Communications in Applied Analysis **11** (2007), no. 2, 269–283

MATHEMATICAL MODEL OF OPTIMUM DISTRIBUTION OF POPULATION INCOMES

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Communicated by S.I. Nenov

ABSTRACT: It is considered the model of the population dynamics in a region under business influence. The people income include a salary and some social payments. This income is distributed on the people: babies, young people, working people and old people. The birth rate and the survive rate depend on the income. The salary of the working people is determined based on maximization of the next year production. The model describes the demographic and economical systems in two variants. The first variant represent the equilibrium state, the second variant describes the system in dynamics. The computations demonstrate that the model is able to estimate the demography within region depending on economic activity.

AMS (MOS) Subject Classification: 49K15, 93A30

1. INTRODUCTION

The models discussed in this paper describe the income distribution impact on all the population groups and its age dynamics. This approach allows us to analyze the regional economic situation in terms of the population needs and income. The model proceeds from an assumption that a long history of family relationships results in income distribution patterns among family members as very close to our optimal one. Given this assumption, the model shows to what extent the income of the working population meets the living standard.

This work started with the paper of Raut and Srinivasan [4], which gave a simplistic model. We substantially updated the model, though the basic idea of analyzing the influence of economic activity on demographic properties in the region remained unchanged. People income determines their living standard, and in particular their health and vital capacity. The employed population receives its own income while the rest of the people have their share of income as a part of working people income. In the

Received February 17, 2007

long run the population dynamics depends on the income and its distribution. The paper describes these processes by two models: the first one presents the processes in real time dynamics while the other focuses on static and, equilibrium processes. The models below expand and develop the model presented by Abakumov and Giricheva [1].

2. DESCRIPTION OF THE MODEL

Section 2 discusses the model of regional development. Let us consider time as discretely measured, time step equals one year: $t = 1, \dots, T$. The people age is characterized by a parameter $\tau \in [0, \bar{\tau}]$, where $\bar{\tau}$ is the oldest possible age. The region population is described by a population density function which depends on parameter t: $x_t(\tau)$. The volume of production created by regional businesses in year t is described by a production function:

$$y_t = f(l_t, k_t). \tag{1}$$

Labor resources amount l_t at every time step t is determined as a total number of those involved in the production:

$$l_t = \int_0^{\bar{\tau}} r(\tau) x_t(\tau) d\tau, \tag{2}$$

where $r(\tau)$ is a function of population involved in production process.

The dynamics of produced capital k_t is described by the equation:

$$k_{t+1} = k_t(1 - \mu) + (c y_t - \alpha_t l_t), \tag{3}$$

where μ is an amortization ratio, c is the products price, and α_t is a salary tariff in year t.

Regional population incomes include a salary and some social payments. Let us take an extra income as a function of a person at the age τ in the year t as $p_t(\tau, \alpha_t)$; then function $q_t(\tau)$ describes the income of a person at the age τ after distribution of an extra income in a household. And population dynamics is described by the equations:

$$x_{t+1}(\tau) = x_{t+1}(0) e^{-\int_0^{\tau} m(\theta, q_t(\theta)) d\theta}, \quad \tau \in [0, 1); \tag{4}$$

$$x_{t+1}(\tau) = x_t(\tau - 1) e^{-\int_{\tau-1}^{\tau} m(\theta, q_t(\theta)) d\theta}, \quad \tau \in [1, \bar{\tau}], \tag{5}$$

where $m(\tau, q_t(\tau))$ is the function of population specific change. Equation (4) describes the population changes whose age does not exceed one year. The size of this population group in the following time step is determined based on a number of newly born

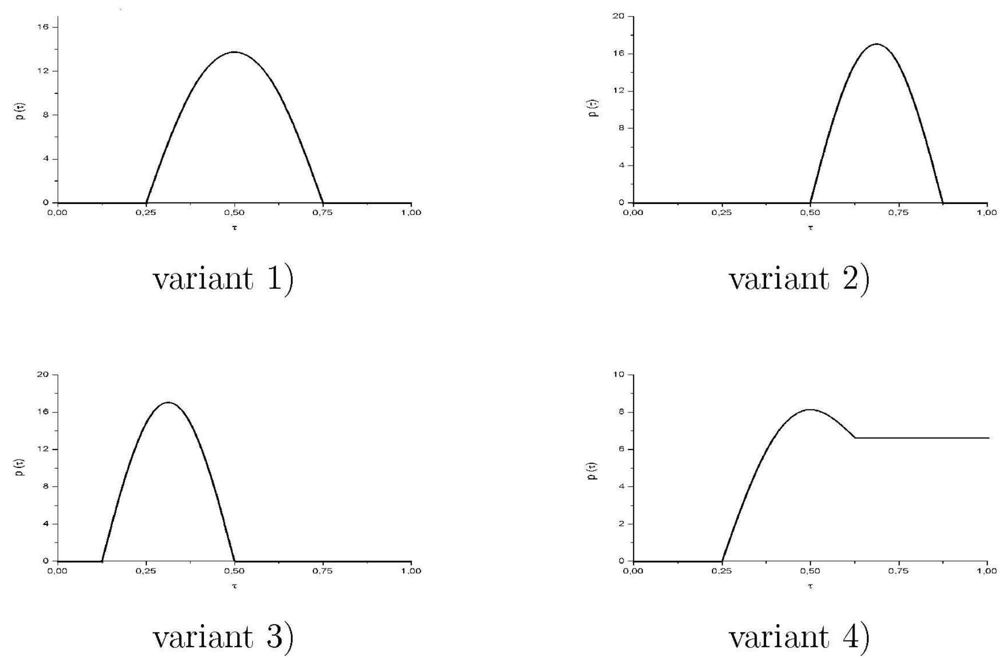

FIGURE 1. Function of an extra income $p(\tau)$

babies that same year:

