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EIGENVALUE PROBLEMS FOR SECOND ORDER DYNAMIC EQUATIONS WITH SIGN CHANGING NONLINEARITIES ON TIME SCALES

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ABSTRACT. In this paper, we consider the eigenvalue problems for second order dynamic equations on time scales with sign changing nonlinearities. By using topological degree theory and constructing suitable operator, we give the eigenvalue intervals in which there exist one or two positive solutions of the problem. An example is also given to illustrate the main results. The results improve and generalize some known results.

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1. INTRODUCTION

The theory of dynamic equations on time scales was introduced by Stefan Hilger in his PhD thesis [9], motivating the subject is that dynamic equations on time scales can build bridges between continuous and discrete equations. It has found a considerable amount of interest and attract many researchers attention. Further, the study of time scales has led to several important applications, for example, in the study of insect population models, phytoremediation of metals, wound healing, and epidemic models [10, 11, 16]. Before introducing the problems of interest for this paper, we present some basic definitions of time scales [4, 5].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. For notation, we shall use the convention that, for each interval J of $\mathbb{R}$, $J_{\mathbb{T}} = J \cap \mathbb{T}$.

The forward jump operator σ and backward jump operator ρ from $\mathbb{T}$ to $\mathbb{T}$ are defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \in \mathbb{T} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \in \mathbb{T},$$

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for all $t \in \mathbb{T}$ with $t < \sup \mathbb{T}$ and $t > \inf \mathbb{T}$. In this definition, $\inf \emptyset := \sup \mathbb{T}$, $\sup \emptyset := \inf \mathbb{T}$. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively.

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided f is continuous at left-dense points in $\mathbb{T}$ and its right-sided limit exists (finite) for right-dense points in $\mathbb{T}$. It is known that if f is ld-continuous, then there exists a function F such that $F^{\nabla}(t) = f(t)$, in this case, we define$\int_a^b f(\tau)\nabla\tau = F(b) - F(a)$.

Very recently, there has been increasing attention pay to the question of positive solutions for multipoint boundary value problems on time scales, see [1, 2, 3, 6, 12, 14, 15] and the references cited therein. Most results so far have been obtained by applying the fixed point theorems in cones, such as Krasnoselskii's fixed point theorem [8], Leggett-williams fixed point theorem [13] and so on. In order to use the concavity of solutions in the proofs, all the existing works were done under the assumption that the nonlinearity is nonnegative. However, little work has been done on the existence of positive solutions for multipoint boundary value problems with sign-changing nonlinearity on time scales.

In particular, in [2], Anderson considered the following problem

$$-u^{\Delta\nabla}(t) = \eta a(t) f(u(t)), \quad t \in (t_1, t_n) \subset \mathbb{T},$$
$$u^{\Delta}(t_1) = 0, u(t_n) = \sum_{i=2}^{n-1} \alpha_i u(t_i) \quad \text{or} \quad u^{\Delta}(t_n) = 0, u(t_1) = \sum_{i=2}^{n-1} \alpha_i u(t_i),$$

where $t_1 < t_2 < \cdots < t_n$ are points in a time scale $\mathbb{T}$, $\eta > 0$ and $\alpha_i \geq 0$ for $i \in \{2, 3, \ldots, n-1\}$ with $0 < \sum_{i=1}^{n-1} \alpha_i < 1$. By using a functional-type cone expansion-compression fixed point theorem, the author established the existence of one positive solution under the assumptions that both $a(t)$ and $f(u)$ are nonnegative.

Dong and Ge [7] studied the existence of two positive solutions to the problem

$$u''(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i),$$

where f is allowed to change sign. The main tool is a new fixed point theorem in cones.

Motivated by the above results, in this paper, we consider the eigenvalue problems for a second order dynamic equations on time scales

$$u^{\Delta\nabla}(t) + \lambda a(t) f(t, u(t)) = 0, \quad t \in [0, T]_{\mathbb{T}}, \tag{1.1}$$

with solutions satisfying

$$u^{\Delta}(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{1.2}$$

where λ is a positive parameter, the nonlinear term f is allowed to change sign, and $0, T \in \mathbb{T}$, $\alpha_i \geq 0$ for $i = 1, 2, \ldots, m-2$, $0 = \xi_0 < \xi_1 < \cdots < \xi_{m-2} < \xi_{m-1} = T$, $\widetilde{d} := 1 - \sum_{i=1}^{m-2} \alpha_i$. By using topological degree theory and constructing suitable operator, we give the eigenvalue intervals in which there exist one or two positive solutions of the problem. An example is also given to illustrate the main results. The results improve and generalize those of in [2, 12].

