfer Theorem 2.1 to the multivariate case. We make

Remark 2.2. Let \(Q\) be a compact and convex subset of \(\mathbb{R}^k, k \geq 2\); \(z := (z_1, \ldots, z_k), x_0 := (x_{01}, \ldots, x_{0k}) \in Q\). Let \(f \in C^n(Q), \quad n \in \mathbb{N}\).

Set

\[
g_z(t) := f(x_0 + t(z - x_0)), 0 \leq t \leq 1; g_z(0) = f(x_0), g_z(1) = f(z). \tag{2.8}
\]
Then
\[ g_z^{(j)}(t) = \left( \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^j f(x_0 + t(z - x_0)), \] 
\( j = 0, 1, 2, \ldots, n, \) and
\[ g_z^{(n)}(0) = \left( \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f(x_0). \] 
If all \( f_\alpha(x_0) := \frac{\partial^\alpha f}{\partial x_\alpha}(x_0) = 0, \) \( \alpha := (\alpha_1, \ldots, \alpha_k), \) \( \alpha_i \in \mathbb{Z}^+, \) \( i = 1, \ldots, k; \) \( |\alpha| := \sum_{i=1}^{k} \alpha_i =: l, \) then \( g_z^{(l)}(0) = 0, \) where \( l \in \{0, 1, \ldots, n\}. \) We quote that
\[ g_z'(t) = \sum_{i=1}^{k} (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)). \] 
Let first \( 1 \leq \nu < 2, \) then here \( n := [\nu] = 1 \) and \( \alpha = \nu - 1; \) \( 1 - \alpha = 2 - \nu. \) Since \( 0 \leq \nu - 1 < 1, \) then \( n^* := [\nu - 1] = 0, \) \( \alpha^* = \nu - 1 - n^* = \nu - 1 \) and \( 1 - \alpha^* = 2 - \nu. \)

Put
\[ (J_{\nu}g_z)(x) := \frac{1}{\Gamma(\nu)} \int_0^x (x - t)^{\nu-1} g_z(t)dt, \] 
\( 0 \leq x \leq 1. \)

Consider
\[ C^\nu([0, 1]) := \{ g \in C^1([0, 1]) : J_{2-\nu}g' \in C^1([0, 1]) \} \subseteq C^1([0, 1]), \] 
\( 1 \leq \nu < 2. \) Assume that as function of \( t : f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1]), \) \( i = 1, \ldots, k, \) then there exists the fractional derivative \( g_z^{(\nu)}, \) \( g_z^{(\nu)} = (J_{2-\nu}g_z)''. \) The last comes by using (2.11) to have
\[ (J_{2-\nu}g_z')(x) = \frac{1}{\Gamma(2 - \nu)} \int_0^x (x - t)^{1-\nu} g_z(t)dt \] 
\[ = \frac{\sum_{i=1}^{k} (z_i - x_{0i})}{\Gamma(2 - \nu)} \int_0^x (x - t)^{1-\nu} \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))dt, \] 
\( 0 \leq x \leq 1. \)

Hence it holds
\[ (J_{2-\nu}g_z'(x))' \]
\[ \sum_{i=1}^{k} (z_i - x_{0i}) \left( \frac{1}{\Gamma(2 - \nu)} \int_0^x (x - t)^{1-\nu} \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0))dt \right)' \] 
\[ = \sum_{i=1}^{k} (z_i - x_{0i})(J_{2-\nu}(f_{x_i}(x_0 + t(z - x_0))))'. \] 
That is
\[ g_z^{(\nu)}(t) = \sum_{i=1}^{k} (z_i - x_{0i}) \left( \frac{\partial f}{\partial x_i}(x_0 + t(z - x_0)) \right)^{(\nu-1)}, \]
\( 0 \leq t \leq 1, \) \( 1 \leq \nu < 2. \)
Thus the remainder turns to
\[(J_\nu g_z^{(\nu)})(t) = \sum_{i=1}^{k} (z_i - x_{0i}) [J_\nu(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}](t). \tag{2.19}\]

By (2.6) applied on \(g_z\) we obtain
\[f(z) = g_z(1) = f(x_0) + (J_\nu g_z^{(\nu)})(1), \tag{2.20}\]
i.e. it holds
\[f(z) = f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) [J_\nu(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)}](1). \tag{2.21}\]

More precisely we get
\[f(z) = f(x_0) + \sum_{i=1}^{k} (z_i - x_{0i}) \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)} dt \tag{2.22}\]

From Remark 2.2 we have established the basic multivariate fractional Taylor formula.