$$x_{t+1}(0) = \int_0^{\bar{\tau}} b(\tau, q_t(\tau)) x_t(\tau) d\tau. \tag{6}$$

Here $b(\tau, q_t(\tau))$ is a birth rate function we assume to depend on the per capita income. Income utility q_t in the year t per capita at the age τ is expressed by the function $u(\tau, q_t(\tau))$.

Functions $x_t(\tau), p_t(\tau, \alpha_t), q_t(\tau), b(\tau, q_t), m(\tau, q_t), u(\tau, q_t)$ is taken as nonnegative and satisfying the following conditions for $\tau \in [0, \bar{\tau}]$, $q_t \in [0, +\infty)$ under $t \in [0, T]$.

Assumption 1. *$p_t(\tau, \alpha)$ and $q_t(\tau)$ are piecewise continuous functions, $x_t(\tau)$ is a piecewise smooth function of the first order, and $b(\tau, q)$ is a smooth function. $m(\tau, q)$, $u(\tau, q)$ functions are continuous, being twice continuously differential (perhaps except $q = 0$ point). Function $m(\tau, q)$ strictly decreases on q and is strictly convex on both arguments, while $\forall q \in [0, +\infty)$ has a minimum against τ at some point $\tilde{\tau} \in (0, \bar{\tau}), \lim\limits_{q \to \infty} m(\tau, q) = 0$. Function $u(\tau, q)$ strictly increases and is strictly concave against q, $u(\tau, 0) = 0$, $\lim\limits_{q \to 0} \frac{\partial u}{\partial q} = +\infty$ and $\lim\limits_{q \to \infty} \frac{\partial u}{\partial q} = 0$. The given functions $p(\tau), b(\tau, q), m(\tau, q), u(\tau, q)$ are not equal to zero almost everywhere.*

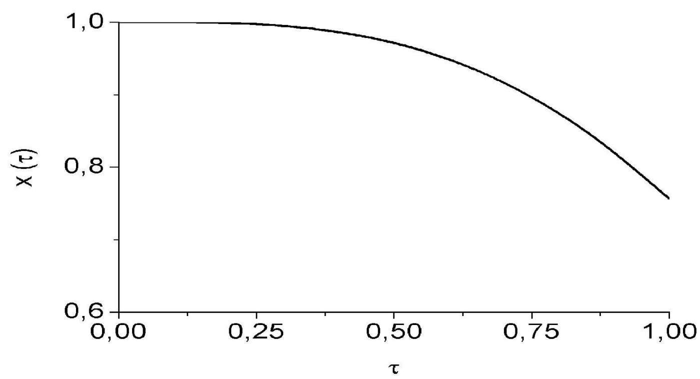

FIGURE 2. Density of population in basic variant

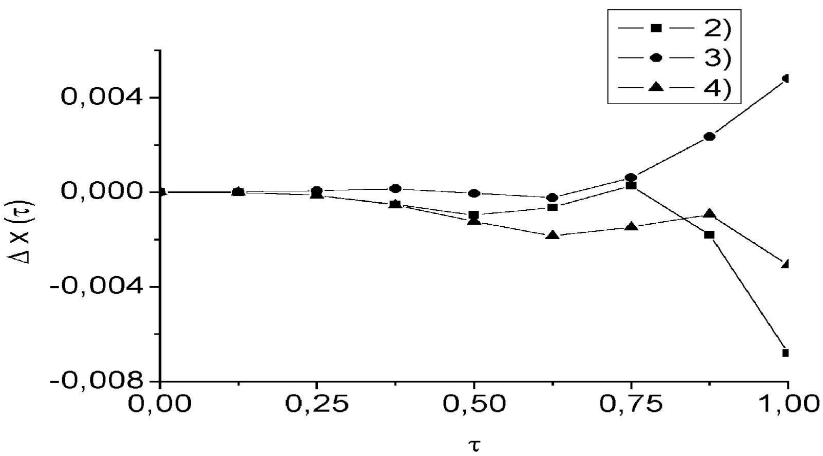

FIGURE 3. Relative deviations of population density in variants 2)-4) from the basic one