The rest of this paper is organized as follows. In Section 2, we state some lemmas which will be needed in proving our main results. Section 3 is devoted to the existence of at least one positive solution for the problem (1.1), (1.2) by using the topological degree theory and constructing available operator. In the final section, we establish the existence of at least two positive solutions of the problem (1.1), (1.2), and an example is also given to illustrate the results.

For the sake of convenience, we list the following hypotheses:

(H_1) $\widetilde{d} > 0$;
(H_2) $a : [0, T]_{\mathbb{T}} \to [0, \infty)$ is ld-continuous with $a(t_0) > 0$ for at least one $t_0 \in (0, T)_{\mathbb{T}}$;
(H_2) $f : [0, T]_{\mathbb{T}} \times [0, \infty) \to (-\infty, \infty)$ is continuous.

2. PRELIMINARIES

Lemma 2.1. *If $\widetilde{d} \neq 0$, then for $h \in C_{ld}([0, T]_{\mathbb{T}})$, the boundary value problem*

$$u^{\Delta\nabla}(t) + h(t) = 0, \qquad t \in [0, T]_{\mathbb{T}}, \tag{2.1}$$

$$u^{\Delta}(0) = 0, \quad u(T) = \sum_{i=1}^{m-2} \alpha_i u(\xi_i), \tag{2.2}$$

has a unique solution

$$\begin{aligned} u(t) &= (T-t)\int_0^t h(s)\nabla s + \int_t^T (T-s)h(s)\nabla s \\ &\quad + \frac{1}{\widetilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T-\xi_i)h(s)\nabla s + \int_{\xi_i}^T (T-s)h(s)\nabla s \right). \end{aligned} \tag{2.3}$$

Proof. Let $u(t)$ be as in (2.3). Using Theorem 2.10 (iii) in [3]

$$\left(\int_a^t g(t,s)\nabla s \right)^{\Delta} = g(\sigma(t), \sigma(t)) + \int_a^t g^{\Delta}(t,s)\nabla s,$$

and the formula

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t),$$

if we take the delta derivative of (2.3), then we get

$$u^{\Delta}(t) = -\int_0^t h(s)\nabla s.$$

Taking the ∇-derivative of this expression yields $u^{\Delta\nabla}(t) = -h(t)$, and routine calculation verify that $u(t)$ satisfies the boundary conditions in (2.2), so that $u(t)$ given in (2.3) is one solution of the problem (2.1), (2.2).

It is easy to see that boundary value problem $x^{\Delta\nabla}(t) = 0$, $x^{\Delta}(0) = 0$, $x(T) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i)$ has only the trivial solution if $\widetilde{d} \neq 0$. Thus $u(t)$ in (2.3) is the unique solution of the problem (2.1), (2.2). □

Lemma 2.2. *Assume* (H_1) *holds. If* $h \in C_{ld}([0,T]_{\mathbb{T}})$ *and* $h \geq 0$, *then the unique solution* $u(t)$ *of the problem* (2.1), (2.2) *satisfies*

$$u(t) \geq 0, \qquad t \in [0,T]_{\mathbb{T}}.$$

Proof. From the fact that $u^{\Delta\nabla}(t) = -h(t) \leq 0$, we know that $u^{\Delta}(t)$ is nonincreasing in $[0,T]_{\mathbb{T}}$, which together with $u^{\Delta}(0) = 0$, we have $u^{\Delta}(t) \leq 0$ for t$\in [0,T]_{\mathbb{T}}$, so we only prove that $u(T) \geq 0$. Observe that

$$u(T) = \frac{1}{\widetilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T - \xi_i) h(s) \nabla s + \int_{\xi_i}^{T} (T - s) h(s) \nabla s \right) \geq 0.$$

The proof is complete. □

Similar to the proof of Lemma 5 [1], we can obtain the following lemma.

