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STRONG NONLINEAR LIMIT-POINT/LIMIT-CIRCLE PROPERTIES FOR A CLASS OF FOURTH ORDER EQUATIONS

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ABSTRACT. The authors consider the nonlinear fourth order differential equation with p-Laplacian

$$\big(a(t)|y''|^{p-1}y''\big)'' + r(t)|y|^{\lambda-1}y = 0 \tag{E}$$

where $p > 0$, $\lambda > 0$, $a(t) > 0$, and $r(t) > 0$. Asymptotic properties of solutions are studied including the nonlinear limit-point/limit-circle and the strong nonlinear limit-point/limit-circle properties. Examples illustrating the results are also included.

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1. INTRODUCTION

Consider the differential equation

$$\big(a(t)|y''|^{p-1}y''\big)'' + r(t)|y|^{\lambda-1}y = 0 \tag{1.1}$$

where a, r are continuous functions, $p > 0$, $\lambda > 0$, $a(t) > 0$, and $r(t) > 0$ on $\mathbb{R}_+ = [0, \infty)$.

Definition 1.1. A function $y \in C^2\big([0,\tau)\big)$, $\tau \le \infty$, is called a *solution* of (1.1) if $a(t)|y''(t)|^{p-1}y''(t) \in C^2\big([0,\tau)\big)$ and (1.1) holds.

We will study solutions on their maximal interval of existence $[0,\tau)$, $\tau \le \infty$. If $\tau < \infty$, then y is called *noncontinuable*. A continuable solution y is called *proper* if it is nontrivial in any neighbourhood of ∞. We denote by $\mathcal{S}$ the set of all solutions of (1.1) that are continuable.

It is easy to see that (1.1) is equivalent to the nonlinear system

$$\begin{cases} y_1' = y_2, \\ y_2' = a^{-\frac{1}{p}}(t)|y_3|^{\frac{1}{p}} \operatorname{sgn} y_3, \\ y_3' = y_4, \\ y_4' = -r(t)|y_1|^{\lambda} \operatorname{sgn} y_1. \end{cases} \tag{1.2}$$

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The relation between a solution y of (1.1) and $(y_1,\dots,y_4)$ of (1.2) is

$$y_1 = y, \quad y_2 = y', \quad y_3 = a(t)|y''|^{p-1}y'', \quad y_4 = \bigl(a(t)|y''|^{p-1}y''\bigr)'. \tag{1.3}$$

Throught this paper as we consider a solution y of (1.1), we will employ the relationships in (1.3) without further mention. The following types of asymptotic properties of solutions will be investigated here.

Definition 1.2. Let y be a continuable solution of (1.1). Then y is called *oscillatory* if there exists a sequence $\{t_k\}_{k=1}^{\infty}$ of zeros of y tending to ∞ and y is nontrivial in any neighborhood of ∞. Otherwise, y is called *nonoscillatory.* In particular, a solution $y \equiv 0$ in a neighbourhood of ∞ is nonoscillatory. A solution y is said to be *strongly oscillatory* if the set of zeros of y has no finite accumulation point in its interval of definition.

Definition 1.3. A continuable solution y of Equation (1.1) is said to be of the *nonlinear limit-circle* type if

$$\int_0^{\infty} \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma < \infty\,, \tag{NLC}$$

and it is said to be of *nonlinear limit-point* type otherwise, i.e., if

$$\int_0^{\prime\infty} \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma = \infty\,. \tag{NLP}$$

Equation (1.1) will be said to be of the *nonlinear limit-circle* type if every continuable solution y of (1.1) satisfies (NLC), and it will be said to be of the *nonlinear limit-point* type if there is at least one continuable solution y for which (NLP) holds.

The nonlinear limit-point/limit-circle problem is described in detail in the monograph by Bartušek, Došlá, and Graef [5]. A discussion of the relationship between these properties and other asymptotic properties of solutions such as oscillation and convergence to zero can be found there as well.

The following definitions are new. The notions of "strong" nonlinear limit-point and "strong" nonlinear limit-circle type solutions were first introduced for second order equations in [7] and [8], respectively, and subsequently used in [6, 9, 10, 11].

Definition 1.4. A continuable solution y of (1.1) is said to be of the *strong nonlinear limit-circle* type if

$$\int_0^{\infty} \bigl|y(\sigma)\bigr|^{\lambda+1}\, d\sigma < \infty \quad \text{and} \quad \int_0^{\infty} \frac{a(\sigma)}{r(\sigma)}\bigl|y''(\sigma)\bigr|^{p+1}\, d\sigma < \infty\,.$$

Equation (1.1) is said to be of the *strong nonlinear limit-circle* type if every continuable solution is of the strong nonlinear limit-circle type.

Definition 1.5. A continuable solution y of (1.1) is said to be of the *strong nonlinear limit-point* type if

$$\int_0^\infty \big|y(\sigma)\big|^{\lambda+1}\,d\sigma=\infty \quad\text{and}\quad \int_0^\infty \frac{a(\sigma)}{r(\sigma)}\big|y''(\sigma)\big|^{p+1}\,d\sigma=\infty\,.$$

Equation (1.1) is said to be of the *strong nonlinear limit-point* type if equation (1.1) has proper solutions and every one of these is of the strong nonlinear limit-point type.

The following section contains some preliminary results while Section 3 contains the main results in the paper. Some examples to illustrate our theorems are also included.

2. PROPERTIES OF SOLUTIONS

In this section, asymptotic properties of solutions of the types listed above will be studied. We begin with the following classification of solutions.

Denote by $\mathcal{O}\subset\mathcal{S}$ ($\mathcal{N}\subset\mathcal{S}$) the set of all oscillatory (nonoscillatory) solutions of (1.1).