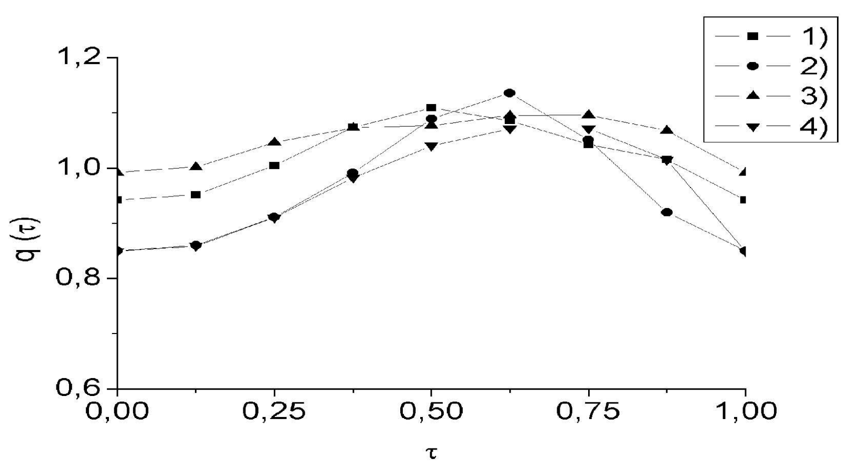

FIGURE 4. Distribution of personal income $\hat{q}(\tau)$ on age

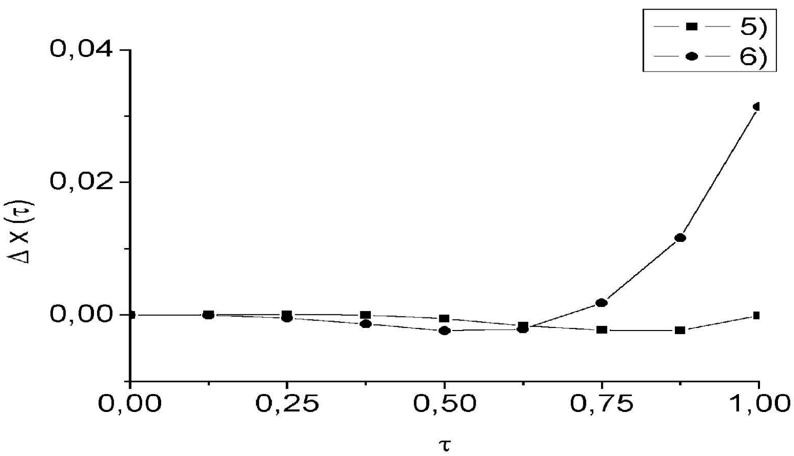

FIGURE 5. Relative deviations of the population density in variants 5)-6) against the basic values

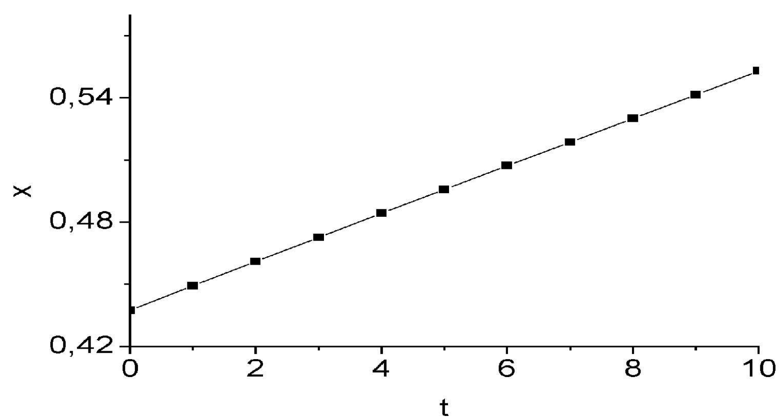

FIGURE 6. Dynamics of population in variant 1)

The optimal distribution of income $\hat{q}_t(\tau)$ at every time step t is determined based on maximization of the total utility

$$\int_0^{\bar{\tau}} u(\tau, q_t(\tau))x_t(\tau)d\tau \rightarrow \sup_{q_t(\tau)} \qquad q_t(\tau) \geq 0 \tag{7}$$

given the total extra and per capita incomes are equal

$$\int_0^{\bar{\tau}} [q_t(\tau) - p_t(\tau, \alpha_t)]x_t(\tau)d\tau = 0. \tag{8}$$

Solution of the problem depends on the parameter α_t: $\hat{q}_t(\tau) = q_t(\tau, \alpha_t)$. Then the population of the following time step depends on this index too: $x_{t+1}(\tau) = \overline{x}_{t+1}(\tau, \alpha)$. The amount of salary is determined based on maximization of the next

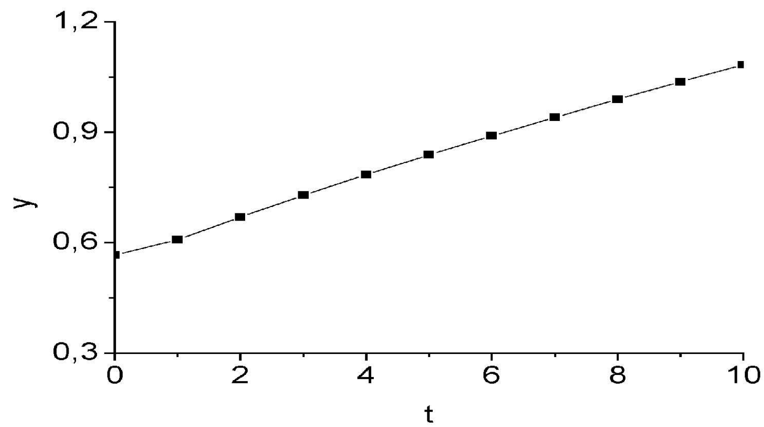

FIGURE 7. Dynamics of production volume in variant 1)

