Communications in Applied Analysis **12** (2008), no. 3, 235–244

POSITIVE PERIODIC SOLUTIONS OF SYSTEM OF FUNCTIONAL DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we deal with the existence of positive periodic solutions of the functional difference system $x(n+1) = A(n)x(n) + F(n, x_n)$. Moreover we characterize the eigenvalue intervals for $x(n+1) = A(n)x(n) + \lambda H(n)G(x_n)$. The technique is based on a fixed point theorem in conical shells.

AMS Subject Classification. 39A10, 39A12

Keywords. Positive periodic solutions; Functional difference systems; Eigenvalue intervals; Fixed point theorem in cones.

1. INTRODUCTION

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers, and $\mathbb{Z}^+$ the nonnegative integers, respectively. In this paper, we study the existence of positive periodic solutions of the following nonlinear functional difference system

$$x(n+1) = A(n)x(n) + F(n, x_n). \tag{1.1}$$

Here $x = [x_1, x_2, \ldots, x_k]^T$ (T stands for the transpose), $F = [f_1, f_2, \ldots, f_k]^T$ and $A(n) = \mathrm{diag}[a_1(n), a_2(n), \ldots, a_k(n)]$. For $j \in \{1, 2, \ldots, k\}$, a_j is $\omega-$periodic, $f_j(n, x) : \mathbb{Z} \times \mathbb{R}^k \to \mathbb{R}$ is continuous in x and $f_j(n, x)$ is $\omega-$periodic in n and x, whenever x is $\omega-$periodic, $\omega \geq 1$ is an integer. Let X denote the class of $\omega-$periodic, continuous functions $x : \mathbb{Z} \to \mathbb{R}^k$ with the norm $\|x\| = \max_{1\leq j\leq k} |x_j|_0$ for $x = (x_1, x_2, \ldots, x_k)$, here $|x_j|_0 = \max_{\theta\in\mathbb{Z}} |x_j(\theta)|$. Then X is a Banach space. If $x \in X$, then $x_n \in X$ for any $n \in \mathbb{Z}$ is defined by $x_n(\theta) = x(n + \theta)$ for $\theta \in \mathbb{Z}$. We also denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ for $a, b \in \mathbb{Z}$ and $a < b$.

The system (1.1) has been proposed as a model for a variety of population dynamics. In fact, we can obtain (1.1) if we consider the multiple-species ecological model and take into account the time delay effect. Recently, the existence of positive periodic solutions for nonlinear functional difference equations and functional

Received January 1, 2007

difference systems have been studied by many authors; see, for example, [4, 12, 14]. Moreover, many authors have considered the continuous case of system (1.1), i.e.,

$$x'(t) = A(t)x(t) + F(t, x_t) \tag{1.2}$$

in the literature [6, 11, 13]. The existence, multiplicity and nonexistence of positive periodic solutions for the general scalar nonlinear nonautonomous delayed differential equation

$$x'(t) = a(t)g(x(t))x(t) - \lambda b(t)f(t, x(t - \tau(t)))$$

has been studied in [15].

In some of the papers mentioned above, the main technique is based on Krasnoselskii fixed point theorem on compression and expansion of cones. In fact, this fixed point theorem have been extensively employed in studying the existence of positive solutions to many kinds of boundary value problems (see for example, [2, 3, 5, 7]). The main results in this paper are proved by employing another fixed point theorem (see Theorem 2.1 in section 2) for compact maps in conical shells. To do this, we extend the ideas introduced by Lan and Webb in [9, 10] to the discrete case, see Lemma 3.1.

We will prove a general existence result for system (1.1) in section 3. As an application, in section 4, we study the following eigenvalue problem

$$x(n + 1) = A(n)x(n) + \lambda H(n)G(x_n). \tag{1.3}$$

Here $H(n) = \mathrm{diag}[h_1(n), h_2(n), \dots, h_k(n)]$, $G(x) = [g^1(x), g^2(x), \dots, g^k(x)]^T$ and $\lambda > 0$ is a positive parameter. We prove that (1.3) has at least one positive periodic solution for each λ in an explicit eigenvalue interval. Recently, several eigenvalue characterizations for different kinds of boundary value problems have appeared and we refer the readers to [1, 2, 3].

