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FIXED-SIGN SOLUTIONS OF VOLTERRA INTEGRAL EQUATIONS: THE SEMIPOSITONE CASE

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ABSTRACT. We consider the following Volterra intergral equation

$$u(t) = \mu \int_0^t g(t,s) f(s,u(s))ds, \;\; t \in [0,T]$$

where $\mu > 0$. Here, the function f may take *'negative'* values, i.e., the 'semipositone' case. Criteria are offered for the existence of one and more *fixed-sign* solutions u of the equation in $C[0,T]$. We say u is of *fixed sign* if $u(t) \geq 0$ or $u(t) \leq 0$ for all $t \in [0,T]$.

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1. INTRODUCTION

In this paper we shall consider the Volterra integral equation

$$u(t) = \mu \int_0^t g(t,s) f(s,u(s))ds, \;\; t \in [0,T], \tag{1.1}$$

where μ is a positive constant. The nonlinearity f need not be *'positive'* in the sense that θf can take *negative* values, where $\theta \in \{1,-1\}$ is fixed.

The Volterra integral equation (1.1) has received a lot of attention in the literature [9–12, 17, 18, 22, 24], since it arises in real-world problems. For example, astrophysical problems (e.g., the study of the density of stars) give rise to the Thomas-Fermi equation

$$u'' - t^p u^q = 0, \;\; t \in [0,T], \;\; u(0) = u'(0) = 0 \quad \text{where} \quad p \geq 0 \text{ and } 0 < q < 1$$

which reduces to (1.1) when $g(t,s) = (t-s)s^p$ and $f(t,u) = u^q$. Other examples occur in nonlinear diffusion and percolation problems (see [10, 11] and the references cited therein) such as

$$u(t) = \int_0^t (t-s)^{\gamma-1} f(u(s))ds, \;\; t \in [0,T],$$

where $\gamma > 1$.

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For the special case $\theta = 1$, the fact that in (1.1) we allow $\theta f = f$ to take negative values is referred to as the *semipositone* case, which arises naturally in chemical reactor theory [13]. The constant μ in (1.1) is called the *Thiele modulus*. It is of physical interest to examine the existence of positive solutions when μ is small. Most results in the literature are devoted to *positone* problems, i.e., when f is nonnegative, see [14–16, 19, 21, 23] and the references cited therein. Only a few results (see [1, Chapter 4] and [5, 6, 8]) are available for *semipositone Fredholm* integral equations, but as far as we know no results are available for semipositone Volterra integral equations. Therefore, in the present work we shall establish the existence of one and more solutions u of the semipositone problem (1.1) in $C[0,T]$. Moreover, we are concerned with *fixed-sign* solutions u, by which we mean $\theta u(t) \geq 0$ for all $t \in [0,T]$, where $\theta \in \{1,-1\}$ is fixed. Note that *positive* solution is a special case of fixed-sign solution when $\theta = 1$. Recently, Agarwal, O'Regan and Wong [2–8] have been interested in the existence of fixed-sign solutions of Fredholm integral equations. We shall tackle the existence of fixed-sign solutions of the semipositone Volterra integral equation (1.1) in Section 2.

2. EXISTENCE RESULTS

Our main tool is Krasnosel'skii's fixed point theorem which we state as follows.

Theorem A. [20] *Let $B = (B, \|\cdot\|)$ be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are bounded open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$ be a continuous and completely continuous operator such that, either*

(a) $\|Su\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Su\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, *or*
(b) $\|Su\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, *and* $\|Su\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then, S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Let the Banach space $B = C[0,T]$ be equipped with the norm $\|u\| = \sup_{t\in[0,T]} |u(t)|$.

Theorem 2.1. *Let $1 \leq p < \infty$ be a constant and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta \in \{1,-1\}$ be fixed. Assume*

(C1)

$$g(t,s) \geq 0, \ \ t \in [0,T], \ \ a.e. \ s \in [0,t],$$

$$g(t,s) > 0, \ \ t \in (0,T], \ \ a.e. \ s \in [0,t],$$

$$g^t(s) \equiv g(t,s) \in L^p[0,t] \text{ for each } t \in [0,T],$$

$$\sup_{t\in[0,T]} \int_0^t [g^t(s)]^p ds < \infty;$$

(C2) *for any* $t, t' \in [0, T]$,

$$\int_0^{t^*} |g(t,s) - g(t',s)|^p ds \to 0 \text{ as } t \to t'$$

where $t^* = \min\{t, t'\}$;