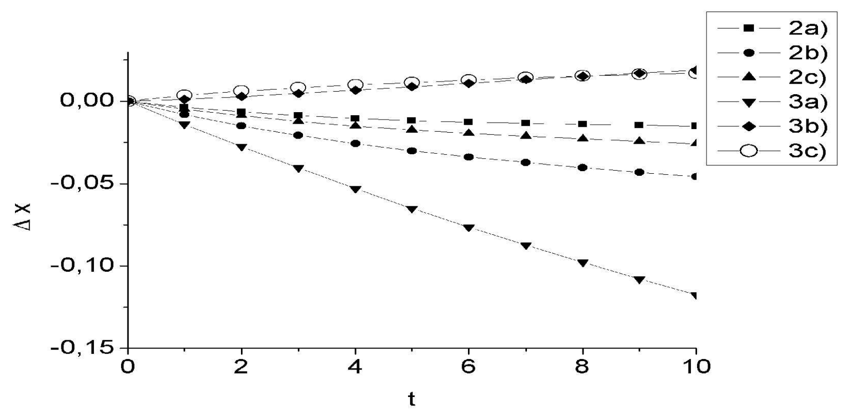

FIGURE 8. Dynamics of population relative deviation in variant 2)-3) vs. optimal population

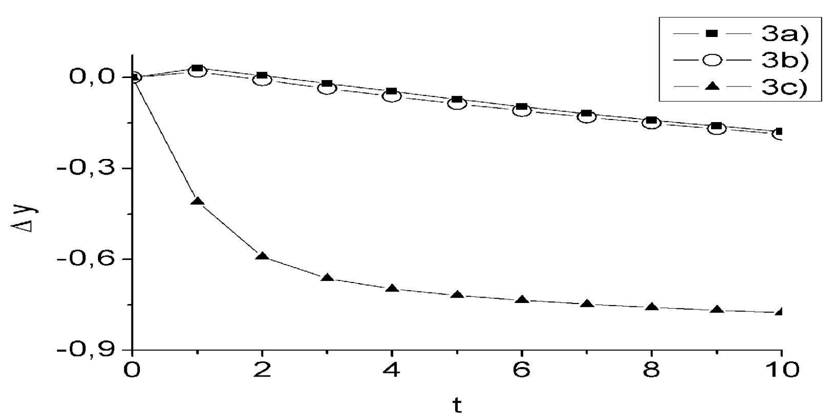

FIGURE 9. Relative deviation dynamics of the production volume in variant 3) vs. optimal volume

year production $y_{t+1} = f(l_{t+1}, k_{t+1})$:

$$\begin{gathered} f(l_{t+1}(\alpha_t), k_{t+1}(\alpha_t)) \to \sup_{\alpha_t} \\ l_{t+1}(\alpha_t) = \int_0^{\bar{\tau}} r(\tau)\bar{x}_{t+1}(\tau, \alpha_t)d\tau; \\ k_{t+1}(\alpha_t) = k_t(1-\mu) + (py_t - \alpha_t l_t) \\ \alpha_t \geq 0. \end{gathered} \tag{9}$$

Solutions $\hat{q}_t(\tau)$, $\hat{\alpha}_t$ of the problems (7)-(8), (9) determine the values of basic demographic and economic indices of region development at the next time step under the system optimal functioning.

Let us consider the equilibrium state of the system. In this case with the salary being a fixed value α, the problem is to find the optimal distribution of income among population $\hat{q}(\tau)$ and its respective equilibrium population distribution against age $\hat{x}(\tau)$ based on the problem of total utility maximization

$$\begin{gathered} \int_0^{\bar{\tau}} u(\tau, q(\tau))x(\tau)d\tau \to \sup_{q(\tau)} \\ q(\tau) \geq 0, \end{gathered} \tag{10}$$

$$\int_0^{\bar{\tau}} [q(\tau) - p(\tau, \alpha)]x(\tau)d\tau = 0, \tag{11}$$

$$\frac{dx}{d\tau} = -m(\tau, q(\tau))x(\tau), \quad x(0) = x_0, (x_0 > 0). \tag{12}$$

3. ALGORITHMS OF SOLVING THE PROBLEMS

Theorem. *Continuously differential functions $\hat{x}(\tau), \hat{q}(\tau)$ represent the optimal solutions of the problem (10)-(12) if and only if the following conditions are implemented under all $\tau \in [0, \bar{\tau}]$ for some continuously differential function $\psi(\tau)$ and a real number $\beta > 0$:*

$$\beta\frac{\partial u(\tau, \hat{q}(\tau))}{\partial q} - \psi\frac{\partial m(\tau, \hat{q}(\tau))}{\partial q} = 1; \tag{a}$$

$$\frac{d\psi}{d\tau} = -\beta u(\tau, \hat{q}(\tau)) + (\hat{q}(\tau) - p(\tau)) + \psi m(\tau, \hat{q}(\tau)), \quad \psi(\hat{\tau}) = 0; \tag{b}$$

$$\int_0^{\hat{\tau}} [\hat{q}(\tau) - p(\tau)]\hat{x}(\tau)d\tau = 0. \tag{c}$$

$$\frac{dx}{d\tau} = -m(\tau, q(\tau))x(\tau), \quad x(0) = x_0; \tag{d}$$

At that $\forall \tau \in [0, \bar{\tau}]$, $\hat{q}(\tau) > 0$ and the solution is unique.