Lemma 2.3. *Suppose that* $\widetilde{d} < 0$. *If* $h \in C_{ld}([0,T]_{\mathbb{T}})$ *and* $h \geq 0$, *then the problem* (2.1), (2.2) *has no positive solution.*

Lemma 2.4. *Assume* (H_1) *holds. If* $h \in C_{ld}([0,T]_{\mathbb{T}})$ *and* $h \geq 0$, *then the unique solution* $u(t)$ *of the problem* (2.1), (2.2) *satisfies*

$$\min_{t \in [0,T]_{\mathbb{T}}} u(t) \geq \gamma \left\| u \right\|,$$

where $\gamma = \frac{\sum_{i=1}^{m-2} \alpha_i (T - \xi_i)}{T - \sum_{i=1}^{m-2} \alpha_i \xi_i}$ *and* $\|u\| = \sup_{t \in [0,T]_{\mathbb{T}}} |u(t)|$.

Proof. From $u^{\Delta\nabla}(t) = -h(t) \leq 0$ and the boundary condition $u^{\Delta}(0) = 0$, we have $u(0) = \|u\|$, $u(T) = \min_{t \in [0,T]_{\mathbb{T}}} u(t)$. So

$$\frac{u(T) - u(\xi_i)}{T - \xi_i} \leq \frac{u(T) - u(0)}{T}, \; i = 1, 2, \ldots, m-2,$$

that is

$$u(\xi_i) - \frac{\xi_i u(T)}{T} \geq \left(1 - \frac{\xi_i}{T}\right) u(0).$$

Then

$$\sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} \frac{a_i \xi_i u(T)}{T} \geq \sum_{i=1}^{m-2} a_i \left(1 - \frac{\xi_i}{T}\right) u(0).$$

Hence by (2.2), we get

$$u(T) \geq \frac{\sum_{i=1}^{m-2} \alpha_i (T - \xi_i)}{T - \sum_{i=1}^{m-2} \alpha_i \xi_i} u(0).$$

This completes the proof. □

Let the Banach space $X = C_{ld}([0,T]_{\mathbb{T}})$ be endowed with the sup norm. By Lemma 2.1, it is easy to see that the boundary value problem (1.1), (1.2) has a solution $u(t)$ if and only if u is a fixed point of the operator equation

$$(Au)(t) = \lambda(T-t)\int_0^t a(s)f(s,u(s))\nabla s + \lambda\int_t^T (T-s)a(s)f(s,u(s))\nabla s \tag{2.4}$$
$$+\frac{\lambda}{\tilde{\tilde{d}}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)f(s,u(s))\nabla s + \int_{\xi_i}^T (T-s)a(s)f(s,u(s))\nabla s\right).$$

Denote

$$K = \{u \in X : u(t) \geq 0 \text{ for } t \in [0,T]_{\mathbb{T}}\},$$

and

$$K' = \{u \in K : u^{\Delta\nabla}(t) \leq 0 \text{ for } t \in [0,T]_{\mathbb{T}}, u^{\Delta}(0) = 0 \text{ and } \min_{t\in[0,T]_{\mathbb{T}}} u(t) \geq \gamma\|u\|\},$$

where γ is defined in Lemma 2.4. It is obvious that K, K' are cones in X and $K' \subset K$.

For convenience, we give the following operators and lemma which will be used later.

For any $u \in X$, we define the integral operators $B : X \to X, C : X \to X$, respectively,

$$(Bu)(t) = \Bigg[\lambda(T-t)\int_0^t a(s)f(s,u(s))\nabla s + \lambda\int_t^T (T-s)a(s)f(s,u(s))\nabla s \tag{2.5}$$
$$+\frac{\lambda}{\tilde{\tilde{d}}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)f(s,u(s))\nabla s + \int_{\xi_i}^T (T-s)a(s)f(s,u(s))\nabla s\right)\Bigg]^+,$$

$$(Cu)(t) = \lambda(T-t)\int_0^t a(s)f^+(s,u(s))\nabla s + \lambda\int_t^T (T-s)a(s)f^+(s,u(s))\nabla s \tag{2.6}$$
$$+\frac{\lambda}{\tilde{\tilde{d}}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)f^+(s,u(s))\nabla s + \int_{\xi_i}^T (T-s)a(s)f^+(s,u(s))\nabla s\right),$$

where $t \in [0,T]_{\mathbb{T}}, \varphi^+ = \max\{\varphi, 0\}$.