Definition 2.1. (i) Let $\mathcal{O}_1\subset\mathcal{O}$ be the set of those solutions $y\in\mathcal{O}$ for which there are sequences $\{t_k^i\}_{k=1}^\infty$, $i=1,2,3,4$, such that

$$\begin{gathered} t_k^1<t_k^4<t_k^3<t_k^2<t_{k+1}^1\,,\quad \lim_{k\to\infty}t_k^1=\infty\,,\\ y_j(t_k^j)=0\,,\quad j=1,2,3,4\,,\\ y_i(t_k^j)\neq 0\,,\quad j=1,2,3,4\,,\quad i\neq j,\\ y_i(t)y_1(t)>0\quad\text{for}\quad t\in(t_k^1,t_k^i)\,,\quad i=2,3,4\,,\quad\text{and}\\ y_i(t)y_1(t)<0\quad\text{for}\quad t\in(t_k^i,t_{k+1}^1)\,,\quad i=2,3,4\,. \end{gathered}\tag{2.1}$$

(ii) Let $\mathcal{O}_2\subset\mathcal{O}$ be the set of those solutions $y\in\mathcal{O}$ for which there are sequences $\{t_k^i\}_{k=1}^\infty$, $i=1,2,3,4$, such that

$$\begin{gathered} t_k^1<t_k^2<t_k^3<t_k^4<t_{k+1}^1\,,\quad \lim_{k\to\infty}t_k^1=\infty\,,\\ y_j(t_k^j)=0\,,\quad j=1,2,3,4\,,\\ y_i(t_k^j)\neq 0\,,\quad j=1,2,3,4\,,\quad i\neq j,\\ (-1)^i y_i(t)y_1(t)>0\quad\text{for}\quad t\in(t_k^1,t_k^i),\quad i=2,3,4\,,\quad\text{and}\\ (-1)^i y_i(t)y_1(t)<0\quad\text{for}\quad t\in(t_k^i,t_{k+1}^1)\,,\quad i=2,3,4\,. \end{gathered}\tag{2.2}$$

Definition 2.2. The complete classification of nonoscillatory solutions of (1.1) is as follows.

(i) Let $\mathcal{N}_1\subset\mathcal{N}$ be the set of those solutions $y\in\mathcal{N}$ for which there exists $t_y\in\mathbb{R}_+$ with the property

$$y_i(t)y_1(t)>0\quad\text{on}\quad[t_y,\infty)\quad\text{for}\quad i=2,3,4\,.\tag{2.3}$$

(ii) Let $\mathcal{N}_2 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_j(t)y_1(t) > 0 \quad &\text{for} \quad j = 2, 3, \quad \text{and} \\ y_1(t)y_4(t) < 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned}$$

(iii) Let $\mathcal{N}_3 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_i(t)y_1(t) > 0 \quad &\text{for} \quad i = 2, 4\,, \quad \text{and} \\ y_3(t)y_1(t) < 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned} \tag{2.4}$$

(iv) Let $\mathcal{N}_4 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_j(t)y_1(t) < 0 \quad &\text{for} \quad j = 3, 4, \quad \text{and} \\ y_1(t)y_2(t) > 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned}$$

(v) Let $\mathcal{N}_5 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_i(t)y_1(t) > 0 \quad &\text{for} \quad i = 3, 4\,, \quad \text{and} \\ y_2(t)y_1(t) < 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned} \tag{2.5}$$

(vi) Let $\mathcal{N}_6 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_j(t)y_1(t) < 0 \quad &\text{for} \quad j = 2, 4, \quad \text{and} \\ y_1(t)y_3(t) > 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned}$$

(vii) Let $\mathcal{N}_7 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$\begin{aligned} y_j(t)y_1(t) < 0 \quad &\text{for} \quad j = 2, 3, \quad \text{and} \\ y_1(t)y_4(t) > 0 \quad &\text{for} \quad t \in [t_y, \infty) . \end{aligned}$$

(viii) Let $\mathcal{N}_8 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ for which there exists $t_y \in \mathbb{R}_+$ with the property

$$y_j(t)y_1(t) < 0 \quad \text{for} \quad j = 2, 3, 4, \quad \text{and} \quad t \in [t_y, \infty) .$$

(ix) Let $\mathcal{N}_9 \subset \mathcal{N}$ be the set of those solutions $y \in \mathcal{N}$ that are trivial in a neighborhood of ∞.

Our first two lemmas show, among other things, that some of the sets in our classifications of solutions are empty.

Lemma 2.3. *The following statements hold.*

(i) $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ *and* $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \mathcal{N}_4 \cup \mathcal{N}_5 \cup \mathcal{N}_9$.

(ii) *If* $\int\limits_0^\infty a^{-\frac{1}{p}}(s)\,ds = \infty$, *then* $\mathcal{N}_4 = \emptyset = \mathcal{N}_5$, *so* $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_3 \cup \mathcal{N}_9$.

(iii) *If* $\int\limits_0^\infty a^{-\frac{1}{p}}(s)\,ds = \infty$ *and* $\int\limits_0^\infty r(s)\,ds = \infty$, *then* $\mathcal{N} = \mathcal{N}_9$.

Proof. (i) According to Theorem 3 in [3], for any $y \in \mathcal{O}$ there exists a neighborhood of ∞ in which the zeros of y have no accumulation point. Hence, Theorem 2 in [3] implies y is strongly oscillatory, and so $\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2$ by the Lemma in [1]. To show that $\mathcal{N}_2 = \emptyset = \mathcal{N}_6$, assume that $y(t) > 0$ for $t \ge t_y$ and integrate y_3' and y_4' to obtain

$$y_3(t) = y_3(t_y) + \int_{t_y}^t \left[y_4(t_y) - \int_{t_y}^s r(\sigma)|y_1(\sigma)|^\lambda\,d\sigma \right] ds$$
$$\le y_3(t_y) + y_4(t_y) \int_{t_y}^t d\sigma \to -\infty$$

as $t \to \infty$. This contradicts the fact that $y_3(t) > 0$ for both of these classes. Using y_1' and y_2' in a similar way, we can show that $\mathcal{N}_7 = \emptyset = \mathcal{N}_8$.

The proof of (ii) is similar to that of part (i) only using y_2' and y_3'. Part (iii) follows from Theorem 1 in [2]. □

For any solution y of (1.1), we define the function F by

$$F(t) = -y_4(t)y_1(t) + y_2(t)y_3(t)\,. \tag{2.6}$$

Lemma 2.4. (i) *Let* y *be a solution of* (1.1) *defined on* $[0, \tau)$. *Then*

$$F'(t) = r(t)|y(t)|^{\lambda+1} + a(t)|y''(t)|^{p+1}\,, \quad t \in [0, \tau)\,;$$

hence, F *is a nondecreasing function for any solution* y *of* (1.1).