The algorithm of finding the optimal solution $\hat{q}(\tau), \hat{x}(\tau)$ of the problem (7) follows from the conditions (a)-(d) of the theorem under fixed t and under equilibrium distribution of population based on age (10). Let us reduce relation (a) to the following form:

$$\psi(\tau) = \frac{\beta \frac{\partial u(\tau, \hat{q}(\tau))}{\partial q} - 1}{\frac{\partial m(\tau, \hat{q}(\tau))}{\partial q}}.$$

After differentiating the given relationship against τ, we get

$$\frac{d\psi}{d\tau} = \frac{\beta [\frac{\partial^2 u}{\partial q \partial \tau} + \frac{\partial^2 u}{\partial q^2} \frac{dq}{d\tau}] \frac{\partial m}{\partial q} - [\frac{\partial^2 m}{\partial q \partial \tau} + \frac{\partial^2 m}{\partial q^2} \frac{dq}{d\tau}] (\beta \frac{\partial u}{\partial q} - 1)}{(\frac{\partial m}{\partial q})^2}. \tag{13}$$

With consideration the equations (a), (13) we can rewrite the equation (b) in the following form:

$$\left.\begin{aligned} \frac{dq}{d\tau} = \frac{\beta \left[m \frac{\partial m}{\partial q} \frac{\partial u}{\partial q} - u (\frac{\partial m}{\partial q})^2 - \frac{\partial^2 u}{\partial q \partial \tau} \frac{\partial m}{\partial q} + \frac{\partial^2 m}{\partial q \partial \tau} \frac{\partial u}{\partial q} \right]}{\beta \left[\frac{\partial^2 u}{\partial q^2} \frac{\partial m}{\partial q} - \frac{\partial^2 m}{\partial q^2} \frac{\partial u}{\partial q} \right] + \frac{\partial^2 m}{\partial q^2}} + \\ + \frac{\left(\frac{\partial m}{\partial q} \right)^2 (q - p) - m \frac{\partial m}{\partial q} - \frac{\partial^2 m}{\partial q \partial \tau}}{\beta \left[\frac{\partial^2 u}{\partial q^2} \frac{\partial m}{\partial q} - \frac{\partial^2 m}{\partial q^2} \frac{\partial u}{\partial q} \right] + \frac{\partial^2 m}{\partial q^2}}, \\ \left. \frac{\partial u}{\partial q} \right|_{\tau = \bar{\tau}} = \frac{1}{\beta} \end{aligned} \right\}. \tag{14}$$

The density of population $x(\tau)$ is found from the condition (d):

$$x(\tau) = x_0 e^{-\int_0^\tau m(\theta, \hat{q}(\theta)) d\theta}. \tag{15}$$

Therefore to solve the problem means to find the solution $\hat{q}(\tau), \hat{x}(\tau)$ which satisfies conditions (14), (15) and the following equation:

$$I(\hat{q}, \hat{x}) = 0, \tag{16}$$

where

$$I(q, x) = \int_0^{\bar{\tau}} [q(\tau) - p(\tau)] x(\tau) d\tau. \tag{17}$$

We solve this problem by the method of successive approximations using the Runge-Kutta method (see Samarskiy and Gulin [5]) when solving differential equations. We use a similar approach while solving the dynamic problems (4)-(8) and (9).

4. COMPUTATIONAL EXPERIMENT

The age of people is determined in such conventional units that $\tau \in [0, 1]$ where 1 correspondents to 80 years. Functions of utility and population specific change are selected based on the traditional concepts. Utility function is selected to be as follows:

$$u(\tau, q) = u_0 q^\nu, \quad \nu \in (0, 1),$$

Variant	1	2	3	4	5	6
Total utility	0.952	0.926	0.965	0.925	0.951	0.941
Total population	0.938	0.937	0.939	0.937	0.937	0.940

Table 1. Total utility and population

while function of population specific change is selected as follows:

$$m(\tau, q) = m_0[(\tau - \tau_0)^{\delta_1} + m_1](q + q_0)^{-\delta_2} \tag{18}$$

with $\delta_1 = 2.0$, $\delta_2 = 0.5$, $q_0 = 0.01$, $m_0 = 1$, $m_1 = 10^{-6}$, $\tau_0 = 0.05$. Extra income, personal income and utility are determined in specific units. Function $p(\tau)$ is selected in several variants under the condition

$$\int_0^{\bar{\tau}} p(\tau)d\tau = 1. \tag{19}$$

Basic variant is 1). It assumes the age of working people to be in the range of 20 to 60 years old. The variant 2) considers the age to be in the range of 40 to 70, and the variant 3) covers the working people age from 10 to 40 years old. These variants are focused on income received as a salary. The variant 4) also accounts for additional income such as pensions, dividends, etc. (Figure 1). Then population is determined in such specific units that $x_0 = 1$.

To compare against optimal solution of variant 1), we considered two cases of non-optimal distribution of income $q(\tau)$ in terms of the basic variant:

5) $q(\tau) = q_1$;

6) $q(\tau) = q_2 + k\tau$.