(C3) *for any* t_1, t_2 *satisfying* $0 < t_1 \leq t_2 \leq T$,

$$g(t_2, s) - g(t_1, s) \geq 0, \text{ a.e. } s \in [0, t_1];$$

(C4) $f : [0,T] \times [0,\infty)^* \to \mathbb{R}$ *is a Carathéodory function, and there exists constant* $M > 0$ *such that*

$$\theta f(t,x) + M \geq 0, \ (t,x) \in [0,T] \times [0,\infty)^*$$

where

$$[0,\infty)^* = \begin{cases} [0,\infty), & \theta = 1, \\ (-\infty, 0], & \theta = -1, \end{cases}$$

(note that $(0,\infty)^*$ *is similarly defined);*

(C5)

$$b(t,x) \leq \theta f(t,x) + M \leq c(t,x), \ (t,x) \in [0,T] \times [0,\infty)^*$$

where $b, \ c : [0,T] \times [0,\infty)^* \to [0,\infty)$ *are continuous, and*

$$b(t,x) > 0, \ (t,x) \in (0,T] \times (0,\infty)^* \ ;$$

moreover, b *and* c *are 'nondecreasing' in the sense that if* $\theta x \leq y$, *then*

$$b(t,x) \leq b(t, \theta y), \ t \in [0,T]$$

$$c(t,x) \leq c(t, \theta y), \ t \in [0,T];$$

(C6) *there exists a constant* $L > 0$, *and a function* $a \in C[0,T]$ *with* $a(0) = 0$ *and* $0 < a(t) \leq 1, \ t \in (0,T]$ *such that*

$$\int_0^t g(t,s) ds \leq L a(t), \ t \in [0,T];$$

(C7) *for any* $R > \mu M L > 0$,

$$\int_0^t g(t,s) b\left(s, \theta(R - \mu M L) a(s)\right) ds \geq a(t) \cdot \int_0^T g(T,s) c(s, \theta R) ds, \ t \in [0,T];$$

(C8) *for any* $R > 0$, *if* $p > 1$ *then*

$$\int_0^T |c(s, \theta R)|^q ds < \infty;$$

if $p = 1$, *then*

$$\text{ess} \sup_{t \in [0,T]} |c(t, \theta R)| < \infty;$$

(C9) *there exists* $\alpha > \mu M L > 0$ *such that*

$$\mu \int_0^T g(T,s) c(s, \theta \alpha) ds \leq \alpha;$$

(C10) *there exists* $\beta > \mu ML > 0$, $\beta \neq \alpha$, *such that*

$$\mu \int_0^T g(T,s) b\left(s, \theta(\beta - \mu ML) a(s)\right) ds \geq \beta.$$

Then, (1.1) has at least one fixed-sign solution $u \in C[0,T]$ *such that*

$$\theta u(t) \geq 0, \ t \in [0,T] \qquad \textit{and} \qquad \theta u(t) > 0, \ t \in (0,T]. \tag{2.1}$$

Moreover, we have

(a) $0 < \alpha - \|\phi\| \leq \|u\| \leq \beta$ *and* $\theta u(t) \geq a(t)\alpha - \mu M \int_0^t g(t,s)ds, \ t \in [0,T]$ *if* $\alpha < \beta$;
(b) $0 < \beta - \|\phi\| \leq \|u\| \leq \alpha$ *and* $\theta u(t) \geq a(t)\beta - \mu M \int_0^t g(t,s)ds, \ t \in [0,T]$ *if* $\beta < \alpha$;

where $\|\phi\| = \mu M \int_0^T g(T,s)ds$.

Proof. To show that (1.1) has a fixed-sign solution, we consider the system

$$y(t) = \mu \int_0^t g(t,s) f^*(s, y(s) - \phi(s)) ds, \ t \in [0,T], \tag{2.2}$$

where

$$\phi(t) = \theta \mu M \int_0^t g(t,s) ds, \ t \in [0,T] \tag{2.3}$$

and

$$f^*(t,x) = f(t,x) + \theta M, \ (t,x) \in [0,T] \times [0,\infty)^*. \tag{2.4}$$

We shall show that (2.2) has a fixed-sign solution y^* satisfying

$$\theta y^*(t) \geq \theta\phi(t), \ t \in [0,T] \qquad \text{and} \qquad \theta y^*(t) > \theta\phi(t), \ t \in (0,T]. \tag{2.5}$$

Then, it is easy to see that $u = y^* - \phi$ is a fixed-sign solution of (1.1) satisfying (2.1).