3. ONE POSITIVE SOLUTION

For convenience, we denote

$$M = \int_0^T (T-s)a(s)\nabla s + \frac{1}{\tilde{\tilde{d}}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)\nabla s + \int_{\xi_i}^T (T-s)a(s)\nabla s\right).$$

Now we give our results on the existence of one positive solution of the problem (1.1), (1.2).

Theorem 3.1. *Assume* $(H_1)-(H_3)$ *hold. Suppose that* $f(t,0)\geq 0$ *for* $t\in[0,T]_{\mathbb{T}}$ *and* $a(t)f(t,0)\not\equiv 0$ *on any subinterval of* $[0,T]_{\mathbb{T}}$. *If there exists* $c>0$ *such that*

$$\lambda\leq b=\frac{c}{M\sup_{t\in[0,T]_{\mathbb{T}},u\in[0,c]}f(t,u)},\tag{3.1}$$

then the boundary value problem (1.1), (1.2) *has at least one positive solution* $u(t)$ *satisfying* $0<\|u\|\leq c$.

Proof. First of all, by the definition of the operator B, it is easy to see that $B(K)\subset K$. Moreover, by the continuity of f and a typical application of the Arzela-Ascoli theorem on time scale, it is easy to see that the operator $A:K\to X$ is completely continuous. So, similar as Lemma 2.2 in [6], we get B is completely continuous.

Denote $K_r=\{u\in K:\|u\|<r\}$, $\partial K_r=\{u\in K:\|u\|=r\}$, for any $r>0$. Now we show that the operator B has a fixed point $u\in K$ with $0<\|u\|\leq c$. For $\lambda\in(0,b]$, by (2.5) and Lemma 2.2, we have

$$\begin{aligned}
\|Bu\|=&\sup_{t\in[0,T]_{\mathbb{T}}}\max\Bigg\{\lambda(T-t)\int_0^t a(s)f(s,u(s))\nabla s+\lambda\int_t^T(T-s)a(s)f(s,u(s))\nabla s\\
&+\frac{\lambda}{\widetilde{d}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)f(s,u(s))\nabla s+\int_{\xi_i}^T(T-s)a(s)f(s,u(s))\nabla s\right),0\Bigg\}\\
\leq&\lambda\sup_{t\in[0,T]_{\mathbb{T}},u\in[0,c]}f(t,u)\Bigg[(T-t)\int_0^t a(s)\nabla s+\int_t^T(T-s)a(s)\nabla s\\
&+\frac{1}{\widetilde{d}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)\nabla s+\int_{\xi_i}^T(T-s)a(s)f(s,u(s))\nabla s\right)\Bigg]\\
\leq&\lambda\sup_{t\in[0,T]_{\mathbb{T}},u\in[0,c]}f(t,u)\Bigg[\int_0^T(T-s)a(s)\nabla s\\
&+\frac{1}{\widetilde{d}}\sum_{i=1}^{m-2}\alpha_i\left(\int_0^{\xi_i}(T-\xi_i)a(s)\nabla s+\int_{\xi_i}^T(T-s)a(s)\nabla s\right)\Bigg]\\
\leq&bM\sup_{t\in[0,T]_{\mathbb{T}},u\in[0,c]}f(t,u)=c.
\end{aligned}$$

If there is $u\in\partial K_c$ such that $Bu=u$, then the operator B has a fixed point in $\overline{K}_c$. If not, for any $u\in\partial K_c, Bu\neq u$, the above implies that

$$\deg_K\{I-B,K_c,0\}=1,$$

where $\deg_K$ denotes the topological degree on cone K. Then the operator B has a fixed point in K_c. So in both case the operator B has a fixed point in $\overline{K}_c$.

Next, we claim that u is also a fixed point of the operator A in $\overline{K}_c$. Suppose this is not true, then there exists $t_0\in[0,T]_{\mathbb{T}}$, such that

$$(Au)(t_0)\neq u(t_0)=(Bu)(t_0)=\max\{(Au)(t_0),0\}.$$

So this forces

$$(Au)(t_0) < 0 = u(t_0).$$

Let I be the maximal interval that contains t_0 such that $(Au)(t) < 0$ for all $t \in I$. Note that

$$u(t) = (Bu)(t) = \max\{(Au)(t), 0\} = 0, \quad t \in I.$$

We claim that $I \neq [0,T]_{\mathbb{T}}$, otherwise, we would have $u(t) = 0$ for $t \in [0,T]_{\mathbb{T}}$, which contradicts to the assumption that $a(t)f(t,0) \not\equiv 0$ on any subinterval of $[0,T]_{\mathbb{T}}$. So we should show either $T \notin I$ or $0 \notin I$.