**Theorem 2.2.** Let \(f \in C^1(Q), Q \text{ compact and convex } \subseteq \mathbb{R}^k, k \geq 2. \) For fixed \(x_0, z \in Q, \) assume that as a function of \(t: f_{x_i}(x_0 + t(z - x_0)) \in C^{\nu-1}([0, 1]), 1 \leq \nu < 2, \) all \(i = 1, \ldots, k.\) Then

(i) \[f(z_1, \ldots, z_k) = f(x_{01}, \ldots, x_{0k}) + \sum_{i=1}^{k} \frac{(z_i - x_{0i})}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)} dt. \tag{2.23}\]

(ii) Given \(f(x_0) = 0,\) then
\[f(z) = \sum_{i=1}^{k} \frac{(z_i - x_{0i})}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_i}(x_0 + t(z - x_0)))^{(\nu-1)} dt. \tag{2.24}\]

We make

**Remark 2.3.** Continuing from Remark 2.2. Here \(f \in C^2(Q), Q \subseteq \mathbb{R}^2, \) we have
\[g_z''(t) = (z_1 - x_{01})^2 \frac{\partial^2 f}{\partial x_1^2}(x_0 + t(z - x_0)) + 2(z_1 - x_{01})(z_2 - x_{02}) \frac{\partial^2 f}{\partial x_1 \partial x_2}(x_0 + t(z - x_0)) + (z_2 - x_{02})^2 \frac{\partial^2 f}{\partial x_2^2}(x_0 + t(z - x_0)). \tag{2.25}\]

Let \(2 \leq \nu < 3,\) then \(n := [\nu] = 2, \alpha := \nu - n = \nu - 2, 1 - \alpha = 3 - \nu.\) Set \(\nu^* := \nu - 2,\) then \(n^* := [\nu - 2] = 0, \alpha^* = (\nu - 2) - n^* = \nu - 2, 1 - \alpha^* = 3 - \nu.\)

We have \((0 \leq x \leq 1)\)
\[(J_{3-\nu}g_z''(x)) = \frac{1}{\Gamma(3 - \nu)} \int_0^x (x - t)^{2-\nu} g_z''(t) dt \]
\[ = (z_1 - x_{01})^2 \frac{1}{\Gamma(3 - \nu)} \int_0^x (x - t)^{2-\nu} f_{x_1x_1}(x_0 + t(z - x_0)) dt \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(3 - \nu)} \int_0^x (x - t)^{2-\nu} f_{x_1x_2}(x_0 + t(z - x_0)) dt \]

\[ + (z_2 - x_{02})^2 \frac{1}{\Gamma(3 - \nu)} \int_0^x (x - t)^{2-\nu} f_{x_2x_2}(x_0 + t(z - x_0)) dt, \] (2.26)

i.e. it holds.

\[ (J_{3-\nu} g_z^\nu)(x) = (z_1 - x_{01})^2 (J_{3-\nu}(f_{x_1x_1}(x_0 + t(z - x_0))))(x) \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02})(J_{3-\nu}(f_{x_1x_2}(x_0 + t(z - x_0))))(x) \]

\[ + (z_2 - x_{02})^2 (J_{3-\nu}(f_{x_2x_2}(x_0 + t(z - x_0))))(x). \] (2.27)

Assuming now that \( f_{x_1x_1}(x_0 + t(z - x_0)), \ f_{x_1x_2}(x_0 + t(z - x_0)), \ f_{x_2x_2}(x_0 + t(z - x_0)), \) as functions of \( t \) belong to \( C^{(\nu-2)}([0,1]) \) we obtain that it exists

\[ g_z^\nu(t) = (z_1 - x_{01})^2 (f_{x_1x_1}(x_0 + t(z - x_0)))^{(\nu-2)} \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02})(f_{x_1x_2}(x_0 + t(z - x_0)))^{(\nu-2)} \]

\[ + (z_2 - x_{02})^2 (f_{x_2x_2}(x_0 + t(z - x_0)))^{(\nu-2)}. \] (2.28)

Next we observe that

\[ (J_\nu g_z^\nu)(x) = (z_1 - x_{01})^2 (J_\nu(f_{x_1x_1}(x_0 + t(z - x_0))))^{(\nu-2)}(x) \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02})(J_\nu(f_{x_1x_2}(x_0 + t(z - x_0))))^{(\nu-2)}(x) \]

\[ + (z_2 - x_{02})^2 (J_\nu(f_{x_2x_2}(x_0 + t(z - x_0))))^{(\nu-2)}(x). \] (2.29)

We have proved via (2.6) the next Taylor type result.