Throughout this paper, we assume the following two conditions on $A(n)$ and $F(n, x)$.

(H$_1$) $0 < a_j(n) < 1$ for all $n \in [0, \omega - 1]$, $f_j(n, x_n) \geq 0$ for all $n \in \mathbb{Z}$ and $x : \mathbb{Z} \to \mathbb{R}^k_+$, $j = 1, 2, \dots, k$.

(H$_2$) For any $L > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $[x,\ y \in X,\ \|x\| \leq L,\ \|y\| \leq L,\ \|x - y\| < \delta,\ 0 \leq s \leq \omega]$ imply

$$|F(s, x_s) - F(s, y_s)| < \varepsilon.$$

2. PRELIMINARIES

In this section, we state some preliminary results, which are essential to the proofs of our main results in section 3.

First we recall that a completely continuous operator means a continuous operator which transforms every bounded set into a relatively compact set. If D is a subset X, we write $D_K = D \cap K$ and $\partial_K D = (\partial D) \cap K$.

Theorem 2.1 ([8]). *Let X be a Banach space and K a cone in X. Assume Ω^1, Ω^2 are open bounded subsets of X with $\Omega^1_K \neq \emptyset$, $\overline{\Omega^1}_K \subset \Omega^2_K$. Let*

$$T : \overline{\Omega^2}_K \to K$$

be a continuous and compact operator such that

(i) $\|Tx\| \leq \|x\|$ *for* $x \in \partial_K \Omega^1$, *and*

(ii) *there exists* $e \in K \backslash \{0\}$ *such that* $x \neq Tx + \lambda e$ *for all* $x \in \partial_K \Omega^2$ *and all* $\lambda > 0$.

Then T has a fixed point in $\overline{\Omega^2}_K \setminus \Omega^1_K$. The same conclusion remains valid if (i) *holds on $\partial_K \Omega^2$ and* (ii) *holds on $\partial_K \Omega^1$.*

Lemma 2.2 ([14]). *Assume* (H_1) *holds. Then $x_j(n) \in X$ is a solution of*

$$x_j(n+1) = a_j(n)x_j(n) + f_j(n, x_n), \qquad j = 1, 2, \ldots, k,$$

if and only if

$$x_j(n) = \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u, x_u), \qquad j = 1, 2, \ldots, k,$$

where

$$G_j(n,u) = \frac{\displaystyle\prod_{s=u+1}^{n+\omega-1} a_j(s)}{1 - \displaystyle\prod_{s=n}^{n+\omega-1} a_j(s)}, \quad u \in [n, n+\omega-1], \quad j = 1, 2, \ldots, k. \tag{2.1}$$

Remark 2.3. It is clear that, for $j = 1, 2, \ldots, k$,

$$G_j(n,u) = G_j(n+\omega, u+\omega), \qquad for\ all\ (n,u) \in \mathbb{Z}^2.$$

To define the desired cone, we observe that

$$\frac{\displaystyle\prod_{s=0}^{\omega-1} a_j(s)}{1 - \displaystyle\prod_{s=n}^{n+\omega-1} a_j(s)} \leq G_j(n,u) \leq \frac{\displaystyle\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \displaystyle\prod_{s=n}^{n+\omega-1} a_j(s)} \tag{2.2}$$

for all $u \in [n, n+\omega-1]$.