We chose $q_2 = 0.5$ in these variants while parameters q_1 and k are determined based on condition (16).

Optimal distribution of population on age $\hat{x}_1(\tau)$ of basic variant 1) is represented in specific units (Figure 2). For other variants the density of population $x_i(\tau)$ ($i = 2, \ldots, 6$) is given as relative deviations from the basic one (Figures 3 and 5).

$$\triangle x_i(\tau) = \frac{\hat{x}_i(\tau) - \hat{x}_1(\tau)}{\hat{x}_1(\tau)}. \tag{20}$$

For all the variants we calculated a total utility $\int_0^{\bar{\tau}} u(\tau, q(\tau))x(\tau)d\tau$ and total population $\int_0^{\bar{\tau}} x(\tau)d\tau$ (Table 1).

In the optimal variants 1)-4) the age reduction of people receiving incomes favorably influences on the total utility and population. This impact agrees with equation (c). All cases show optimal distribution of personal income $\hat{q}(\tau)$ with age to have a more uniform distribution than that of an extra income $p(\tau)$ (Figure 4). In two non-optimal variants 5)-6) values of total utility are less than optimal. Variant 6) demonstrates the total population value to be more than optimal due to an increase of the aged people under the given income distribution (Figure 5). All the variants discussed show the population density $x(\tau)$ as deviating from the basic values which agrees with the income $q(\tau)$ deviations against the basic one.

Let us consider behavior of the system in dynamics. Economic and demographic characteristics are determined in specific units. Let production be described by the Cobb-Douglas function

$$f(l,k) = f_0 l^\gamma k^{1-\gamma}, \quad \gamma = 3/4.$$

Function of specific population change is chosen to have the form (18) similar to that of equilibrium case. Extra income is determined as $p(\tau,\alpha) = \alpha r(\tau)$, where population involved in the production $r(\tau)$ corresponds to the basic variant of the equilibrium model. Birth rate function is described in the following way:

$$b(\tau,q) = \begin{cases} 0, & \tau < \tau_1; \\ b_0(\tau-\tau_1)^2(\tau_2-\tau)^2, & \tau_1 \le \tau \le \tau_2; \\ 0, & \tau > \tau_2, \end{cases} \tag{21}$$

where $\tau_1 = \dfrac{1}{4}$, $\tau_2 = \dfrac{1}{2}$, and parameter b_0 is calculated from the condition

$$\int_0^{\bar\tau} b(\tau, q(\tau))d\tau = \tilde b, \quad \tilde b = 0.015. \tag{22}$$

Where $\tilde b$ is the number of children born by one individual over a year. Income utility is described by the function

$$u(\tau,q) = u_0 q^{\nu(\tau)}, \quad \nu(\tau) = \nu_1 - \nu_2\tau, \tag{23}$$

$\nu_1 = 0.3$, $\nu_2 = 0.06$. Let us assume that density of population $x^0(\tau)$ at the initial time $t = 0$ is equal to the optimal distribution $\hat x_1(\tau)$ for the basic variant in the equilibrium model.

Let us consider the following variants of the system functioning. An optimal variant 1) proceeds from the assumption that income distribution among the population and the salary tariff at every time step t are chosen as the solutions of respective optimization problems. Variant 2) determines the salary tariff $\hat\alpha_t$ based on the terms of production maximization, while the income distribution among the population is the following. Variant 2a) corresponds to a uniform distribution of income among all the population; 2b) is a variant with children under the age of 15 getting the maximum of an income, and the remaining income being equally divided among the rest of the people; variant 2c) provides the largest share of income for the working people, aged 30 to 60 while the remaining part of income is equally distributed among children, young and retired people. Variant 3) determines the income distribution among the population as a solution of optimization problem, wile the salary tariff is the following. In variant 3a) this ratio is much lower than an optimal level; in variants 3b) and 3c) it amount to 10 and 90 percent of the capital, respectively. Dynamics of population and the volume of production for 10 years are given in Figures 6-7, variant 1), while Figures 8-9 show variants 2)-3) in their relative deviations from the ratio values of variant 1).

In the case of non-optimal distribution of income total population is lower than in variant 1). The case 2b) shows most deviations from the optimal value then other variants (Figure 8). Under this distribution the individuals at a fertile age get minimal income. It results in lower number of newborns and makes the demographic situation worse as a whole. Positive deviation from optimal values of population correspond to variants 3b), 3c). In these cases salaries are much higher than the optimal value, which results in population increase, though, leads to production recession (Figure 9).

5. CONCLUSION

Two models describing the population age structure against its economic activity are developed. Typical peculiarities of the models are the describing of the distribution of the incomes received by working people among all the groups of population. This results in optimization problems with a few criteria. The first model describes the dynamics of the processes while the second one focuses on their equilibrium state.

The calculations are given for equilibrium and dynamic cases. The results obtained allow us to prove the feasibility of optimization criteria we assumed. The models demonstrate the strategies of management effects for the demography improvement within the region.