We shall employ Theorem A. Without any loss of generality, let $\beta < \alpha$. To proceed, we define a cone C_a and open subsets Ω_α, Ω_β in B as

$$C_a = \left\{ y \in B \;\middle|\; \theta y \text{ is nondecreasing on } [0,T], \text{ and } \theta y(t) \geq a(t)\|y\| \text{ for } t \in [0,T] \right\}, \tag{2.6}$$

$$\Omega_\alpha = \{y \in B \mid \|y\| < \alpha\} \qquad \text{and} \qquad \Omega_\beta = \{y \in B \mid \|y\| < \beta\}. \tag{2.7}$$

Note that for $y \in C_a$, we have

$$\|y\| = \theta y(T). \tag{2.8}$$

Let the operator $S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to B$ be defined by

$$Sy(t) = \mu \int_0^t g(t,s) f^*(s, y(s) - \phi(s)) ds, \ t \in [0,T]. \tag{2.9}$$

Clearly, a fixed point of the operator S is a solution of (2.2). Indeed, a fixed point of S obtained in C_a will be a *fixed-sign solution* of (2.2). Since a solution y^* of (2.2) satisfies $y^*(0) = 0$, from (2.6) we must have $a(0) = 0$ if we require y^* to be in C_a. Moreover, from the definition of C_a, we should have $a(t) \in [0,1]$ for $t \in [0,T]$. All these are fulfilled noting (C6).

We shall show that the operator $S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a$ is continuous and completely continuous. First, we shall prove that

$$S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C[0,T] \text{ is well defined.} \tag{2.10}$$

Let $y \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Then, $\|y\| = R \in [\beta, \alpha]$ and so

$$0 \le a(t)\beta \le a(t)R \le \theta y(t) \le R \le \alpha, \ t \in [0,T],$$

and

$$\theta y(t) \ge a(t)R \ge a(t)\beta > 0, \ t \in (0,T].$$

Thus, together with (C6), we obtain for $t \in (0,T]$,

$$\theta[y(t) - \phi(t)] = \theta y(t) - \mu M \int_0^t g(t,s)ds \ge a(t)R - \mu M L a(t) \ge (\beta - \mu M L)a(t) > 0. \tag{2.11}$$

Moreover, it is obvious that

$$0 \le \theta[y(t) - \phi(t)] \le \theta y(t) \le \|y\| = R \le \alpha, \ t \in [0,T]. \tag{2.12}$$

It follows from (C4) and (C5) that for $t \in (0,T]$,

$$0 < |f^*(t, y(t) - \phi(t))| = \theta f(t, y(t) - \phi(t)) + M \le c(t, y(t) - \phi(t)) \le c(t, \theta R) \tag{2.13}$$

and

$$\begin{aligned} |f^*(t, y(t) - \phi(t))| &= \theta f(t, y(t) - \phi(t)) + M \\ &\ge b(t, y(t) - \phi(t)) \\ &\ge b(t, \theta(R - \mu M L)a(t)). \end{aligned} \tag{2.14}$$

Now, for $t, t' \in [0,T]$ and $t' < t$, we employ Hölder's inequality and (2.13) to obtain

$$\begin{aligned} &|Sy(t) - Sy(t')| \\ &\le \mu \int_0^{t'} |g(t,s) - g(t',s)| \cdot |f^*(s, y(s) - \phi(s))| ds + \mu \int_{t'}^t |g(t,s)| \cdot |f^*(s, y(s) - \phi(s))| ds \\ &\le \mu \left[\left(\int_0^{t'} |g(t,s) - g(t',s)|^p ds \right)^{\frac{1}{p}} + \left(\int_{t'}^t |g(t,s)|^p ds \right)^{\frac{1}{p}} \right] \left(\int_0^T |f^*(s, y(s) - \phi(s))|^q ds \right)^{\frac{1}{q}} \\ &\le \mu \left[\left(\int_0^{t'} |g(t,s) - g(t',s)|^p ds \right)^{\frac{1}{p}} + \left(\int_{t'}^t |g(t,s)|^p ds \right)^{\frac{1}{p}} \right] \left(\int_0^T [c(s, \theta R)]^q ds \right)^{\frac{1}{q}}. \end{aligned}$$

Then, in view of (C1), (C2) and (C8), it follows that

$$|Sy(t) - Sy(t')| \to 0 \text{ as } t \to t'. \tag{2.15}$$

This proves (2.10).