(ii) *If* $y \in \mathcal{O}_1$, *then* $\lim\limits_{t\to\infty} F(t) \in (0, \infty]$.

(iii) *If* $y \in \mathcal{O}_2$, *then* $\lim\limits_{t\to\infty} F(t) \in (-\infty, 0]$.

(iv) *If* $y \in \mathcal{N}_3 \cup \mathcal{N}_5 \cup \mathcal{N}_9$, *then* $\lim\limits_{t\to\infty} F(t) \in (-\infty, 0]$.

Proof. (i) Clearly, $F'(t) \ge 0$ by (1.2) and (1.3).

(ii) In view of (2.6) and (2.1), $F(t_k^1) = y_2(t_k^1)y_3(t_k^1) > 0$ for $k \in \{2, 3, \dots\}$, and the conclusion follows from case (i).

(iii) Similarly, (2.6) and (2.2) yield $F(t_k^1) = y_2(t_k^1)y_3(t_k^1) < 0$ for $k = 2, 3, 4, \dots$, and since F is nondecreasing, we have $\lim\limits_{t\to\infty} F(t) \le 0$.

(iv) If $y \in \mathcal{N}_3 \cup \mathcal{N}_5$, then (2.6), (2.4), and (2.5) yield $F(t) < 0$ for $t \ge t_y$, and the conclusion follows. If $y \in \mathcal{N}_9$, then $\lim\limits_{t\to\infty} F(t) = 0$. □

The next lemma gives some preliminary results on the strong nonlinear limit-circle property.

Lemma 2.5. *Let $y \in \mathcal{S}$ satisfy $\lim_{t\to\infty} F(t) < \infty$.*

(i) *Then*

$$\int_0^\infty r(t)|y(t)|^{\lambda+1}\,dt < \infty \quad \text{and} \quad \int_0^\infty a(t)|y''(t)|^{p+1}\,dt < \infty\,. \tag{2.7}$$

(ii) *If there exist positive constants r_0 and a_0 such that*

$$r_0 \le r(t) \quad \text{and} \quad a_0 \le a(t) \quad \text{on} \quad \mathbb{R}_+\,, \tag{2.8}$$

then

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty} y'(t) = 0\,, \tag{2.9}$$

$$\int_0^\infty |y(t)|^{\lambda+1}\,dt < \infty, \quad \text{and} \quad \int_0^\infty \frac{a(t)}{r(t)}|y''(t)|^{p+1}\,dt < \infty\,. \tag{2.10}$$

Proof. (i) The conclusion follows directly from Lemma 2(i) by an integration on $\mathbb{R}_+$.

(ii) Property (2.10) follows from (2.7) and (2.8). Furthermore, (2.7) and (2.8) yield $y \in L_{\lambda+1}(\mathbb{R}_+)$ and $y'' \in L_{p+1}(\mathbb{R}_+)$. Hence, Lemma 1.5 in [15] yields (2.9). □

Our next lemma gives some additional properties of solutions of (1.1) that belong to the class $\mathcal{O}_1$.

Lemma 2.6. *Assume that $\lambda \le p$ and there are positive constants $\varepsilon \le p$, a_0, r_0, and M_0, and $t_0 \in \mathbb{R}_+$ such that*

$$a_0 \le a(t) \le M_0 t^{p-\varepsilon} \quad \text{and} \quad r_0 \le r(t) \quad \text{for} \quad t \in [t_0,\infty)\,. \tag{2.11}$$

Then $\lim_{t\to\infty} F(t) = \infty$ for any solution $y \in \mathcal{O}_1$.

Proof. Let $y \in \mathcal{O}_1$ and assume to the contrary that $\lim_{t\to\infty} F(t) = M \in (0,\infty)$. Then, (2.7) and (2.8) imply $y \in L_{\lambda+1}(\mathbb{R}_+)$ and $y'' \in L_{p+1}(\mathbb{R}_+)$ as before. Since $\lambda \le p$, Theorem 2 in [12, §V.3] implies $y' \in L_{p+1}(\mathbb{R}_+)$. Using Hölder's inequality, (2.7) and (2.11), we have

$$\begin{aligned}
\int_{t_0}^t a(s)|y'(s)|\,|y''(s)|^p &\le \Big(\int_{t_0}^t |y'(s)|^{p+1}\,ds\Big)^{\frac{1}{p+1}} \Big(\int_{t_0}^t a^{\frac{p+1}{p}}(s)|y''(s)|^{p+1}\,ds\Big)^{\frac{p}{p+1}}\\
&\le \Big(\int_{t_0}^\infty |y'(s)|^{p+1}\,ds\Big)^{\frac{1}{p+1}} \Big(\int_{t_0}^\infty a(s)|y''(s)|^{p+1}\,ds\Big)^{\frac{p}{p+1}} M_0^{\frac{1}{p}} t^{\frac{p-\varepsilon}{p}}\\
&\le C t^{1-\frac{\varepsilon}{p}} \qquad (2.12)
\end{aligned}$$

for $t \in [t_0,\infty)$, where C is a suitable positive constant. Now define

$$Z(t) = -y_3(t)y_1(t) + 2\int_{t_1}^t a(s)y'(s)|y''(s)|^{p-1}y''(s)\,ds\,, \tag{2.13}$$

where $t_1 \in [t_0, \infty)$ satisfies

$$y_1(t_1) = 0 \quad \text{and} \quad F(t) \geq M/2 \quad \text{on} \quad [t_1, \infty)\,.$$

Since $Z'(t) = F(t)$ on $[t_1, \infty)$, if we take an increasing sequence $\{s_k\}_{k=1}^{\infty}$ of zeros of y_1 with $s_1 \geq t_1$, then (2.13) and (2.12) yield

$$\frac{M}{2}(s_k - t_1) \leq \int_{t_1}^{s_k} F(s)\, ds = \int_{t_1}^{s_k} Z'(s)\, ds = Z(s_k) \leq C s_k^{1-\frac{\varepsilon}{p}}.$$

This is a contradiction for large k and so the conclusion follows from Lemma 2(ii). $\square$

Remark 2.7. If $a \equiv 1$, $p = 1$, and $y \in \mathcal{O}_1$, then $\lim_{t\to\infty} F(t) = \infty$ without any additional assumptions on r and λ (see [4, Lemma 3.4]).