**Theorem 2.3.** Let \( f \in C^2(Q), \) \( Q \) compact and convex \( \subseteq \mathbb{R}^2 \). For fixed \( x_0, z \in Q \) assume that as functions of \( t : f_{x_1x_1}(x_0 + t(z - x_0)), \ f_{x_1x_2}(x_0 + t(z - x_0)), \ f_{x_2x_2}(x_0 + t(z - x_0)) \in C^{(\nu-2)}([0,1]), \) where \( 2 \leq \nu < 3 \). Then

(i)

\[ f(z_1, z_2) = f(x_0, x_0) + (z_1 - x_{01}) \frac{\partial f}{\partial x_1}(x_0) + (z_2 - x_{02}) \frac{\partial f}{\partial x_2}(x_0) \]

\[ + (z_1 - x_{01})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_1x_1}(x_0 + t(z - x_0)))^{(\nu-2)} dt \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_1x_2}(x_0 + t(z - x_0)))^{(\nu-2)} dt \]

\[ + (z_2 - x_{02})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_2x_2}(x_0 + t(z - x_0)))^{(\nu-2)} dt. \] (2.30)

(ii) When \( f(x_0) = \frac{\partial f}{\partial x_1}(x_0) = \frac{\partial f}{\partial x_2}(x_0) = 0 \), then

\[ f(z_1, z_2) = (z_1 - x_{01})^2 \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_1x_1}(x_0 + t(z - x_0)))^{(\nu-2)} dt \]

\[ + 2(z_1 - x_{01})(z_2 - x_{02}) \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu-1}(f_{x_1x_2}(x_0 + t(z - x_0)))^{(\nu-2)} dt. \]
The following general multivariate fractional Taylor formula is valid.

**Theorem 2.4.** Let \( f \in C^n(Q), Q \) compact and convex \( \subseteq \mathbb{R}^k \), \( k \geq 2 \); here \( \nu \geq 1 \) such that \( n = \lfloor \nu \rfloor \). For fixed \( x_0, z \in Q \) assume that as functions of \( t : f_\alpha(x_0 + t(z - x_0)) \in C^{(\nu-n)}([0,1]) \), for all \( \alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = n \). Then

\[
(i)\quad f(z_1, \ldots, z_k) = f(x_{01}, \ldots, x_{0k}) + \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial f}{\partial x_i}(x_{01}, \ldots, x_{0k}) + \frac{1}{(n-1)!} \left[ \left( \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_{01}, \ldots, x_{0k})
\]

\[ + \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left\{ \left[ \left( \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\} dt. \tag{2.32} \]

(ii) If all \( f_\alpha(x_0) = 0 \), \( \alpha := (\alpha_1, \ldots, \alpha_k), \alpha_i \in \mathbb{Z}^+, i = 1, \ldots, k; |\alpha| := \sum_{i=1}^k \alpha_i = l, l = 0, \ldots, n - 1 \), then

\[
 f(z_1, \ldots, z_k) = \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1} \left\{ \left[ \left( \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right)^n f \right] (x_0 + t(z - x_0)) \right\} dt. \tag{2.33} \]

**Proof.** Use of (2.6).

(Note that fractional differentiation is a linear operation.)

We make

**Remark 2.4.** Continuing from the previous remarks. Let here \( 0 < \nu < 1 \). Assume that \( f(x_0 + t(z - x_0)) \in C^{\nu}([0,1]) \) as function of \( t \). Then by \( g_z(t) := f(x_0 + t(z - x_0)) \) we have

\[
g_z^{(\nu)}(t) = (f(x_0 + t(z - x_0)))^{(\nu)}, \quad \text{and} \quad (J_\nu g_z^{(\nu)})(t) = (J_\nu(f(x_0 + t(z - x_0)))^{(\nu)})(t),
\]

\( t \in [0,1] \). Hence

\[
(J_\nu g_z^{(\nu)})(1) = \frac{1}{\Gamma(\nu)} \int_0^1 (1-t)^{\nu-1}(f(x_0 + t(z - x_0)))^{(\nu)}dt. \tag{2.34} \]
We have established the next multivariate fractional Taylor formula when $0 < \nu < 1$.