For all $(n,s) \in \mathbb{Z}^2$ and $j = 1, 2, \ldots, k$, let $\sigma_j = (\prod_{s=0}^{\omega-1} a_j(s))^2$, then $\sigma_j \in (0,1)$, $j = 1, 2, \ldots, k$. Now we define

$$K = \{x \in X : \ x(n) \geq 0, \ n \in \mathbb{Z} \text{ and } x_j(n) \geq \sigma_j |x_j|_0, \ \forall\ j = 1, 2, \ldots, k\}. \tag{2.3}$$

One can easily verify that K is a cone in X. Moreover, let $T : K \to X$ be a map with components $(T^1, \dots, T^k)$, where $T^j, j = 1, 2, \dots, k$, is defined by

$$(T_i x)(n) = \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u, x_u), \quad j = 1, 2, \dots, k; \tag{2.4}$$

here $G_j(n,u)$ is given as (2.1).

Lemma 2.4. *Assume that* (H$_1$) *and* (H$_2$) *hold. Then* T *is well defined and maps* K *into* K*. Moreover,* T *is continuous and completely continuous.*

Proof Since (H$_2$) holds, using a standard argument, one can show that T is continuous and completely continuous. Moreover, the periodicity properties of the functions F and $A(n)$ guarantee that $(T_j x)(n) = (T_j x)(n + \omega)$ for all $j = 1, 2, \dots, k$.

Next, to show that T maps K into K. Let $x \in K$, so we have

$$(T_j x(n)) \geq 0, \quad j = 1, 2, \dots, k.$$

By using (2.4), (2.2), we see that

$$(T_j x)(n) \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} f_j(u, x_u),$$

and this implies that

$$|T_j x|_0 = \max_{n\in[0,\omega-1]} |(T_j x)(n)| \leq \frac{\prod_{s=0}^{\omega-1} a_j^{-1}(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} f_j(u, x_u).$$

Therefore,

$$\begin{aligned}(T_j x)(n) &= \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u, x_u) \geq \frac{\prod_{s=0}^{\omega-1} a_j(s)}{1 - \prod_{s=n}^{n+\omega-1} a_j(s)} \sum_{u=n}^{n+\omega-1} f_j(u, x_u) \\ &\geq (\prod_{s=0}^{\omega-1} a_j(s))^2 |T_j x|_0 = \sigma_j |T_j x|_0.\end{aligned}$$

That is, $T(K) \subset K$. $\square$

3. MAIN RESULTS

In this section we establish the existence of positive periodic solutions for system (1.1) and characterize the eigenvalue intervals for system (1.3). First we extend the ideas introduced by Lan and Webb in [9, 10] to the discrete case.

For $r > 0$, we define the open sets

$$\Omega^r = \{x \in X : \min_{n\in[0,\omega-1]} x_j(n) < \sigma_j r \text{ for all } j = 1, 2, \dots, k\},$$

$$B^r = \{x \in X : \|x\| < r\}.$$

Lemma 3.1. $\Omega^r,\ B^r$ *defined above have the following properties:*

(a) Ω^r_K *and* B^r_K *are open relative to* K.

(b) $B^{\sigma r}_K \subset \Omega^r_K \subset B^r_K$, *here* $\sigma = \min\{\sigma_j,\ j = 1, 2, \dots, k\}$.

(c) $x \in \partial_K \Omega^r$ *if and only if* $x \in K$ *and* $\min_{n\in[0,\omega-1]} x_i(n) = \sigma_i r$ *for some* $i \in \{1, 2, \dots, k\}$ *and* $\min_{n\in[0,\omega-1]} x_j(n) \le \sigma_j r$ *for each* $j \in \{1, 2, \dots, k\}$.

(d) *If* $x \in \partial_K \Omega^r$, *then* $\sigma_i r \le x_i(n) \le r,\ n \in [0, \omega-1]$ *for some* $i \in \{1, 2, \dots, k\}$ *and* $0 \le x_j(n) \le r,\ n \in [0, \omega-1]$ *for each* $j \in \{1, 2, \dots, k\}$. *Moreover,* $|x_j|_0 \le r$.