The next lemma shows that, in the case $\lambda \leq p$, all solutions of equation (1.1) are defined on $\mathbb{R}_+$.

Lemma 2.8. *Let $\lambda \leq p$. Then all solutions of* (1.1) *are continuable.*

Proof. Let y be a noncontinuable solution of (1.1) defined on $[0, \tau)$ with $\tau < \infty$. Then,

$$\limsup_{t\to\tau_-} \big|y_4(t)\big| = \infty\,. \tag{2.14}$$

Furthermore, (1.2) yields

$$\begin{aligned} \big|y_1(t)\big| &\leq C_1 + \int_0^t \big|y_2(s)\big|\, ds\,, \\ \big|y_2(t)\big| &\leq C_2 + \int_0^t a^{-\frac{1}{p}}(s)\big|y_3(s)\big|^{\frac{1}{p}}\, ds\,, \\ \big|y_3(t)\big| &\leq C_3 + \int_0^t \big|y_4(s)\big|\, ds\,, \end{aligned} \tag{2.15}$$

and

$$\big|y_4(t)\big| \leq C_4 + \int_0^t r(s)\big|y_1(s)\big|^{\lambda}\, ds\,,$$

with $C_i = \big|y_i(0)\big|$, $i = 1,2,3,4$, and $t \in [0,\tau)$. Define v by $v(0) = \big|y_4(0)\big|$, $v(t) = \max_{0\leq s\leq t} \big|y_4(s)\big|$ for $t \in (0,\tau)$. Note that in view of (2.14), $\lim_{t\to\tau_-} v(t) = \infty$. Hence, (2.15) yields

$$\begin{aligned} \big|y_4(t)\big| &\leq C_4 + \int_0^t r(s)\Big[C_1 + \int_0^s \Big[C_2 + \int_0^{s_1} a^{-\frac{1}{p}}(s_2) \\ &\qquad \times \Big[C_3 + \int_0^{s_2} \big|y_4(s_3)\big|\, ds_3\Big]^{\frac{1}{p}} ds_2\Big]\, ds_1\Big]^{\lambda} ds \\ &\leq C_4 + \int_0^t r(s)\Big[C_1 + \int_0^s \Big[C_2 + \big(C_3 + v(s)\tau\big)^{\frac{1}{p}} \int_0^{\tau} a^{-\frac{1}{p}}(s_2)\, ds_2\Big]\, ds_1\Big]^{\lambda} ds \\ &\leq C_4 + \int_0^t r(s)\Big[C_1 + \tau\Big(C_2 + \big(C_3 + v(s)\tau\big)^{\frac{1}{p}} \int_0^{\tau} a^{-\frac{1}{p}}(s_2)\, ds_2\Big)\Big]^{\lambda} ds\,. \end{aligned}$$

Thus,

$$v(t) \le C_4 + \int_0^t r(s)\big[C_5 + C_6 v^{\frac{1}{p}}(s)\big]^{\lambda}\, ds$$

$$\le C_4 + \int_0^t r(s) 2^{\lambda}\big(C_5^{\lambda} + C_6^{\lambda} v^{\frac{\lambda}{p}}(s)\big)\, ds \tag{2.16}$$

for suitable constants C_5 and C_6. Since $\lim\limits_{t\to\tau_-} v(t) = \infty$, there exists $t_0 \in [0,\tau)$ such that $v(t) \ge 1$ on $[t_0,\tau)$, and so (2.16) and the fact that $\lambda/p \le 1$ yield

$$v(t) \le C_7 + C_8 \int_{t_0}^t v(s)\, ds\,, \quad t \in [t_0,\infty), \tag{2.17}$$

where $C_7 = C_4 + 2^{\lambda} C_5^{\lambda} \int\limits_0^{\tau} r(s)\, ds$ and $C_8 = 2^{\lambda} C_6^{\lambda} \max\limits_{0\le s\le\tau} r(s)$. Now (2.17) and Gronwall's inequality imply v is bounded on $[0,\tau)$ contradicting (2.14). □

Our final lemma gives bounds on the solutions of the system (1.2).

Lemma 2.9. *Let $[a,b] \subset \mathbb{R}_+$ and a_0, a_1, r_0, and r_1 be positive constants. Let y be a solution of (1.2) such that y_i, $i = 1,2,3,4$, have zeros on $[a,b]$, and let $\nu_i = \max\limits_{a\le t\le b} \big|y_i(t)\big|$, $i = 1,2,3,4$.*

(i) *If $a_0 \le a(t) \le a_1$ and $r(t) \le r_1$ on $[a,b]$, then*

$$\nu_2 \le K_0 \nu_1^{\delta}, \quad \nu_3 \le K_1 \nu_1^{\delta_1}, \quad \text{and} \quad \nu_4 \le K_2 \nu_1^{\delta_2}, \tag{2.18}$$

with

$$\delta = \frac{1}{2} + \frac{\lambda+1}{2(p+1)}, \quad \delta_1 = \frac{p(\lambda+1)}{p+1}, \quad \delta_2 = \frac{1}{2}\big(\lambda + \frac{p(\lambda+1)}{p+1}\big),$$

$$K_1 = \big(4a_0^{-\frac{1}{2p}} a_1^{\frac{1}{p}} r_1^{\frac{1}{2}}\big)^{2p/p+1}, \quad K_0 = \big(2a_0^{-\frac{1}{p}} K_1^{\frac{1}{p}}\big)^{1/2}, \quad \text{and} \quad K_2 = (2r_1K_1)^{1/2}.$$