If $T \notin I$, let $\beta = \min_{t\in I}\{s \in [0,T]_{\mathbb{T}} : s > t\}$, then $(Au)(\beta) \geq 0$. Since $(Au)(t) < 0$ for $t \in I$, we have $(Au)^{\Delta}(\beta) \geq 0$. Hence $(Au)(t) < 0$ and is bounded away from 0 everywhere in I, this forces $0 \in I$, we have $u(t) = 0$ for $t \in I$. Taking into account $(Au)^{\Delta\nabla}(t) = -a(t)f(t,0) \leq 0$ for all $t \in I$, we see that $(Au)^{\Delta}(t)$ is decreasing on I, and so $(Au)^{\Delta}(t) > 0$ for $t \in I = [0,\beta)_{\mathbb{T}}$. In particular, $(Au)^{\Delta}(0) > 0$, which contradicts with the first condition of (1.2).

If $0 \notin I$, let $\gamma = \max_{t\in I}\{s \in [0,T]_{\mathbb{T}} : s < t\}$, then $(Au)(\gamma) \geq 0$. Since $(Au)(t) < 0$ for $t \in I$, we have $(Au)^{\Delta}(\gamma) \leq 0$, and $(Au)^{\Delta}(t)$ is decreasing on I, since $(Au)^{\Delta\nabla}(t) = -a(t)f(t,0) \leq 0$ for all $t \in I$. $(Au)^{\Delta}(t) \geq (Au)^{\Delta}(T)$ and $(Au)(t) < 0$ imply that $(Au)(T) < 0$, that is $T \in I$. If $(Au)(\xi_i) \geq 0$, for any $i \in \{1,2,\ldots,m-2\}$, which contradicts with the second condition in (1.2), on the other hand, if $(Au)(\xi_j) < 0$, we must have $\xi_j \in I$. Therefore $(Au)(\xi_i) > (Au)(T)$ for $i \in \{1,2,\ldots,m-2\}$ and so

$$(Au)(T) = \sum_{i=1}^{m-2} \alpha_i (Au)(\xi_i) \geq \sum_{i=1}^{m-2} \alpha_i (Au)(T),$$

which is also impossible, since $\sum_{i=1}^{m-2} \alpha_i < 1$ and $(Au)(T) < 0$. Combining with these cases, we conclude that u is also a fixed point of A. Therefore, $u(t)$ is one positive solution of the problem (1.1), (1.2) with $0 < \|u\| \leq c$. □

Corollary 3.2. *Assume* $(H_1) - (H_3)$ *hold. Suppose that* $f(t,0) \geq 0$ *for* $t \in [0,T]_{\mathbb{T}}$ *and* $a(t)f(t,0) \not\equiv 0$ *for* $t \in [0,T]_{\mathbb{T}}$. *If*

$$\lim_{u\to\infty} \frac{\sup_{t\in[0,T]_{\mathbb{T}}} f(t,u)}{u} = 0,$$

then for any $\lambda \in (0,\infty)$, *the boundary value problem* (1.1), (1.2) *has at least one positive solution* $u(t)$ *with* $0 < \|u\| < \infty$.

4. MULTIPLE POSITIVE SOLUTIONS

Theorem 4.1. *Assume* $(H_1) - (H_3)$ *hold. Suppose that* $f(t,0) \geq 0$ *for* $t \in [0,T]_{\mathbb{T}}$, $a(t)f(t,0) \not\equiv 0$ *for* $t \in [0,T]_{\mathbb{T}}$, *and there exist numbers* b, c *and* d *such that* $0 < \frac{b}{\gamma} <$

$c < \gamma d < d$, $f(t,u) \geq 0$ for $(t,u) \in [0,T]_{\mathbb{T}} \times [b,d]$. If λ satisfies

$$\frac{d}{M \min_{t\in[0,T]_T, u\in[\gamma d,d]} f(t,u)} \leq \lambda \leq \frac{c}{M \max_{t\in[0,T]_{\mathbb{T}}, u\in[0,c]} f(t,u)}, \tag{4.1}$$

then the boundary value problem (1.1), (1.2) has at least two positive solutions $u_1(t)$ and $u_2(t)$ with $0 < \|u_1\| \leq c \leq \|u_2\| \leq d$.