**Theorem 2.5.** Let bounded $f : Q \to \mathbb{R}$, where $Q$ convex $\subseteq \mathbb{R}^k$, $k \geq 2$, such that as a function of $t : f(x_0 + t(z - x_0)) \in C^\nu([0, 1])$, $0 < \nu < 1$, $x_0, z \in Q$ being fixed.

Then

$$f(z_1, \ldots, z_k) = \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1}(f(x_0 + t(z - x_0)))^{(\nu)} dt. \quad (2.35)$$

**Proof.** Use of (2.7).

We make

**Remark 2.5.** Next we study the ordinary partial derivatives of fractional derivatives. Let $0 < \alpha < 1$, $f \in C^1([0, 1]^2)$, $x \in [0, 1]$ fixed, and consider

$$\gamma(x, z) := \int_0^x (x - t)^{-\alpha} f(t, z) dt, \quad (2.36)$$

$\forall z \in [0, 1]$.

We observe that

$$|\gamma(x, z)| \leq \int_0^x (x - t)^{-\alpha} |f(t, z)| dt \leq \|f\|_\infty \int_0^x (x - t)^{-\alpha} dt$$

$$= \|f\|_\infty \frac{x^{1-\alpha}}{1-\alpha} \leq \frac{\|f\|_\infty}{1-\alpha} < +\infty,$$

i.e. the function

$$\rho(t) := (x - t)^{-\alpha} f(t, z) \quad (2.37)$$

is Lebesgue integrable in $t \in [0, x]$, $\forall z \in [0, 1]$. Thus one can consider integration in (2.36) over $[0, x)$, $\forall z \in [0, 1]$.

Also the function

$$\lambda(z) := (x - t)^{-\alpha} f(t, z) \quad (2.38)$$

is differentiable in $z \in [0, 1]$, $\forall t \in [0, x)$, i.e. we have

$$\lambda'(z) = (x - t)^{-\alpha} \frac{\partial f(t, z)}{\partial z}, \forall t \in [0, x). \quad (2.39)$$

Moreover

$$|\lambda'(z)| \leq (x - t)^{-\alpha} \left\| \frac{\partial f}{\partial z} \right\|_\infty, \quad (2.40)$$

$\forall (t, z) \in [0, x) \times [0, 1]$.

The R.H.S. (2.40) is integrable in $t \in [0, x]$ and nonnegative. Hence by H. Bauer [3], pp. 103-104 we obtain that $(x - t)^{-\alpha} \frac{\partial f(t, z)}{\partial z}$ is integrable in $t \in [0, x)$ and

$$\frac{\partial \gamma(x, z)}{\partial z} = \int_0^x (x - t)^{-\alpha} \frac{\partial f(t, z)}{\partial z} dt, \quad (2.41)$$

$\forall z \in [0, 1]$.

We have proved
Lemma 2.1. Let $0 < \alpha < 1$, $f \in C^1([0, 1]^2)$, $0 \leq x \leq 1$. Then
\[
\frac{\partial}{\partial z} \left( \int_0^x (x - t)^{-\alpha} f(t, z) dt \right) = \int_0^x (x - t)^{-\alpha} \frac{\partial f(t, z)}{\partial z} dt,
\]
\[\forall z \in [0, 1].\]

We make

Remark 2.6. Assume now $0 < \alpha < 1$, $f \in C^{n+1}([0, 1]^2)$, $n \in \mathbb{N}$. Then by Lemma 2.1 we get
\[
\frac{\partial}{\partial z} \left( \int_0^x (x - t)^{-\alpha} \frac{\partial^n f}{\partial t^n}(t, z) dt \right) = \int_0^x (x - t)^{-\alpha} \frac{\partial^{n+1} f}{\partial t^{n+1}}(t, z) dt.
\]
Let now $\nu > 0$, $n := [\nu]$, $\alpha := \nu - n$.

We suppose existence of
\[
g^{(\nu)}(x, z) := \frac{\partial^n g(x, z)}{\partial x^n} = \frac{\partial}{\partial x} \left( J_{1-\alpha} \left( \frac{\partial^n g}{\partial t^n}(\cdot, z) \right) \right)(x, z).
\]
We also assume here that $g \in C^{n+1}([0, 1]^2)$, and $(g^{(\nu)}(x, z))_z$, $g^{(\nu)}_z(x, z)$ exist and are jointly continuous in $(x, z) \in [0, 1]^2$, $[\nu] = n \in \mathbb{N}$.