(ii) *If $a_0 \le a(t)$ and $r_0 \le r(t) \le r_1$ on $[a,b]$, then*

$$\nu_1 \le K_3 \nu_3^{\delta_3}\,, \quad \nu_2 \le K_4 \nu_3^{\delta_4}, \quad \text{and} \quad \nu_4 \le K_5 \nu_3^{\delta_5},$$

with

$$\delta_3 = \frac{p+1}{p(\lambda+1)}\,, \quad \delta_4 = \frac{\lambda+p+2}{2p(\lambda+1)}\,, \quad \delta_5 = \frac{1}{2}\Big(1 + \frac{\lambda(p+1)}{p(\lambda+1)}\Big)\,,$$

$$K_3 = \big(4r_0^{-1} r_1^{\frac{1}{2}} a_0^{-\frac{1}{2p}}\big)^{\frac{2}{\lambda+1}}\,, \quad K_4 = \big(2a_0^{-\frac{1}{p}} K_3\big)^{\frac{1}{2}}, \quad \text{and} \quad K_5 = (2r_1 K_3^{\lambda})^{\frac{1}{2}}\,.$$

Proof. (i) We can choose intervals $J_i \subset [a,b]$, $i = 1,2,3$ such that

$$\min_{t\in J_i} \big|y_i(t)\big| = 0 \quad \text{and} \quad \nu_i = \max_{t\in J_i} \big|y_i(t)\big|\,,$$

where ν_i occurs at one of the endpoints of the interval J_i and y_i does not change its sign on J_i, $i = 1,2,3$. Then (1.2) and (1.3) yield

$$\nu_2^2 \le 2\int_{J_1} \big|y_2(s)\, y_2'(s)\big|\, ds = 2\int_{J_1} a^{-\frac{1}{p}}(s)\big|y_2(s)\big|\, \big|y_3(s)\big|^{\frac{1}{p}}\, ds$$

$$\le 2a_0^{-\frac{1}{p}} \nu_1 \nu_3^{\frac{1}{p}}\,, \tag{2.19}$$

$$\begin{aligned}\nu_3^2 &\le 2\int_{J_2} \big|y_3(s)\,y_3'(s)\,ds\big| \le 2\int_{J_2} a^{\frac{1}{p}}(s)\big|y_2'(s)\big|\,\big|y_3(s)\big|^{\frac{p-1}{p}}\big|y_4(s)\big|\,ds \\ &\le 2a_1^{\frac{1}{p}}\nu_2\nu_3^{\frac{p-1}{p}}\nu_4\,, \end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\nu_4^2 &\le 2\int_{J_3} \big|y_4(s)\,y_4'(s)\big|\,ds \le 2\int_{J_3} r(s)\big|y_1(s)\big|^{\lambda}\big|y_3'(s)\big|\,ds \\ &\le 2r_1\nu_1^{\lambda}\nu_3\,. \end{aligned} \tag{2.21}$$

From (2.19)–(2.21), we have

$$\nu_3^2 \le 2a^{\frac{1}{p}}\big(2a_0^{-\frac{1}{p}}\nu_1\nu_3^{\frac{1}{p}}\big)^{\frac{1}{2}}(2r_1\nu_1^{\lambda}\nu_3)^{\frac{1}{2}}\nu_3^{\frac{p-1}{p}}$$

and

$$\nu_3 \le K_1\nu_1^{\frac{p(\lambda+1)}{p+1}}\,. \tag{2.22}$$

Hence, the second inequality in (2.18) is proved. Furthermore, (2.19) and (2.22) yield $\nu_2 \le K_0\nu_1^{\delta}$, so the first inequality in (2.18) holds. The last inequality in (2.18) follows from (2.21) and (2.22).

(ii) System (1.2) can be rewritten as the system $Z = (Z_1, \dots, Z_4)$

$$\begin{aligned} Z_1' &= Z_2, & Z_3' &= Z_4, \\ Z_2' &= -r(t)|Z_3|^{\lambda}\operatorname{sgn} Z_3, & Z_4' &= a^{-\frac{1}{p}}(t)|Z_1|^{\frac{1}{p}}\operatorname{sgn} Z_1, \end{aligned} \tag{2.23}$$

with $Z_1 = y_3$, $Z_2 = y_4$, $Z_3 = y_1$, and $Z_4 = y_2$. As the proof of (i) does not depend on the signs of the coefficients $a^{\frac{1}{p}}$ and $-r$ in (1.2), the results follow from part (i) applied to (2.23). □

3. MAIN RESULTS

This section contains our main results on the behavior of solutions of equation (1.1).

Theorem 3.1. (i) *If*

$$\int_0^{\infty} a^{-\frac{1}{p}}(\sigma)\,d\sigma = \infty \quad \text{and} \quad \int_0^{\infty} r(\sigma)\,d\sigma = \infty,$$

then $\mathcal{N} = \mathcal{N}_9$.

(ii) *Assume there are positive constants a_0 and r_0 such that*

$$a_0 \le a(t) \quad \text{and} \quad r_0 \le r(t) \quad \text{on} \quad \mathbb{R}_+\,. \tag{3.1}$$

If $y \in \mathcal{O}_2 \cup \mathcal{N}_9$, then

$$\int_0^{\infty}\big|y(t)\big|^{\lambda+1}\,dt < \infty \quad \text{and} \quad \int_0^{\infty}\frac{a(t)}{r(t)}\big|y''(t)\big|^{p+1}\,dt < \infty\,.$$

Proof. Part (i) follows from Lemma 2.3(ii). Part (ii) follows from Lemma 2.4(iii) and Lemma 2.5(ii) if $y \in \mathcal{O}_2$; if $y \in \mathcal{N}_9$, the result is obvious. □

The next theorem is concerned with the solutions of (1.1) that belong to the class $\mathcal{N}_1$.

Theorem 3.2. *Let $y \in \mathcal{N}_1$. Then $\int\limits_0^\infty |y(s)|^{\lambda+1}\,ds = \infty$. Moreover,*

(i) *if $\int\limits_0^\infty a^{-\frac{1}{p}}(s) r^{-1}(s)\,ds = \infty$, then $\int\limits_0^\infty \frac{a(s)}{r(s)} |y''(s)|^{p+1}\,ds = \infty$;*

(ii) *if*

$$\int_{t_0}^\infty r(s)\Big(\int_{t_0}^s \int_{t_0}^\sigma a^{-\frac{1}{p}}(\sigma_1)\,d\sigma_1\,d\sigma\Big)^\lambda ds = \infty \quad \text{for} \quad t_0 \in \mathbb{R}_+\,, \tag{3.2}$$

then $\mathcal{N}_1 = \emptyset$.