(e) *For each* $\delta > r$, *the following relations hold:*

$$\Omega^r_K = (\Omega^r \cap B^\delta)_K \quad \text{and} \quad \overline{\Omega^r}_K = (\overline{\Omega^r \cap B^\delta})_K.$$

Proof (a) is true since $\min_{n\in[0,\omega-1]} x_j(n)$ is continuous (discrete topology) for each $j \in \{1, 2, \dots, k\}$. (c) is clear. Let $x \in \partial_K \Omega^r$, so we have from (c) that there exists $i \in \{1, 2, \dots, k\}$ such that

$$\sigma_i |x_i|_0 \le \min_{n\in[0,\omega-1]} x_i(n) = \sigma_i r.$$

Thus $|x_i|_0 \le r$ and $\sigma_i r \le x_i(n) \le r,\ n \in [0, \omega-1]$. In addition notice for each $j \in \{1, 2, \dots, k\}$ that $\sigma_j |x_j|_0 \le \min_{n\in[0,\omega-1]} x_j(n) \le \sigma_j r$, so $|x_j|_0 \le r$ and $0 \le x_j(n) \le r$ for $n \in [0, \omega-1]$, i.e., (d) holds.

Now we prove (b). Let $x \in B^{\sigma r}_K$, then for each $j \in \{1, 2, \dots, k\}$, we have $|x_j|_0 < \sigma r$, so $\min_{n\in[0,\omega-1]} x_j(n) < \sigma r \le \sigma_j r$ and $x \in \Omega^r_K$. If $x \in \Omega^r_K$, then for each $j \in \{1, 2, \dots, k\}$, we have $\min_{n\in[0,\omega-1]} x_j(n) < \sigma_j r$ and $x_j(n) \ge \sigma_j |x_j|_0$ for $n \in [0, \omega-1]$. This implies $|x_j|_0 < r$, i.e., $\Omega^r_K \subset B^r_K$. Hence (b) holds.

Finally we prove (e). The first equality follows immediately from (b). For the second let $x \in \overline{\Omega^r}_K$, then from (c), we have that

$$\sigma_j |x_j|_0 \le \min_{n\in[0,\omega-1]} x_j(n) \le \sigma_j r < \sigma_j \delta, \quad j = 1, 2, \dots, k.$$

Thus $|x_j|_0 < \delta,\ j = 1, 2, \ldots, k$, and this implies that $x \in (\overline{\Omega^r} \cap B^\delta) \cap K$. Now, since Ω^r and B^δ are open sets we have $\overline{\Omega^r} \cap B^\delta \subset \overline{\Omega^r \cap B^\delta}$. Thus $x \in (\overline{\Omega^r \cap B^\delta})_K$, and therefore $\overline{\Omega^r}_K \subseteq (\overline{\Omega^r \cap B^\delta})_K$. The reverse inclusion is trivial. □

Remark 3.2. It is clear that the sets Ω^r are unbounded sets for each $r > 0$, so we cannot use Theorem 2.1 with Ω^r directly. However we will be able to apply Theorem 2.1 with Ω^r_K since (e) holds.

Theorem 3.3. *Assume that* (H_1) *and* (H_2) *hold. Furthermore, it is assumed that the following two hypotheses hold:*

(D_1) *For each* $j = 1, 2, \ldots, k$, *there exist a constant* $\alpha > 0$ *and a continuous function* $\psi_j : \mathbb{Z} \to (0, \infty)$ *such that*

$$f_j(n, x) \geq \sigma_j \alpha \psi_j(n),\ \text{for all } n \in [0, \omega - 1],\ 0 \leq x_l \leq \alpha\ (l \in \{1, 2, \ldots, k\} \backslash \{j\})$$

and $\sigma_j \alpha \leq x_j \leq \alpha$; *and*

$$\min_{n \in [0, \omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n, u) \psi_j(n) \geq 1;$$

(D_2) *For each* $j = 1, 2, \ldots, k$, *there exist a constant* $\beta > 0$ *and a continuous function* $\chi_j : [0, \omega - 1] \to (0, \infty)$ *such that*

$$f_j(n, x) \leq \beta \chi_j(n) \qquad \text{for all } n \in [0, \omega - 1],\ 0 < x_j \leq \beta$$

and

$$\max_{n \in [0, \omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n, u) \chi_j(n) \leq 1.$$