Next, we shall check that

$$S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a. \tag{2.16}$$

Once again let $y \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Noting (C1) and (C4), we obtain

$$\theta(Sy)(t) = \mu \int_0^t g(t,s)[\theta f(s, y(s) - \phi(s)) + M]ds \geq 0, \ t \in [0,T]. \tag{2.17}$$

Now, let $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$. Then,

$$\begin{aligned} \theta(Sy)(t_2) - \theta(Sy)(t_1) &= \int_0^{t_1} [g(t_2,s) - g(t_1,s)] \cdot [\theta f(s, y(s) - \phi(s)) + M]ds \\ &\quad + \int_{t_1}^{t_2} g(t_2,s)[\theta f(s, y(s) - \phi(s)) + M]ds \\ &\geq 0 \end{aligned}$$

where we have used (C3), (C1) and (C4) in the last inequality. Hence, $\theta(Sy)$ is nondecreasing on $[0,T]$. It remains to show that $\theta(Sy)(t) \geq a(t)\|Sy\|$ for $t \in [0,T]$. Noting (2.14), it is clear that for $t \in [0,T]$,

$$\theta(Sy)(t) \geq \mu \int_0^t g(t,s) b(s, \theta(R - \mu ML)a(s))ds \tag{2.18}$$

where $R = \|y\|$. On the other hand, using (2.13) we get

$$\theta(Sy)(T) \leq \mu \int_0^T g(T,s) c(s, \theta R)ds. \tag{2.19}$$

Since $\theta(Sy)$ is nondecreasing, we obtain

$$\|Sy\| = \theta(Sy)(T) \leq \mu \int_0^T g(T,s) c(s, \theta R)ds \equiv A. \tag{2.20}$$

Applying (2.20) in (2.18), we get for $t \in [0,T]$,

$$\theta(Sy)(t) \geq \mu \int_0^t g(t,s) b(s, \theta(R - \mu ML)a(s))ds \cdot \frac{\|Sy\|}{A} \geq a(t)\|Sy\|$$

where we have used (C7) in the last inequality. This completes the proof of (2.16).

Next, we shall show that

$$S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a \text{ is compact.} \tag{2.21}$$

Once again let $y \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Using the monotonicity of $\theta(Sy)$, (2.19), Hölder's inequality, (C1) and (C8), we obtain for $t \in [0,T]$,

$$\begin{aligned} \theta(Sy)(t) \leq \theta(Sy)(T) &\leq \mu \int_0^T g(T,s) c(s, \theta R)ds \\ &\leq \mu \|g^T\|_p \left(\int_0^T [c(s, \theta R)]^q ds \right)^{\frac{1}{q}} \equiv A_0 < \infty. \end{aligned}$$

Thus, $S(C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta))$ is uniformly bounded. Moreover, (2.15) guarantees the continuity of Sy. Hence, the compactness of $S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a$ follows from the Arzéla-Ascoli theorem. Having established (2.10), (2.16) and (2.21), we have shown that $S : C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta) \to C_a$ is continuous and completely continuous.

We shall now show that (i) $\|Sy\| \le \|y\|$ for $y \in C_a \cap \partial\Omega_\alpha$, and (ii) $\|Sy\| \ge \|y\|$ for $y \in C_a \cap \partial\Omega_\beta$. To verify (i), let $u \in C_a \cap \partial\Omega_\alpha$. Then, $\|y\| = R = \alpha$. Applying $(2.20)|_{R=\alpha}$ and (C9), we obtain

$$\|Sy\| \le \mu \int_0^T g(T,s)c(s,\theta\alpha)ds \le \alpha = \|y\|.$$

Next, to prove (ii), let $y \in C_a \cap \partial\Omega_\beta$. So $\|y\| = R = \beta$. Now $\|Sy\| = \theta(Sy)(T)$ Thus, using $(2.18)|_{R=\beta}$ and (C10) we find

$$\|Sy\| = \theta(Sy)(T) \ge \mu \int_0^T g(T,s)b(s,\theta(\beta - \mu ML)a(s))ds \ge \beta = \|y\|.$$

Having obtained (i) and (ii), it follows from Theorem A that S has a fixed point $y^* \in C_a \cap (\overline{\Omega}_\alpha \backslash \Omega_\beta)$. Thus,

$$\beta \le \|y^*\| \le \alpha \qquad \text{and} \qquad \theta y^*(t) \ge a(t)\beta,\ t \in [0,T]. \tag{2.22}$$

Using the fact that $\beta > \mu ML > 0$, (C6) and (2.3), we find for $t \in (0,T]$,

$$\theta y^*(t) \ge a(t)\|y^*\| \ge \beta a(t) > \mu MLa(t) \ge \mu M \int_0^t g(t,s)ds = \theta\phi(t).$$

Also, it is clear that $\theta y^*(t) \ge \theta\phi(t)$ for $t \in [0,T]$. Hence, y^* satisfies (2.5).