Proof. Let $y \in \mathcal{N}_1$ and suppose for simplicity that $y_i(t) > 0$ for $i = 1, 2, 3, 4$ and $t \ge t_y$. In view of (1.2), $y_1(t) = y(t) \ge y_1(t_y) > 0$, so we have

$$\int_0^\infty |y(t)|^{\lambda+1}\,dt = \infty\,.$$

Moreover, since y_4 is positive and decreasing, we have

$$\begin{aligned} y_3(t_y) \le y_3(t) &= y_3(t_y) + \int_{t_y}^t y_3'(s)\,ds \\ &\le y_3(t_y) + y_4(t_y)(t - t_y) \le 2y_4(t_y)t \end{aligned} \tag{3.3}$$

for t large enough, say for $t \ge t_1 \ge t_y$. Hence,

$$Ca^{-\frac{1}{p}}(t) \le y''(t) = \big(y_3(t)a^{-1}(t)\big)^{\frac{1}{p}} \tag{3.4}$$

with $C = y_3^{\frac{1}{p}}(t_y)$. Part (i) then follows immediately.

To prove (ii), notice that the first inequality in (3.4) yields

$$y_1(t) \ge C \int_{t_y}^t \int_{t_y}^s a^{-\frac{1}{p}}(\sigma)\,d\sigma\,ds\,. \tag{3.5}$$

It follows from (1.2) that

$$y_4(t) = y_4(t_y) - \int_{t_y}^t r(s)\big(y_1(s)\big)^\lambda ds > 0 \quad \text{on} \quad [t_y, \infty),$$

so (3.5) yields

$$C^\lambda \int_{t_y}^\infty r(s)\Big[\int_{t_y}^s \int_{t_y}^\sigma a^{-\frac{1}{p}}(\sigma_1)\,d\sigma_1\,d\sigma\Big]^\lambda ds \le \int_{t_y}^\infty r(s) y_1^\lambda(s)\,ds < \infty\,.$$

This contradicts (3.2) and completes the proof. □

Notation. Let $h \in C^0(\mathbb{R}_+)$. If $\int\limits_t^\infty h(s)\,ds = \infty$ for $t \in \mathbb{R}_+$, then we set

$$\int\limits_0^\infty \int\limits_s^\infty h(\sigma)\,d\sigma\,ds = \infty.$$

Next, we turn our attention to solutions in the class $\mathcal{N}_3$.

Theorem 3.3. *Let $y \in \mathcal{N}_3$. Then:*

(i) $\int\limits_0^\infty \big|y(s)\big|^{\lambda+1}\,ds = \infty.$

(ii) *If*

$$\int_0^\infty a^{-\frac{1}{p}}(t) r^{-1}(t) \Big(\int_t^\infty \int_s^\infty r(\sigma)\,d\sigma\,ds \Big)^{\frac{p+1}{p}} dt = \infty\,,$$

then

$$\int_0^\infty \frac{a(s)}{r(s)} \big|y''(s)\big|^{p+1}\,ds = \infty\,.$$

Proof. Let $y \in \mathcal{N}_3$ be such that $y_1 > 0$, $y_2 > 0$, $y_3 < 0$, and $y_4 > 0$ on $[t_y, \infty)$. (If $y_1 < 0$, the proof is similar.) From (1.2), we have $|y_i|$ is decreasing for $i = 2, 3, 4$ and y_1 is increasing on $[t_y, \infty)$.

Part (i) follows from the fact that $y(t) \geq y(t_y) > 0$ on $[t_y, \infty)$. To prove (ii), note that using (1.2), we have

$$\begin{aligned} \big|y_3(t)\big| &\geq \int_t^\infty y_4(s)\,ds \geq \int_t^\infty \int_s^\infty r(\sigma) y^\lambda(\sigma)\,d\sigma\,ds \\ &\geq y^\lambda(t_y) \int_t^\infty \int_s^\infty r(\sigma)\,d\sigma\,ds\,, \end{aligned}$$

and thus

$$\big|y''(t)\big| \geq y^{\frac{\lambda}{p}}(t_y) a^{-\frac{1}{p}}(t) \Big(\int_t^\infty \int_s^\infty r(\sigma)\,d\sigma\,ds \Big)^{\frac{1}{p}}$$

for $t \geq t_y$. Hence,

$$\begin{aligned} &\int_{t_y}^\infty \frac{a(s)}{r(s)} \big|y''(s)\big|^{p+1}\,ds \\ &\qquad \geq C \int_{t_y}^\infty a^{-\frac{1}{p}}(t) r^{-1}(t) \Big(\int_t^\infty \int_s^\infty r(\sigma)\,d\sigma\,ds \Big)^{\frac{p+1}{p}} dt = \infty \end{aligned}$$

with a suitable constant $C > 0$, and the statement holds. $\square$

For our next result, we consider those solutions that belong to the class $\mathcal{O}_1$.

Theorem 3.4. *Let $\lambda \leq p$, let a_0, a_1, r_0, and r_1 be positive constants, and let $y \in \mathcal{O}_1$.*

(i) *If $a_0 \leq a(t) \leq a_1$ and $r(t) \leq r_1$ on $\mathbb{R}_+$, then*

$$\int_0^\infty \big|y(t)\big|^{\lambda+1} = \infty\,.$$

(ii) *If* $a_0 \le a(t) \le a_1$ *and* $r_0 \le r(t) \le r_1$ *on* $\mathbb{R}_+$, *then*

$$\int_0^\infty \frac{a(t)}{r(t)} \big|y''(t)\big|^{p+1}\,dt = \infty\,.$$

Proof. Let $y \in \mathcal{O}_1$.