Then, the following results hold:

(a) *if* $\beta < \sigma\alpha$, *then problem* (1.1) *has at least one positive periodic solution* x *satisfying*

$$\beta \leq \|x\| = \max_{j \in \{1, 2, \ldots, k\}} \max_{n \in [0, \omega-1]} |x_j(n)| \leq \alpha;$$

(b) *if* $\alpha < \beta$, *then problem* (1.1) *has at least one positive periodic solution* x *satisfying*

$$\sigma\alpha \leq \|x\| \leq \beta.$$

Proof As was indicated in the introduction, the proof is based on Theorem 2.1. We show that :

(i) $\|Tx\| \leq \|x\|$ for $x \in \partial_K B^\beta$, and
(ii) there exists $e \in K \backslash \{0\}$ such that $x \neq Tx + \lambda e$, for all $x \in \partial_K \Omega^\alpha$ and all $\lambda > 0$.

We start with (i). Now for any $x \in \partial_K B^\beta$, we have $|x_j|_0 \le \beta$ for each $j \in \{1, ..., k\}$. Fix $j \in \{1, ..., k\}$. Then from (D_2) we obtain, for each $n \in [0, \omega - 1]$,

$$\begin{aligned}(T_j x)(n) &= \sum_{u=n}^{n+\omega-1} G_j(n,u) f_j(u, x_u) \le \beta \sum_{u=n}^{n+\omega-1} G_j(n,u)\chi_j(u) \\ &\le \beta \max_{n\in[0,\omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n,u)\chi_j(u) \le \beta.\end{aligned}$$

Hence, $|T_j x|_0 \le \|x\|$ for each $j \in \{1, ..., k\}$. This implies that (i) holds.

Next we consider part (ii). Let $e(t) \equiv 1$, so $e \in K \setminus \{0\}$. Next, suppose that there exists $x \in \partial_K \Omega^\alpha$ and $\lambda > 0$ such that $x = Tx + \lambda e$. Since $x \in \partial_K \Omega^\alpha$, then from Lemma 3.1 (d) there exists $i \in \{1, 2, \ldots, k\}$ with $\sigma_i \alpha \le x_i(n) \le \alpha,\ n \in [0, \omega - 1]$, and $0 \le x_j(n) \le \alpha$ for $n \in [0, \omega - 1]$ and $j \in \{1, 2, \ldots, k\}\setminus\{i\}$.

From (D_1) we have, for $n \in [0, \omega - 1]$, that

$$\begin{aligned}x_i(n) &= (T_i x)(n) + \lambda = \sum_{u=n}^{n+\omega-1} G_i(n,u) f_i(u, x_u) + \lambda \\ &\ge \sigma_i \alpha \sum_{u=n}^{n+\omega-1} G_i(n,u)\psi_i(u) + \lambda \\ &\ge \sigma_i \alpha \min_{n\in[0,\omega-1]} \sum_{u=n}^{n+\omega-1} G_i(n,u)\psi_i(u) + \lambda \ge \sigma_i\alpha + \lambda.\end{aligned}$$

Hence $\min_{n\in[0,\omega-1]} x_i(n) \ge \sigma_i\alpha + \lambda > \sigma_i\alpha$, contradicting the statement of Lemma 3.1 (c). This contradiction proves part (ii) above.

Now suppose that $\beta < \sigma\alpha$. Then one has from Lemma 3.1 that $\overline{B^\beta}_K \subset B_K^{\sigma\alpha} \subset \Omega_K^\alpha$ and therefore it follows from Theorem 2.1 that T has at least one fixed point $x \in \overline{\Omega_K^\alpha} \setminus B_K^\beta$. Hence $\|x\| \ge \beta$ and $\sigma_j\beta \le \min_{n\in[0,\omega-1]} x_j(n) \le \sigma_j\alpha,\ j \in \{1, 2, \ldots, k\}$. On the other hand, $\sigma_j |x_j|_0 \le \min_{n\in[0,\omega-1]} x_j(n) \le \sigma_j\alpha$ and therefore $|x_j|_0 \le \alpha$ for each $j \in \{1, 2, \ldots, k\}$. This implies that $\|x\| \le \alpha$.