Since the fixed-sign solution u of (1.1) is given by $u = y^* - \phi$, we have (2.1) and also, in view of (2.22),

$$\theta u(t) = \theta y^*(t) - \theta\phi(t) \ge a(t)\beta - \mu M \int_0^t g(t,s)ds,\ t \in [0,T]$$

and

$$\beta - \|\phi\| \le \|y^*\| - \|\phi\| \le \|u\| \le \|y^*\| \le \alpha.$$

Now, noting (C3) we find

$$\|\phi\| = \sup_{t\in[0,T]} \mu M \int_0^t g(t,s)ds \le \sup_{t\in[0,T]} \mu M \int_0^t g(T,s)ds = \mu M \int_0^T g(T,s)ds.$$

Clearly,

$$\|\phi\| = \sup_{t\in[0,T]} \mu M \int_0^t g(t,s)ds \ge \mu M \int_0^T g(T,s)ds.$$

Thus, we get $\|\phi\| = \mu M \int_0^T g(T,s)ds$. Finally, to see that $\beta - \|\phi\| > 0$, using (C6) we get

$$\beta > \mu ML \ge \mu MLa(t) \ge \mu M \int_0^t g(t,s)ds = \theta\phi(t),\ t \in [0,T]$$

and so $\beta > \|\phi\|$. Therefore, conclusion (b) follows immediately. The proof is complete. □

Remark 2.1. If (C2) is changed to

(C2)′ for any $t, t' \in [0, T]$,

$$\int_0^{t^*} |g(t,s) - g(t',s)|^p ds + \int_{t^*}^{t^{**}} [g(t^{**}, s)]^p ds \to 0 \text{ as } t \to t'$$

where $t^* = \min\{t, t'\}$ and $t^{**} = \max\{t, t'\}$,

then automatically we have $\sup_{t\in[0,T]} \left(\int_0^t [g^t(s)]^p ds \right) < \infty$ which appears in (C1).

Remark 2.2. In (C9) if we have *strict* inequality instead, i.e.,

$$\mu \int_0^T g(T,s) c(s, \theta\alpha) ds < \alpha,$$

then from the latter part of the proof of Theorem 2.1 we see that a fixed point y^* of S must satisfy $\|y^*\| \neq \alpha$. Similarly, if the inequality in (C10) is *strict*, i.e.,

$$\mu \int_0^T g(T,s) b\left(s, \theta(\beta - \mu ML) a(s)\right) ds > \beta,$$

then a fixed point y^* of S must fulfill $\|y^*\| \neq \beta$. Hence, with *strict* inequalities in (C9) and (C10), the conclusion of Theorem 2.1 becomes:

The system (1.1) has at least one fixed-sign solution $u \in C[0,T]$ such that (2.1) holds. Moreover, we have

(a) $0 < \alpha - \|\phi\| < \|u\| < \beta$ and $\theta u(t) > a(t)\alpha - \mu M \int_0^t g(t,s) ds,\ t \in [0,T]$ if $\alpha < \beta$;
(b) $0 < \beta - \|\phi\| < \|u\| < \alpha$ and $\theta u(t) > a(t)\beta - \mu M \int_0^t g(t,s) ds,\ t \in [0,T]$ if $\beta < \alpha$;

where $\|\phi\| = \mu M \int_0^T g(T,s) ds$.

The next result generalizes Theorem 2.1 and gives the existence of *multiple* fixed-sign solutions of (1.1).

Theorem 2.2. *Let $1 \leq p < \infty$ be a constant and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta \in \{1, -1\}$ be fixed. Assume (C1)–(C8) hold. Let (C9) be satisfied for $\alpha = \alpha_\ell,\ \ell = 1, 2, \ldots, k$, and (C10) be satisfied for $\beta = \beta_\ell,\ \ell = 1, 2, \ldots, m$. Let $\|\phi\| = \mu M \int_0^T g(T,s) ds$.*

(a) *Let $m = k + 1$ and $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k < \beta_{k+1}$.*
  (i) *If $\alpha_i < \beta_{i+1} - \|\phi\|,\ 1 \leq i \leq k - 1$, then (1.1) has (at least) k fixed-sign solutions $u^1, \ldots, u^k \in C[0,T]$ such that*

$$\beta_i - \|\phi\| \leq \|u^i\| \leq \alpha_i,\ 1 \leq i \leq k.$$

  (ii) *If $\beta_i < \alpha_i - \|\phi\|,\ 2 \leq i \leq k$, then (1.1) has (at least) k fixed-sign solutions $u^1, \ldots, u^k \in C[0,T]$ such that*

$$\alpha_i - \|\phi\| \leq \|u^i\| \leq \beta_{i+1},\ 1 \leq i \leq k.$$

(b) *Let* $m = k$ *and* $0 < \beta_1 < \alpha_1 < \cdots < \beta_k < \alpha_k$.