(i) We will first prove that y is unbounded on $\mathbb{R}_+$, so suppose that y is bounded there. Then Lemma 2.9 and (2.23) imply that $y_i(t)$, $i = 2,3,4$, are bounded, so $\lim_{t\to\infty} F(t) < \infty$. This contradicts Lemma 2.6 and shows that

$$\limsup_{t\to\infty} \big|y_1(t)\big| = \infty\,.$$

Thus, there is a subsequence of $\{t_k^2\}_{k=1}^\infty$, which for simplicity we denote by $\{t_k^2\}_{k=1}^\infty$ itself, such that

$$\lim_{k\to\infty} \big|y_1(t_k^2)\big| = \infty\,. \tag{3.6}$$

Let $\Delta_k = [t_k^1, t_k^2]$ and $\Delta_k^1 = [\tau_k, t_k^2]$ where τ_k is defined by $\tau_k \in \Delta_k$, $\big|y_1(\tau_k)\big| = \frac{1}{2}\big|y_1(t_k^2)\big|$, $k = 1,2,\dots$. Observe that y_2, y_3 and y_4 have zeros on Δ_k (see (2.1)) and all conditions in Lemma 2.9(i) are satisfied on Δ_k for $k \in \{1,2,\dots\}$. Furthermore, $\frac{1}{2}\big|y_1(t_k^2)\big| \le \big|y_1(t)\big| \le \big|y_1(t_k^2)\big|$, and the first inequality in (2.18) yields

$$\frac{1}{2}\big|y_1(t_k^2)\big| = \big|y_1(t_k^2)\big| - \big|y_1(\tau_k)\big| = \int_{\Delta_k^1} \big|y_2(s)\big|\,ds \le K_0(t_k^2 - \tau_k)\big|y_1(t_k^2)\big|^\delta$$

where K_0 is given in Lemma 2.9 and $\delta = \frac{1}{2} + \frac{\lambda+1}{2(p+1)} \le 1$. From this and (3.6), we see that there is a positive constant M such that

$$t_k^2 - \tau_k \ge M\,, \quad k = 1,2,\dots$$

We then have

$$\int_0^\infty \big|y(t)\big|^{\lambda+1}\,dt \ge \sum_{k=1}^\infty \int_{\Delta_k^1} \big|y_1(t)\big|^{\lambda+1}\,dt \ge \frac{M}{2^{\lambda+1}} \sum_{k=1}^\infty \big|y_1(t_k^2)\big|^{\lambda+1} = \infty\,.$$

(ii) Let $\Delta_k = [t_k^3, t_{k+1}^4]$ and $\Delta_k^1 = [\tau_k, t_{k+1}^4]$ where τ_k is defined by $\tau_k \in \Delta_k$ and $\big|y_3(\tau_k)\big| = \frac{1}{2}\big|y_1(t_{k+1}^4)\big|$, $k = 1,2,\dots$ Then, similar to what we did in the proof of part (i), we can show that $\lim_{k\to\infty} \big|y_3(t_{k+1}^4)\big| = \infty$, $t_{k+1}^4 - \tau_k \ge M_1$, $k = 1,2,\dots$, for a suitable constant M_1 by using Lemma 2.9(ii) and the fact that $\delta_5 \le 1$. Hence,

$$\begin{aligned}\int_0^\infty \frac{a(t)}{r(t)}\big|y''(t)\big|^{p+1}\,dt &= \int_0^\infty \frac{\big|y_3(t)\big|^{\frac{p+1}{p}}}{a^{\frac{1}{p}}(t)r(t)}\,dt \ge a_1^{-\frac{1}{p}} r_1^{-1} \sum_{k=1}^\infty \int_{\Delta_k^1} \big|y_3(t)\big|^{\frac{p+1}{p}}\,dt \\ &\ge a_1^{-\frac{1}{p}} r_1^{-1} \frac{M}{2^{\lambda+1}} \sum_{k=1}^\infty \big|y_3(t_{k+1}^4)\big| = \infty\,.\end{aligned}$$

This completes the proof of the theorem. □

The next theorem summarizes many of the results obtained here on solutions of (1.1).

Theorem 3.5. *Let $\lambda \le p$ and let a_0, a_1, r_0, and r_1 be positive constants such that*

$$a_0 \le a(t) \le a_1\,, \quad r_0 \le r(t) \le r_1 \quad \text{on} \quad \mathbb{R}_+\,.$$

(i) *All solutions of* (1.1) *are continuable, $\mathcal{S} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{N}_9$, and $\mathcal{O}_1 \neq \emptyset$.*

(ii) *For any solution y of* (1.1)*, the following statements are equivalent:*
   (a) *y is of the nonlinear limit-circle type;*
   (b) *y is of the strong nonlinear limit-circle type;*
   (c) *$y \in \mathcal{O}_2 \cup \mathcal{N}_9$.*

(iii) *For any solution y of* (1.1)*, the following statements are equivalent:*
   (a) *y is of the nonlinear limit-point type;*
   (b) *y is of the strong nonlinear limit-point type;*
   (c) *$y \in \mathcal{O}_1$.*

Proof. Parts (i) and (iii) of Lemma 2.3 together with Lemma 2.8 imply that all solutions of (1.1) are continuable and $\mathcal{S} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{N}_9$. Now if y is a solution of (1.1) such that $F(0) > 0$, then parts (iii) and (iv) of Lemma 2.4 imply $y \notin \mathcal{O}_2 \cup \mathcal{N}_9$. Hence, $y \in \mathcal{O}_1$, and so part (i) holds. Parts (ii) and (iii) follow from Theorems 3.1 and 3.4. □

It is reasonable to ask if it is possible to obtain results similar to those in Theorem 3.5 for $\lambda > p$. To show the difficulties involved, we consider the case where $p = 1$, $\lambda > 1$, and $a(t) \equiv 1$ in equation (1.1), i.e.,

$$y^{(4)} + r(t)|y|^{\lambda-1}y = 0. \tag{3.7}$$

We will need the following lemma.