Finally, if $\alpha < \beta$ one has $\overline{\Omega^\alpha}_K \subset B_K^\beta$, and then Theorem 2.1 guarantees the existence of at least one fixed point $x \in \overline{B^\beta}_K \setminus \Omega_K^\alpha$ of T. Hence we obtain the inequality $\sigma\alpha \le \|x\| \le \beta$. □

4. EIGENVALUE INTERVALS OF (1.3)

In this section, we employ Theorem 3.3 to characterize the eigenvalue intervals of the system (1.3). First we establish one existence result for the following functional difference system

$$x(n+1) = A(n)x(n) + H(n)G(x_n), \tag{4.1}$$

here $H(n) = \mathrm{diag}[h_1(n), h_2(n), \ldots, h_k(n)]$, $G(x) = [g^1(x), g^2(x), \ldots, g^k(x)]^T$.

For each $j = 1, 2, \ldots, k$, we assume that:

(H$_3$) $g^j : \mathbb{R}_+^k \to \mathbb{R}_+$ is continuous with $g^j(x) > 0$ for $\|x\| > 0$.

(H$_4$) $h_j(n) : \mathbb{Z} \to \mathbb{R}_+$ is continuous and $\sum_{u=n}^{n+\omega-1} G_j(n,u)h_j(u) > 0$.

Theorem 4.1. *Suppose that conditions* (H$_1$), (H$_3$) *and* (H$_4$) *hold. Then problem* (4.1) *has at least one positive periodic solution x with $x(n) \not\equiv 0$ for $n \in [0, \omega-1]$ if one of the following conditions holds.*

(h_1) $0 \le g_0^j < A_j^{-1}$ *and* $B_j^{-1} < g_\infty^j \le \infty,\ j = 1, 2, \ldots, k;$

(h_2) $0 \le g_\infty^j < A_j^{-1}$ *and* $B_j^{-1} < g_0^j \le \infty,\ j = 1, 2, \ldots, k;$

here $g_0^j = \lim_{x \to 0^+} \dfrac{g(x)}{\|x\|}, \quad g_\infty^j = \lim_{x\to\infty} \dfrac{g(x)}{\|x\|},\ j = 1, 2, \ldots, k,$ *and*

$$A_j = \max_{n\in[0,\omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n,u)h_j(u), \quad B_j = \min_{n\in[0,\omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n,u)h_j(u).$$

Proof To see this, we will apply Theorem 3.3 with $f_j(n,x) = h_j(n)g^j(x),\ j = 1, 2, \ldots, k$. We assume that ($h_1$) holds. The case when ($h_2$) holds is similar.

From the first part of (h_1), there exists $\beta > 0$ such that $g^j(x) \le A_j^{-1}\beta$ for $0 < \|x\| \le \beta$. Choose $\chi_j(n) = A_j^{-1}h_j(n)$ for $j = 1, 2, \ldots, k$. Fix $j \in \{1, ..., k\}$. Then

$$f_j(n,x) = h_j(n)g^j(x) \le A_j^{-1}\beta h_j(n) = \beta\chi_j(n) \qquad \text{if } \ n \in [0, \omega-1] \text{ and } \ 0 < x_j \le \beta$$

and

$$\sum_{u=n}^{n+\omega-1} G_j(n,u)\chi_j(u) = A_j^{-1} \sum_{u=n}^{n+\omega-1} G_j(n,u)h_j(u)$$
$$\le A_j^{-1} \max_{n\in[0,\omega-1]} \sum_{u=n}^{n+\omega-1} G_j(n,u)h_j(u) = 1.$$

Thus hypothesis (D$_2$) holds.