Proof. First of all, in view of the second inequality and Theorem 3.1, we obtain that the problems (1.1), (1.2) has a positive solution u_1 with $0 < \|u_1\| \leq c$.

Next we show that the existence of another positive solution of the problem (1.1), (1.2). By the continuity of f and the definition of the operator C, it is easy to see that $C : K' \to K'$ is completely continuous. For $u \in \partial K'_c$, from the definition of C and the first inequality of (4.1) we have

$$\begin{aligned}
\|Cu\| &= Cu(0) = \lambda \int_0^T (T-s)a(s)f^+(s,u(s))\nabla s \\
&\quad + \frac{\lambda}{\tilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T-\xi_i)a(s)f^+(s,u(s))\nabla s + \int_{\xi_i}^T (T-s)a(s)f^+(s,u(s))\nabla s \right) \\
&\leq \frac{c}{M} \left[\int_0^T (T-s)a(s)\nabla s + \frac{1}{\tilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T-\xi_i)a(s)\nabla s + \int_{\xi_i}^T (T-s)a(s)\nabla s \right) \right] \\
&= c.
\end{aligned}$$

For $u \in \partial K'_d$, we have $\|u\| = d$ and in view of Lemma 2.4, we have

$$\min_{t\in[0,T]_{\mathbb{T}}} u(t) = u(t) \geq \gamma \|u\| = \gamma d,$$

so $u(t) \in [\gamma d, d]$ for $t \in [0,T]_{\mathbb{T}}$. By (2.6) and (4.1) we obtain

$$\begin{aligned}
\|Cu\| &= \lambda \int_0^T (T-s)a(s)f^+(s,u(s))\nabla s \\
&\quad + \frac{\lambda}{\tilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T-\xi_i)a(s)f^+(s,u(s))\nabla s + \int_{\xi_i}^T (T-s)a(s)f^+(s,u(s))\nabla s \right) \\
&\geq \frac{d}{M} \left[\int_0^T (T-s)a(s)\nabla s + \frac{1}{\tilde{d}} \sum_{i=1}^{m-2} \alpha_i \left(\int_0^{\xi_i} (T-\xi_i)a(s)\nabla s + \int_{\xi_i}^T (T-s)a(s)\nabla s \right) \right] \\
&= d.
\end{aligned}$$

It follows that

$$\deg_{K'} \{I - C, K'_c, 0\} = 1, \ \deg_{K'} \{I - C, K'_d, 0\} = 0.$$

Thus

$$\deg_{K'} \left\{I - C, K'_d \setminus \overline{K'_c}, 0\right\} = -1,$$

and C has a fixed point u_2 in $K'_d \setminus \overline{K'_c}$.

Finally we claim that u_2 is also a fixed point of A in $K'_d \setminus \overline{K'_c}$. In fact, for $u_2 \in \left(K'_d \setminus \overline{K'_c}\right) \cap \{u : Cu = u\}$, it is clear that $\|u_2\| > c$, and by Lemma 2.4

$$u_2(T) = \min_{t\in[0,T]_{\mathbb{T}}} u_2(t) \geq \gamma\,|u_2| \geq \gamma c > b,$$

so $b \leq u_2(t) \leq d$ for $t \in [0,T]_{\mathbb{T}}$. By assumption we know that $f^+(t,u) = f(t,u)$ for $(t,u) \in [0,T]_{\mathbb{T}} \times [b,d]$. This implies that $Au_2 = Cu_2$ for $u_2 \in \left(K'_d \setminus \overline{K'_c}\right) \cap \{u : Cu = u\}$. so u_2 is a fixed point of the operator A on K. Therefore, the problem (1.1), (1.2) has at least two positive solutions u_1 and u_2 with $0 < \|u_1\| \leq c \leq \|u_2\| \leq d$. □

Example 4.2. Let $\mathbb{T} = \{1 - (\frac{1}{2})^{\mathbb{N}_0}\} \cup \{1\}$. Taking $T = 1, \alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3}, \xi_1 = \frac{1}{2}$, $\xi_2 = \frac{7}{8}$, we have $\gamma = \frac{7}{11}$. If we let $a(t) \equiv 1$, then by some simple calculation, we get $M = \frac{13}{8}$.