Lemma 3.6. *Let $[T, T_1] \subset \mathbb{R}_+$, $T \le T_1 - 2$, $C_i \in \mathbb{R}$ for $i = 1, 2, 3, 4$, $\lambda > 1$, $\mu = (3\lambda^2 + 8\lambda + 5)/(3\lambda^2 + 6\lambda + 7)$, and*

$$0 < r_0 \le r(t) \le r_1 \quad \text{on} \quad [T, T_1].$$

Then there exist constants $\delta > 0$ and $K > 0$, which do not depend on T, T_1, r_0, or r_1, such that a solution of the Cauchy initial value problem consisting of equation (1.1) *and the conditions*

$$y^{(i)} = C_{i+1}, \quad i = 0, 1, 2, 3, \tag{3.8}$$

is noncontinuable if

$$-C_1C_4 + C_2C_3 > K r_0^{-\frac{1}{\mu-1}} r_1^{\frac{3(\lambda+1)\mu}{(3\lambda+5)(\mu-1)}} (T_1 - T)^{-\frac{2\mu}{(\lambda+1)(\mu-1)}}$$

and

$$|C_1| + |C_2| + |C_3| + |C_4| > 4^{\frac{\mu+1}{\mu-1}} r_0^{-\frac{\lambda}{\lambda-1}} (\delta r_0 + r_1)(T_1 - T)^{-\frac{4}{\mu-1}}.$$

Proof. This follows from the proof of Theorem 1 of Chanturia [13]. Observe that $\mu > 1$. □

We then have the following result.

Theorem 3.7. *Assume that* $\lambda > 1$, $r_0 < r_1$, *and*

$$r_0 \le r(t) \le r_1 t^\sigma \quad \textit{for} \quad t \in \mathbb{R}_+, \tag{3.9}$$

with $\sigma < \frac{2}{\lambda+1}\frac{3\lambda+5}{3\lambda+3}$. *Then equation* (3.7) *is of the strong nonlinear limit-circle type.*

Proof. By Lemma 2.3(ii), $\mathcal{N} = \mathcal{N}_9$. We will prove that $\mathcal{O}_1 = \emptyset$. Suppose $y \in \mathcal{O}_1 \neq \emptyset$. Then Lemma 2.4(ii) implies $\lim_{t\to\infty} F(t) > 0$, and there exists $T \in \mathbb{R}$ such that

$$F(T) = -y(T)y'''(T) + y'(T)y''(T) > 0.$$

Choose $T_1 \in [T+2, \infty)$ such that

$$F(T) > K r_0^{-\frac{1}{\mu-1}} (r_1 T_1)^{\frac{3(\lambda+1)\mu\sigma}{(3\lambda+5)(\mu-1)}} (T_1 - T)^{-\frac{2\mu}{(\lambda+1)(\mu-1)}}$$

and

$$\sum_{i=0}^{3} |y^{(i)}(T)| > 4^{\frac{\mu+1}{\mu-1}} r_0^{-\frac{\lambda}{\lambda-1}} (\delta r_0 + r_1 T_1^\sigma)(T_1 - T)^{-\frac{4}{\mu-1}},$$

where δ, K, and μ are given in Lemma 3.6; this choice is possible since $\mu > 1$,

$$\frac{2\mu}{(\lambda+1)(\mu-1)} > \frac{3(\lambda+1)\mu\sigma}{(3\lambda+5)(\mu-1)},$$

and $4/(\mu-1) > \sigma$. Now all the conditions of Lemma 3.6 hold with $r_0 = r_0$, $T = T$, $T_1 = T_1$, $r_1 = r_1 T_1^\sigma$, and so the solution y is noncontinuable. This contradiction proves that $\mathcal{O}_1 = \emptyset$. Thus, $\mathcal{O} = \mathcal{O}_2$ and the conclusion of the theorem follows from Theorem 3.1(ii). ☐

Remark 3.8. By Lemma 3.4 in [4], for equation (3.7), we have $\lim_{t\to\infty} F(t) \in \{0, \infty\}$. If (3.9) holds, then we only have $F(t) \to 0$ as $t \to \infty$. Moreover, $F(t) \to 0$ as $t \to \infty$ if and only if y is a nonlinear limit-circle solution. It remains as an open problem whether Theorem 3.7 can be extended to equation (1.1).

To illustrate some of our results, we consider equation (1.1) with $a(t)$ and $r(t)$ being powers of t, namely, the equation

$$\left(t^\alpha |y''|^{p-1} y''\right)'' + t^\beta |y|^{\lambda-1} y = 0. \tag{3.10}$$

We see that, in view of Theorem 3.1(i), if

$$p \ge \alpha \quad \text{and} \quad \beta \ge -1,$$

then equation (3.10) has no proper nonoscillatory solutions. Now if

$$\beta \le (p-\alpha)/p\,,$$

then any solution of (3.10) that belongs to $\mathcal{N}_1$ is of the strong nonlinear limit-point type by Theorem 3.2(i). By part (iii) of Theorem 3.2, if

$$\beta \ge -1,$$

then in fact $\mathcal{N}_1 = \emptyset$. By Theorem 3.3, if

$$\beta \geq -2 \quad \text{or} \quad -2 > \beta > \alpha - 3p - 2,$$

then any solution of equation (3.10) that belongs to $\mathcal{N}_3$ is of the strong nonlinear limit-point type.

It should be clear that the point-wise conditions on the coefficient functions a and r are less than desirable. This is especially true in the case of the function r; equation (1.1) and equations similar to it often appear in the literature with $a(t) \equiv 1$. However, by Theorem 10.3 of Kiguradze and Chanturia [14],

$$\int_0^\infty s^3 r(s)\, ds < \infty \tag{3.11}$$

is a necessary and sufficient condition for proper nonoscillatory solutions of equation (3.7) to exist, that is, Theorem 3.1(i) is not true if (3.11) holds. In this same spirit, if (3.11) holds, then by Theorem 16.9 in [14] there is a solution tending a nonzero constant so that equation (3.10) is of the nonlinear limit-point type contrary to Theorem 3.7. Now condition (3.11) implies $\liminf_{t\to\infty} r(t) = 0$, so in view of the above observations, the condition $r(t) \geq r_0$ may not be as restrictive as it first seems.

In conclusion, we wish to point out that this is the first time that the strong nonlinear limit-point and limit-circle properties have been defined for fourth order equations. Hence, the form that analogous results take for equation (1.1) with $r(t) < 0$ remains as an open problem.

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