(i) *If* $\alpha_i < \beta_{i+1} - \|\phi\|,\ 1 \le i \le k-1$, *then (1.1) has (at least)* k *fixed-sign solutions* $u^1, \ldots, u^k \in C[0,T]$ *such that*

$$\beta_i - \|\phi\| \le \|u^i\| \le \alpha_i,\ 1 \le i \le k.$$

(ii) *If* $\beta_i < \alpha_i - \|\phi\|,\ 2 \le i \le k-1$, *then (1.1) has (at least)* $k-1$ *fixed-sign solutions* $u^1, \ldots, u^{k-1} \in C[0,T]$ *such that*

$$\alpha_i - \|\phi\| \le \|u^i\| \le \beta_{i+1},\ 1 \le i \le k-1.$$

(c) *Let* $k = m+1$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m < \alpha_{m+1}$.

(i) *If* $\beta_i < \alpha_{i+1} - \|\phi\|,\ 1 \le i \le m-1$, *then (1.1) has (at least)* m *fixed-sign solutions* $u^1, \ldots, u^m \in C[0,T]$ *such that*

$$\alpha_i - \|\phi\| \le \|u^i\| \le \beta_i,\ 1 \le i \le m.$$

(ii) *If* $\alpha_i < \beta_i - \|\phi\|,\ 2 \le i \le m$, *then (1.1) has (at least)* m *fixed-sign solutions* $u^1, \ldots, u^m \in C[0,T]$ *such that*

$$\beta_i - \|\phi\| \le \|u^i\| \le \alpha_{i+1},\ 1 \le i \le m.$$

(d) *Let* $k = m$ *and* $0 < \alpha_1 < \beta_1 < \cdots < \alpha_m < \beta_m$.

(i) *If* $\beta_i < \alpha_{i+1} - \|\phi\|,\ 1 \le i \le m-1$, *then (1.1) has (at least)* m *fixed-sign solutions* $u^1, \ldots, u^m \in C[0,T]$ *such that*

$$\alpha_i - \|\phi\| \le \|u^i\| \le \beta_i,\ 1 \le i \le m.$$

(ii) *If* $\alpha_i < \beta_i - \|\phi\|,\ 2 \le i \le m-1$, *then (1.1) has (at least)* $m-1$ *fixed-sign solutions* $u^1, \ldots, u^{m-1} \in C[0,T]$ *such that*

$$\beta_i - \|\phi\| \le \|u^i\| \le \alpha_{i+1},\ 1 \le i \le m-1.$$

Proof. In (a), by applying Theorem 2.1 repeatedly, we find that there are possibly $2k$ (*not* necessarily distinct) fixed-sign solutions to (1.1), namely, $u^1, \ldots, u^{2k} \in C[0,T]$ such that

$$\beta_1 - \|\phi\| \le \|u^1\| \le \alpha_1, \quad \alpha_1 - \|\phi\| \le \|u^2\| \le \beta_2,$$

$$\beta_2 - \|\phi\| \le \|u^3\| \le \alpha_2, \quad \alpha_2 - \|\phi\| \le \|u^4\| \le \beta_3, \quad \ldots,$$

$$\beta_k - \|\phi\| \le \|u^{2k-1}\| \le \alpha_k, \quad \alpha_k - \|\phi\| \le \|u^{2k}\| \le \beta_{k+1}.$$

In case (i), we see that k of these solutions are distinct, namely, $u^1, u^3, \ldots, u^{2k-1}$ with

$$\beta_1 - \|\phi\| \le \|u^1\| \le \alpha_1, \quad \beta_2 - \|\phi\| \le \|u^3\| \le \alpha_2, \quad \ldots, \quad \beta_k - \|\phi\| \le \|u^{2k-1}\| \le \alpha_k.$$

In case (ii), it is clear that the k solutions $u^2, u^4, \ldots, u^{2k}$ are distinct with

$$\alpha_1 - \|\phi\| \le \|u^2\| \le \beta_2, \quad \alpha_2 - \|\phi\| \le \|u^4\| \le \beta_3, \quad \ldots, \quad \alpha_k - \|\phi\| \le \|u^{2k}\| \le \beta_{k+1}.$$

The proof of (b)–(d) is similar. ☐

Remark 2.3. Suppose in Theorem 2.2 we have some *strict* inequalities in (C9) and (C10), say, involving α_i and β_j for some $i \in \{1, 2, \ldots, k\}$ and some $j \in \{1, 2, \ldots, m\}$. Then, noting Remark 2.2, those inequalities in the conclusion involving α_i and β_j will also be *strict*.