Suppose

$$f(u) = (1-u)\sin u.$$

Clearly f is sign changing in $[0,\infty)$.

If we choose $c = 3.25$, $d = 5.93$, $\gamma d = 3.77$ it is easy to get that

$$\min_{t\in[0,T]_T, u\in[\gamma d,d]} f(t,u) = \min_{u\in[3.77,5.93]} (1-u)\sin u \approx 1.66,$$

$$\max_{t\in[0,T]_{\mathbb{T}}, u\in[0,c]} f(t,u) = \max_{u\in[0,3.25]} (1-u)\sin u \approx 0.24.$$

So

$$\frac{d}{M \min_{t\in[0,T]_T, u\in[\gamma d,d]} f(t,u)} \approx 2.20,$$

$$\frac{c}{M \max_{t\in[0,T]_{\mathbb{T}}, u\in[0,c]} f(t,u)} \approx 8.33.$$

Thus, if $2.20 < \lambda < 8.33$, then the problem

$$u^{\Delta\nabla}(t) + \lambda(1-u(t))\sin u(t) = 0,$$

$$u^{\Delta}(0) = 0,\ u(1) = \frac{1}{2}u\left(\frac{1}{2}\right) + \frac{1}{3}u\left(\frac{7}{8}\right)$$

has at least two positive solutions u_1 and u_2 in K with $0 < \|u_1\| \leq 3.25 \leq \|u_2\| \leq 5.93$ by Theorem 4.1.

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REFERENCES

[1] D.R. Anderson, Solution to second-order three-points problem on time scales, *J. Difference Equation Appl.*, **8** (2002), 673-688.

[2] D.R. Anderson, Twin n-point boundary value problems, *Appl. Math. Lett.*, **17** (2004), 1053-1059.

[3] F.M. Atici, G. Guseinov, Green's functions and positive solution for boundary value problem on time scales, *J. Comput. Appl. Math.*, **141** (2002), 75-99.

[4] M. Bohner, A. Peterson, *Dynamic Equation On Time Scales: An Introduction with Applications*, Birkhauser, Boston, 2001.

[5] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhauser Boston, 2003.

[6] W.S. Cheung, J. Ren, Twin positive solution for quasi-linear multipoint boundary value problem, *Nonlinear Anal.*, **62** (2005), 167-177.

[7] S. Dong, W. Ge, Positive solutions of an m-point boundary value problem with sign changing nonlinearities, *Computers Math. Appl.*, **49** (2005), 589-598.

[8] D. Guo, V. Lakshmikantham, *Nonlinear Problem in Abstract Cones*, San Diego, Academic Press, 1988.

[9] S. Hilger, Analysis on measure chains - a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18-56.

[10] V. Jamieson, V. Spedding, Taming nature's numbers, *New Scientist: the global science and technology weekly*, **2404** (2003), 28-31.

[11] M.A. Jones, B. Song, D.M. Thomas, Controlling wound healing through debridement, *Math. Comput. Modelling.*, **40** (2004), 1057-1064.

[12] E.R. Kaufmann, Positive solution of a three-point boundary value problem on a time scale, *Electronic J. Differential Equ.*, **82** (2003), 1-11.

[13] R. Leggett, L. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana. Univ. Math. J.*, **28** (1979), 673-688.

[14] W.T. Li, H.R. Sun, Positive solutions for second-order m-point boundary value problems on time scales, *Acta. Math. Sinica*, **22(6)** (2006), 1797-1804.

[15] H.R. Sun, W.T. Li, Existence of positive solution for nonlinear three-points boundary value problem on time scales, *J. Math. Anal. Appl.*, **299** (2004), 508-524.

[16] D.M. Thomas, L. Vandemuelebroeke, K. Yamaguchi, A mathematical evolution model for phytoremediation of metals, *Discrete Contin. Dyn. Syst. Ser. B*, **5** (2005), 411-422.