From the second part of (h_1), there exists $\alpha > 0$ such that $\sigma_j\alpha > \beta$ and $g^j(x) \ge B_j^{-1}\sigma_j\alpha$ for $x_j \ge \sigma_j\alpha,\ j = 1, 2, \ldots, k$.

Thus $g^j(x) \ge B_j^{-1}\sigma_j\alpha$ for $x_j \ge \sigma_j\alpha,\ j = 1, 2, \ldots, k$. Choose $\psi_j(n) = B_j^{-1}h_j(n)$, then

$$f_j(n,x) = h_j(n)g^j(x) \ge B_j^{-1}\sigma_j\alpha h_j(n) = \sigma_j\alpha\psi_j(n), \text{ if } n \in [0, \omega-1],\ x_j \ge \sigma_j\alpha,$$

(so in particular for $\sigma_j\alpha \le x_j \le \alpha$) and

$$\sum_{n=u}^{n+\omega-1} G_j(n,u)\psi_j(u) = B_j^{-1} \sum_{n=u}^{n+\omega-1} G_j(n,u)h_j(u)$$
$$\ge B_j^{-1} \min_{n\in[0,\omega-1]} \sum_{n=u}^{n+\omega-1} G_j(n,u)h_j(u) = 1.$$

This implies that hypothesis (D_1) holds. The result now follows from Theorem 3.3. □

Theorem 4.2. *Suppose that conditions* ($\mathtt{H}_1$), ($\mathtt{H}_3$) *and* ($\mathtt{H}_4$) *hold. Then problem* (1.3) *has at least one positive periodic solution for each*

$$\lambda \in \Big(\frac{1}{B\min\limits_{j=1,2,\ldots,k}\{g_\infty^j\}}, \frac{1}{A\max\limits_{j=1,2,\ldots,k}\{g_0^j\}}\Big) \tag{4.2}$$

if $\frac{1}{B\min\limits_{j=1,2,\ldots,k}\{g_\infty^j\}} < \frac{1}{A\max\limits_{j=1,2,\ldots,k}\{g_0^j\}}$. *The same result remains valid for each*

$$\lambda \in \Big(\frac{1}{B\min\limits_{j=1,2,\ldots,k}\{g_0^j\}}, \frac{1}{A\max\limits_{j=1,2,\ldots,k}\{g_\infty^j\}}\Big) \tag{4.3}$$

if $\frac{1}{B\min\limits_{j=1,2,\ldots,k}\{g_0^j\}} < \frac{1}{A\max\limits_{j=1,2,\ldots,k}\{g_\infty^j\}}$. *Here*

$$A = \max\{A_j, j = 1, 2, \ldots, k\}, \quad B = \min\{B_j, j = 1, 2, \ldots, k\}$$

and we write $1/g_\alpha^j = 0$ *if* $g_\alpha^j = \infty$ *and* $1/g_\alpha^j = \infty$ *if* $g_\alpha^j = 0$, *where* $\alpha = 0, \infty$.

Proof We consider the case (4.2). The case (4.3) is similar. If λ satisfies (4.2), then

$$\lambda g_0^j \le \lambda \max_{j=1,2,\ldots,k}\{g_0^j\} < \frac{1}{A} \le \frac{1}{A_j}, \qquad j = 1, 2, \ldots, k,$$

and

$$\lambda g_\infty^j \ge \lambda \min_{j=1,2,\ldots,k}\{g_\infty^j\} > \frac{1}{B} \ge \frac{1}{B_j}, \qquad j = 1, 2, \ldots, k.$$

Thus Theorem 4.1 applies directly. □

Remark 4.3. Our results improve those in [12] when $n = 1$.

Remark 4.4. In this paper, if condition (H_1) is replaced by

(H_1)* If $a_j(n) > 1$ for all $n \in [0, \omega-1]$, then $f_j(n, x_n) \le 0$ for all $n \in \mathbb{Z}$ and $x : \mathbb{Z} \to \mathbb{R}_+^k$, $j = 1, 2, \ldots, k$.

Then we can obtain the similar results as we present in Theorem 3.3 and Theorem 4.2.

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