Then it holds
\[
(g^{(\nu)}(x, z))_z = \frac{\partial}{\partial x} (g^{(\nu)}(x, z)) = \frac{\partial}{\partial z} \left( \frac{\partial}{\partial x} \left( J_{1-\alpha} \left( \frac{\partial^n g}{\partial t^n}(\cdot, z) \right) \right) \right)(x, z)
\]
\[= \frac{\partial}{\partial x} \left( J_{1-\alpha} \left( \frac{\partial^n g}{\partial t^n}(\cdot, z) \right) \right)(x, z) \quad \text{(by (2.43))}
\]
\[= \frac{\partial}{\partial x} \left( J_{1-\alpha} \left( \frac{\partial^{n+1}}{\partial z \partial t^n} g(\cdot, z) \right) \right)(x, z)
\]
\[= \frac{\partial}{\partial x} \left( J_{1-\alpha} \left( \frac{\partial^n}{\partial t^n} g_z(\cdot, z) \right) \right) = g^{(\nu)}_z(x, z).
\]
That is
\[
(g^{(\nu)}(x, z))_z = g^{(\nu)}_z(x, z), \quad \forall (x, z) \in [0, 1]^2, \; \nu > 0.
\]
In brief, it holds
\[
(g^{(\nu)})_z = (g^{(\nu)}_z).
\]
Under more similar suitable assumptions one obtains
\[
(g^{(\nu)}_{zz}) = (g^{(\nu)}_{zz}), \quad (g^{(\nu)}_{z1z2}) = (g^{(\nu)}_{z1z2}), \quad (g^{(\nu)}_{z1z2z3}) = (g^{(\nu)}_{z1z2z3}), \quad \text{etc.}
\]
We have established that the order of fractional-ordinary partial differentiation is immaterial.

Theorem 2.6. Let $g \in C^{n+1}([0, 1]^2)$, and $\nu > 0$ such that $[\nu] = n \in \mathbb{N}$. Assume the existence of $g^{(\nu)}(x, z)$, and $(g^{(\nu)}(x, z))_z$, $g^{(\nu)}_z(x, z)$ both exist and are jointly continuous in $(x, z) \in [0, 1]^2$.

Then
\[
(g^{(\nu)}(x, z))_z = g^{(\nu)}_z(x, z),
\]
\[\quad \text{(2.48)}
\]
\( \forall (x, z) \in [0, 1]^2 \).

We make

**Remark 2.7.** Next comes fractional integration by parts. Let \( f, g \in C^\nu([0, 1]), \nu > 0, n := [\nu], \alpha := \nu - n \). Here

\[
g^{(\nu)} = \frac{d(J_{1-\alpha}g^{(n)})}{dx}, \quad f^{(\nu)} = \frac{d(J_{1-\alpha}f^{(n)})}{dx}.
\]

That is \( d(J_{1-\alpha}g^{(n)}) = g^{(\nu)}dx, \quad d(J_{1-\alpha}f^{(n)}) = f^{(\nu)}dx \).

We observe that

\[
\int_0^1 (J_{1-\alpha}f^{(n)})(x)g^{(\nu)}(x)dx = \int_0^1 (J_{1-\alpha}f^{(n)})(x)d(J_{1-\alpha}g^{(n)})(x)
\]

\[
= (J_{1-\alpha}f^{(n)})(1)(J_{1-\alpha}g^{(n)})(1) - \int_0^1 (J_{1-\alpha}g^{(n)})(x)d(J_{1-\alpha}f^{(n)})(x) = \\
(J_{1-\alpha}f^{(n)})(1)(J_{1-\alpha}g^{(n)})(1) - \int_0^1 (J_{1-\alpha}g^{(n)})(x)f^{(\nu)}(x)dx. \tag{2.49}
\]

Next let us take \( g \in C^\nu([0, 1]), 1 \leq \nu < 2, n := [\nu] = 1, \alpha := \nu - n = \nu - 1, 1 - \alpha = 2 - \nu, \) and \( f \in C^1([0, 1]). \)

Then \( g^{(\nu)}(x) = \frac{d(J_{2-\nu}g')(x)}{dx}, \) i.e. \( d(J_{2-\nu}g')(x) = g^{(\nu)}(x)dx. \) Hence

\[
\int_0^1 f(x)g^{(\nu)}(x)dx = \int_0^1 f(x)d(J_{2-\nu}g')(x)
\]

\[
= f(1)(J_{2-\nu}g')(1) - \int_0^1 (J_{2-\nu}g')(x)f'(x)dx. \tag{2.50}
\]

We have established the following fractional integration by parts formulae.