We are now ready to discuss more specific conditions concerning the existence of $a(t)$ in (C6).

Theorem 2.3. *Let $1 \le p < \infty$ be a constant and q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $\theta \in \{1, -1\}$ be fixed. Suppose (C1)–(C5) and (C8) hold. Further, assume*

(C11) *there exists $N > 0$ such that*

$$g(t, s) \ge N > 0, \ t \in (0, T], \ a.e. \ s \in [0, t];$$

(C12) *for any $(t, x) \in [0, T] \times [0, \infty)^*$,*

$$b(t, x) \ge r(t)w(|x|)$$

and

$$c(t, x) \le \rho(t)w(|x|)$$

where $\rho, r : [0, T] \to [0, \infty)$, $r(t) > 0$ for a.e. $t \in [0, T]$, r is continuous, $w : [0, \infty) \to [0, \infty)$ is continuous, $w(s) > 0$ for $s > 0$, $w(st) \ge w(s)w(t)$ for $s, t > 0$, and $\frac{w(x)}{w(y)} \ge \ell > 0$ for $0 < x < y$;

(C13) *the function $J : [0, \infty) \to [0, \infty)$ defined by*

$$J(z) = \int_0^z \frac{dx}{w(x)}$$

satisfies

$$J^{-1}\left(\frac{N\ell}{Q} \int_0^t r(s)ds\right) \le 1, \ t \in [0, T],$$

where $Q = \int_0^T g(T, s)\rho(s)ds$.

Let

$$a(t) = J^{-1}\left(\frac{N\ell}{Q} \int_0^t r(s)ds\right) \qquad \text{and} \qquad L = \max_{t \in (0,T]} \frac{1}{a(t)} \int_0^t g(t, s)ds. \tag{2.23}$$

Further, let (C9) and (C10) hold. Then, the system (1.1) *has at least one fixed-sign solution $u \in C[0, T]$ satisfying* (2.1) *and conclusions (a) and (b) of Theorem* 2.1 *hold.*

Proof. Clearly, (C6) is satisfied. Theorem 2.1 is applicable if we can show that (C7) is fulfilled. To begin, notice the inequality in (C7) is the same as

$$\frac{\int_0^t g(t, s)b(s, \theta(R - \mu ML)a(s))ds}{\int_0^T g(T, s)c(s, \theta R)ds} \ge a(t), \ t \in [0, T]. \tag{2.24}$$

Now, applying (C12) and (C11), we find

$$\begin{aligned}\frac{\int_0^t g(t,s)b(s,\theta(R-\mu ML)a(s))ds}{\int_0^T g(T,s)c(s,\theta R)ds} &\geq \frac{\int_0^t g(t,s)r(s)w(|R-\mu ML|a(s))ds}{\int_0^T g(T,s)\rho(s)w(R)ds}\\ &\geq \frac{\ell\int_0^t g(t,s)r(s)w(a(s))ds}{\int_0^T g(T,s)\rho(s)ds}\\ &\geq \frac{N\ell}{Q}\int_0^t r(s)w(a(s))ds.\end{aligned}$$

Thus, (2.24) is satisfied if we can find some $a \in C[0,T]$ with $a(0)=0$ and $0<a(t)\leq 1,\ t\in(0,T]$, such that

$$a(t)=\frac{N\ell}{Q}\int_0^t r(s)w(a(s))ds. \tag{2.25}$$

We claim that (2.25) is satisfied if

$$a(t)=J^{-1}\left(\frac{N\ell}{Q}\int_0^t r(s)ds\right). \tag{2.26}$$

In fact, from (2.26) we have $J(a(t))=\frac{N\ell}{Q}\int_0^t r(s)ds$, or

$$\int_0^{a(t)}\frac{dx}{w(x)}=\frac{N\ell}{Q}\int_0^t r(s)ds.$$

Next, the above equation is the same as

$$\int_0^t \frac{a'(s)ds}{w(a(s))}=\frac{N\ell}{Q}\int_0^t r(s)ds$$

which upon differentiation gives

$$a'(t)=\frac{N\ell}{Q}r(t)w(a(t)).$$

Integrating the above from 0 to t then yields (2.25). Thus, (2.25) is satisfied if $a(t)$ is defined by (2.26), moreover this $a\in C[0,T]$ fulfills $a(0)=0$ and $0<a(t)\leq 1,\ t\in(0,T]$ (see (C13)).