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6. APPENDIX

6.1. **PROOF OF THEOREM.** Necessity. Conditions c), d) directly correspond to those of equations (11)-(12). Let us prove b). We apply the Pontryagin maximum principle (see Pontryagin et al [3]) to the problem (10)-(12). Hamiltonian has the form $H(\tau, x, q, \beta, \gamma, \psi) = h(\tau, q, \beta, \gamma, \psi)x(\tau)$, where

$$h(\tau, q, \beta, \gamma, \psi) = \beta u(\tau, q(\tau)) + \gamma[q(\tau) - p(\tau)] - \psi(\tau)m(\tau, q(\tau)), \tag{24}$$

with β, γ as constants, and $\beta \geq 0$, $\psi(\tau)$ as the continuous solution of differential equation

$$\frac{d\psi}{d\tau} = -h(\tau, q, \beta, \gamma, \psi) \tag{25}$$

under condition $\psi(\bar{\tau}) = 0$. That is equation b).

Now let us discuss condition a). For any $\tau \in [0, \bar{\tau}]$ the optimal value $q(\tau)$ is determined from the maximization condition of $h(\tau, q, \beta, \gamma, \psi)$ on q.

First we determine the sign of constant γ. Note that $\beta = \gamma = 0$ case is impossible. Otherwise from the equation (25) under $\psi(\bar{\tau}) = 0$ gives us $\psi(\tau) \equiv 0$ for $\tau \in [0, \bar{\tau}]$, which contradicts the non-triviality condition of the Pontryagin maximum principle. If $\gamma \geq 0$ holds, and least one of the numbers $\beta, \gamma \neq 0$, then function $h(\bar{\tau}, q(\bar{\tau}), \beta, \gamma, \psi(\bar{\tau}))$ does not reach the maximum on q under finite value. It is contrary to the condition of optimal solution existence. From this $\gamma < 0$ follows. Using linearity of Hamiltonian H regarding β, γ, ψ, we assume $\gamma = -1$. Formula (24) then takes the form

$$h(\tau, q, \beta, \psi) = \beta u(\tau, q(\tau)) - [q(\tau) - p(\tau)] - \psi(\tau)m(\tau, q(\tau))$$

(in marking h, $\gamma = -1$ is omitted).

Then

$$\frac{\partial h}{\partial q} = \beta\frac{\partial u(\tau, q)}{\partial q} - 1 - \psi(\tau)\frac{\partial m(\tau, q)}{\partial q}, \tag{26}$$

$$\frac{\partial^2 h}{\partial q^2} = \beta\frac{\partial^2 u(\tau, q)}{\partial q^2} - \psi(\tau)\frac{\partial^2 m(\tau, q)}{\partial q^2}. \tag{27}$$

Let us find out properties of $\psi(\tau)$ function. If at some point $\tau_1 \in [0, \bar{\tau}]$ $\psi(\tau_1) < 0$, then $h(\tau_1, \hat{q}, \beta, \psi) \geq h(\tau_1, 0, \beta, \psi) \geq 0$, and from (25): $\frac{d\psi}{d\tau}\Big|_{\tau=\tau_1} \leq 0$. In this case condition $\psi(\bar{\tau}) = 0$ is not true. Therefore $\forall \tau \in [0, \bar{\tau}]$ $\psi(\tau) \geq 0$.

Let us research the case $\beta = 0$. We multiply both parts of equation (b) by $x(\tau)$ and applying equation (c) we receive

$$\frac{d(\psi x)}{d\tau} = -\beta u(\tau, q(\tau))x(\tau) + [q(\tau) - p(\tau)]x(\tau).$$

This equation integration on $\tau \in [0, \bar{\tau}]$ with boundary condition (b) and equation (c), results in relation

$$\psi(0)x_0 = \beta \int\limits_0^{\bar{\tau}} u(\tau, q(\tau))x(\tau)d\tau.$$

Under $\beta = 0$ we get $\psi(0) = 0$ and $h(0, \hat{q}(0), 0, 0) \geq h(0, 0, 0, 0) \geq 0$. Then from equation (25) it follows that $\frac{d\psi}{d\tau} \leq 0$ and in a similar way we get $\psi(\tau) \equiv 0 \; \forall \tau \in [0, \bar{\tau}]$. From this $q(\tau) \equiv 0$ it follows then that condition (c) is not true. Therefore $\beta > 0$. Then function h is strictly concave on q since $\frac{\partial^2 h}{\partial q^2} < 0$ in formula (27). Then from the Assumption 1 we receive $\frac{\partial h}{\partial q} \underset{q \to 0}{\longrightarrow} +\infty$, $\frac{\partial h}{\partial q} \underset{q \to \infty}{\longrightarrow} -1$. From this and from equation $\frac{\partial h(\tau, \hat{q}(\tau), \beta, \psi(\tau))}{\partial q} = 0$ the existence of unique $\hat{q}(\tau) > 0$ for $\forall \tau \in [0, \bar{\tau}]$ follows. Point of maximum $\hat{q}(\tau)$ of the function h on q is an optimal solution. In this case equation $\frac{\partial h}{\partial q} = 0$ is the condition (a).