**Theorem 2.7.** (i) Let \( f, g \in C^\nu([0, 1]), \nu > 0, n := [\nu], \alpha := \nu - n \). Then

\[
\int_0^1 (J_{1-\alpha}f^{(n)})(x)g^{(\nu)}(x)dx
\]

\[
= (J_{1-\alpha}f^{(n)})(1)(J_{1-\alpha}g^{(n)})(1) - \int_0^1 (J_{1-\alpha}g^{(n)})(x)f^{(\nu)}(x)dx. \tag{2.51}
\]

(ii) Let \( g \in C^\nu([0, 1]), 1 \leq \nu < 2, f \in C^1([0, 1]). \) Then

\[
\int_0^1 f(x)g^{(\nu)}(x)dx = f(1)(J_{2-\nu}g')(1) - \int_0^1 (J_{2-\nu}g')(x)f'(x)dx. \tag{2.52}
\]

We make the last

**Remark 2.8.** Here we estimate the Remainder (2.32). By definition in this article, see (2.2), (2.3), the fractional derivatives are continuous functions. So the function

\[
G_\nu(t) := \left\{ \left[ \left( \sum_{i=1}^k (z_i - x_0i) \frac{\partial}{\partial x_i} \right)^n f \right]^{(\nu-n)} \right\}(x_0 + t(z - x_0)), \tag{2.53}
\]
$t \in [0, 1]$, that appears in the remainder of (2.32), is continuous in $t$. We write the remainder (2.32) as

$$R_{\nu} := \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1} \left\{ \left[ \sum_{i=1}^k (z_i - x_{0i}) \frac{\partial}{\partial x_i} \right]^{(\nu-n)} f(x_0 + t(z - x_0)) \right\} dt$$

$$= \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1} G_{\nu}(t) dt, \quad \nu \geq 1. \quad (2.54)$$

We obtain

$$|R_{\nu}| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1} |G_{\nu}(t)| dt \leq \frac{1}{\Gamma(\nu)} \int_0^1 |G_{\nu}(t)| dt$$

$$= \frac{1}{\Gamma(\nu)} \|G_{\nu}\|_{L_1([0,1])}. \quad (2.55)$$

Also for $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, we get

$$|R_{\nu}| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1} |G_{\nu}(t)| dt \leq \frac{1}{\Gamma(\nu)} \left( \int_0^1 ((1 - t)^{\nu - 1})^p dt \right)^{1/p} \left( \int_0^1 |G_{\nu}(t)|^q dt \right)^{1/q}$$

$$= \frac{1}{\Gamma(\nu)} \frac{1}{(p(\nu - 1) + 1)^{1/p}} \|G_{\nu}\|_{L_p([0,1])}. \quad (2.56)$$

In case $p = q = 2$ we have

$$|R_{\nu}| \leq \frac{1}{\Gamma(\nu)} \frac{1}{\sqrt{2\nu - 1}} \|G_{\nu}\|_{L_2([0,1])}. \quad (2.57)$$

Finally we get that

$$|R_{\nu}| \leq \frac{1}{\Gamma(\nu)} \int_0^1 (1 - t)^{\nu - 1} |G_{\nu}(t)| dt \leq \frac{\|G_{\nu}\|_{\infty}}{\Gamma(\nu + 1)}. \quad (2.58)$$

We have established the following remainder estimate.

**Theorem 2.8.** All here as in Theorem 2.4. Let $R_{\nu}$ be the remainder in (2.32), see (2.54), and $G_{\nu}$ as in (2.53). Then

$$|R_{\nu}| \leq \min \left\{ \frac{\|G_{\nu}\|_{L_1([0,1])}}{\Gamma(\nu)}, \frac{\|G_{\nu}\|_{L_p([0,1])}}{\Gamma(\nu)(p(\nu - 1) + 1)^{1/p}}, \frac{\|G_{\nu}\|_{L_2([0,1])}}{\Gamma(\nu)\sqrt{2\nu - 1}}, \frac{\|G_{\nu}\|_{\infty}}{\Gamma(\nu + 1)} \right\}. \quad (2.59)$$

where $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$.

**Comment.** The Chain Rule is not possible in fractional differentiation. That limits us a lot from using the multivariate fractional Taylor formula, as we employ the usual one involving only ordinary partial derivatives of functions.
REFERENCES