We have shown that the condition (C7) is satisfied and so the conclusion follows from Theorem 2.1. □

By using Theorems 2.3 repeatedly, we obtain the existence of *multiple* fixed-sign solutions of (1.1).

Theorem 2.4. *Let* $1\leq p<\infty$ *be a constant and* q *be such that* $\frac{1}{p}+\frac{1}{q}=1$. *Let* $\theta\in\{1,-1\}$ *be fixed. Assume (C1)–(C5), (C8) and (C11)–(C13) hold. Let* $a(t)$ *and* L *be defined in* (2.23). *Let (C9) be satisfied for* $\alpha=\alpha_\ell,\ \ell=1,2,\ldots,k$, *and (C10) be satisfied for* $\beta=\beta_\ell,\ \ell=1,2,\ldots,m$. *Let* $\|\phi\|=\mu M\int_0^T g(T,s)ds$. *Then, conclusions (a)–(d) of Theorem* 2.2 *hold.*

Remark 2.4. Remarks similar to those of Remarks 2.1–2.3 also hold for Theorems 2.3 and 2.4.

We shall now present an example to illustrate the results obtained.

Example 2.1. Consider the Volterra integral equation (1.1) with

$$g(t,s) = t - s + 1, \qquad f(t,u) = -t \qquad \text{and} \qquad T > 0. \tag{2.27}$$

Suppose we are interested in positive solutions, thus we set $\theta = 1$.

We shall apply Theorem 2.3. Clearly, the conditions (C1)–(C3) are satisfied. Also, (C4) is satisfied if we choose $M = 2T$. Hence, in (C5) we have

$$T \le \theta f(t,u) + M = 2T - t \le 2T,$$

and we can choose $b = T$ and $c = 2T$. The condition (C8) is clearly fulfilled. Also, since

$$g(t,s) = t - s + 1 \ge t - t + 1 = 1 \equiv N,$$

the condition (C11) is satisfied with $N = 1$.

Next, in (C12) we can pick

$$r = T, \qquad \rho = 2T \qquad \text{and} \qquad w = 1 \text{ (thus } \ell = 1).$$

Hence, we find

$$Q = \int_0^T g(T,s)\rho(s)ds = \int_0^T 2T(T - s + 1)ds = T^2(T+2),$$

$$J(z) = \int_0^z \frac{dx}{w(x)} = \int_0^z dx = z \qquad \text{and} \qquad J^{-1}(z) = z.$$

It follows that

$$J^{-1}\left(\frac{N\ell}{Q}\int_0^t r(s)ds\right) = \frac{N\ell}{Q}\int_0^t r(s)ds = \frac{1}{Q}\int_0^t T ds = \frac{t}{T(T+2)} < 1$$

and so the condition (C13) is satisfied. Further, as in (2.23) we let

$$a(t) = J^{-1}\left(\frac{N\ell}{Q}\int_0^t r(s)ds\right) = \frac{t}{T(T+2)}$$

and

$$L = \max_{t\in(0,T]} \frac{1}{a(t)}\int_0^t g(t,s)ds = \max_{t\in(0,T]} \frac{1}{2}\, T(T+2)(t+2) = \frac{1}{2}\, T(T+2)^2.$$

The inequality in (C9) then reduces to

$$\mu \int_0^T g(T,s)c(s,\theta\alpha)ds = \mu T^2(T+2) \le \alpha$$

which is true for $\alpha > \mu ML$. Similarly, condition (C10) also holds for some $\beta(\neq \alpha)$.

We now conclude from Theorem 2.3 that the Volterra integral equation with (2.27) has at least one positive solution $u \in C[0,T]$ such that

$$u(t) \ge 0, \ t \in [0,T] \qquad \text{and} \qquad u(t) > 0, \ t \in (0,T]. \tag{2.28}$$

Moreover, we have

(a) $0 < \alpha - \|\phi\| \leq \|u\| \leq \beta$ and $u(t) \geq a(t)\alpha - \mu M \int_0^t g(t,s)ds = \frac{t}{T(T+2)}\ \alpha - 2T\mu\left(\frac{t^2}{2} + t\right),\ t \in [0,T]$ if $\alpha < \beta$;

(b) $0 < \beta - \|\phi\| \leq \|u\| \leq \alpha$ and $u(t) \geq a(t)\beta - \mu M \int_0^t g(t,s)ds = \frac{t}{T(T+2)}\ \beta - 2T\mu\left(\frac{t^2}{2} + t\right),\ t \in [0,T]$ if $\beta < \alpha$;

where $\|\phi\| = \mu M \int_0^T g(T,s)ds = \mu T^2(T+2)$.

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