Sufficiency. While proving sufficiency we use the technique indicated by Mazalov and Rettieva [2]. Conditions (c), (d) correspond to (11), (12). We have to prove (10). Let $q(\tau)$, $x(\tau)$ is any other pair of acceptable functions satisfying (10)-(12). It follows from (a), an Assumption 1 and formula (26) that $h(\tau, q, \alpha, \psi)$ reaches maximum on q under $q(\tau) = \hat{q}(\tau)$ for any $\tau \in [0, \bar{\tau}]$. Then (keep in mind that $x(\tau) \geq 0$)

$$h(\tau, q, \beta, \psi)x \leq h(\tau, \hat{q}, \beta, \psi)x = h(\tau, \hat{q}, \beta, \psi)\hat{x}$$
$$+h(\tau, \hat{q}, \beta, \psi)(x - \hat{x}) = h(\tau, \hat{q}, \beta, \psi)\hat{x} - \frac{d\psi}{d\tau}(x - \hat{x}).$$

The last equality uses equation (b) or a similar equation (25) which in fact makes no difference.

Integrating on $\tau \in [0, \tau]$ the initial and finite parts of the unequality obtained we get

$$\beta \int\limits_0^{\bar{\tau}} u(\tau, q(\tau))x(\tau)d\tau - \int\limits_0^{\bar{\tau}} \psi(\tau)m(\tau, q(\tau))x(\tau)d\tau$$

$$\leq \beta \int\limits_0^{\bar{\tau}} u(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau - \int\limits_0^{\bar{\tau}} \psi(\tau)m(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau$$

$$- \int\limits_0^{\bar{\tau}} \frac{d\psi}{d\tau}(x(\tau) - \hat{x}(\tau))d\tau = \beta \int\limits_0^{\bar{\tau}} u(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau$$

$$- \int\limits_0^{\bar{\tau}} \psi(\tau)m(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau$$

$$-[\psi(\tau)(x(\tau) - \hat{x}(\tau))] \Big|_0^{\bar{\tau}} + \int\limits_0^{\bar{\tau}} \psi(\tau) \left(\frac{dx}{d\tau} - \frac{d\hat{x}}{d\tau} \right) d\tau$$

$$= \beta \int_0^{\bar\tau} u(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau - \int_0^{\bar\tau} \psi(\tau)m(\tau, q(\tau))x(\tau)d\tau,$$

(the last equality uses conditions (b) and (d)).

Since $\beta > 0$, then it follows from initial and finite parts of these transformations that

$$\int_0^{\bar\tau} u(\tau, q(\tau))x(\tau)d\tau \le \int_0^{\bar\tau} u(\tau, \hat{q}(\tau))\hat{x}(\tau)d\tau,$$

which means then the condition (10). Sufficiency has been proved.

Uniqueness. Let us assume that two optimal solutions $q_1(\tau)$, $x_1(\tau)$ and $q_2(\tau)$, $x_2(\tau)$ exist:

$$\int_0^{\bar\tau} u(\tau, q_1(\tau))x_1(\tau)d\tau = \int_0^{\bar\tau} u(\tau, q_2(\tau))x_2(\tau)d\tau. \tag{28}$$

Then under some β_1 and $\psi_1(\tau)$ satisfying the conditions (a), (b) under $\hat{q}(\tau) = q_1(\tau)$

$$\begin{aligned}\int_0^{\bar\tau} h(\tau, q_1(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau = \beta_1 \int_0^{\bar\tau} u(\tau, q_1(\tau))x_1(\tau)d\tau \\ - \int_0^{\bar\tau} \psi_1(\tau)m(\tau, q_1(\tau))x_1(\tau)d\tau.\end{aligned} \tag{29}$$

Applying the same approach we used while proving sufficiency,

$$h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))x_1(\tau) = h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))x_2(\tau)$$
$$-\frac{d\psi_1(\tau)}{d\tau}(x_1(\tau) - x_2(\tau)),$$

we have

$$\begin{aligned}\int_0^{\bar\tau} h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau = \beta_1 \int_0^{\bar\tau} u(\tau, q_2(\tau))x_2(\tau)d\tau \\ - \int_0^{\bar\tau} \psi_1(\tau)m(\tau, q_1(\tau))x_1(\tau)d\tau.\end{aligned} \tag{30}$$

Comparing (29) and (30) and taking into account (28) we get

$$\int_0^{\bar\tau} h(\tau, q_1(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau = \int_0^{\bar\tau} h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau. \tag{31}$$

Let $\tau_1 \in [0, \bar\tau]$ is a point in which $q_1(\tau_1) \ne q_2(\tau_1)$. Due to its continuity this inequality is also true for the points belonging to some neighborhood of point τ_1. Then in this neighborhood $h(\tau, q_1(\tau), \beta_1, \psi_1(\tau)) > h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))$ holds as based on what we proved above, the maximum of function $h(\tau, q(\tau), \beta_1, \psi_1(\tau))$ is unique on q. Under this condition $\forall\tau \in [0, \bar{\tau}]$ $h(\tau, q_1(\tau), \beta_1, \psi_1(\tau)) \geq h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))$. Therefore

$$\int\limits_0^{\bar{\tau}} h(\tau, q_1(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau > \int\limits_0^{\bar{\tau}} h(\tau, q_2(\tau), \beta_1, \psi_1(\tau))x_1(\tau)d\tau,$$

which contradicts relationship (31). It also means that $\forall\tau \in [0, \bar{\tau}]$ $q_1(\tau) = q_2(\tau)$. The theorem has been proved.