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# RATE OF CONVERGENCE F TWO-DIMENSIONAL ANALOGOUE OF BASKAKOV OPERATORS

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**ABSTRACT.** The aim of this paper is to study the rate of convergence of the two-dimensional generalization of the Baskakov operators.

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## 1. Introduction

Approximation properties of Szász-Mirakyan operators

$$A_{m,n}(f;a_m,b_n;x,y) := \sum_{j=0}^{[m(x+a_m)]} \sum_{k=0}^{[n(y+b_n)]} \binom{m-1+j}{j} x^j (1+x)^{-m-j} \times \binom{n-1+k}{k} y^k (1+y)^{-n-k} f\left(\frac{j}{m},\frac{k}{n}\right), \quad f \in C_{p,q}, \quad (x,y) \in R_0^2 := R_0 \times R_0, \tag{1}$$

where $(a_m)_1^\infty$ and $(b_n)_1^\infty$ are given sequences of positive numbers such that $\lim_{m\to\infty} \sqrt{m} a_m = \infty$ and $\lim_{n\to\infty} \sqrt{n} b_n = \infty$, in polynomial weighted spaces $C_{p,q}$ were examined in [8]. The space $C_{p,q}$, $p,q \in N_0 := \{0,1,2,\dots\}$, considered in [8] is associated with the weighted function

$$w_{p,q}(x,y) := w_p(x) w_q(y), \qquad (x,y) \in R_0^2, \tag{2}$$

where $w_p(\cdot)$ is defined by

$$w_0(x) := 1, \quad w_p(x) := (1+x^p)^{-1}, \qquad \text{if} \quad p \geq 1, \tag{3}$$

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and consists of all real-valued functions $f$ continuous on $R_0^2$ for which $w_{p,q}f$ is uniformly continuous and bounded on $R_0^2$. The norm on $C_{p,q}$ is defined by

$$\|f\|_{p,q} \equiv \|f(\cdot,\cdot)\|_{p,q} := \sup_{(x,y)\in R_0^2} w_{p,q}(x,y)\,|f(x,y)|. \tag{4}$$

The operators (1) are related to the well-known Baskakov operators

$$V_{m,n}(f;;x,y) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{m-1+j}{j}x^j(1+x)^{-m-j}\binom{n-1+k}{k}y^k(1+y)^{-n-k}f\left(\frac{j}{m},\frac{k}{n}\right). \tag{5}$$

Note, that this two-dimensional analogue of the Baskakov operators

$$V_n(f;x) = \sum_{k=0}^{\infty}\binom{n-1+k}{k}x^k(1+x)^{-n-k}f\left(\frac{k}{n}\right), \quad x\in R_0,\ n\in N. \tag{6}$$

In view of [2] it is known that:

A. $V_n(1;x)=1$, $V_nf$ preserves constants.

B. For every fixed $2\le q\in N$ there exist algebraic polynomials $P_{j,q}$, $0\le j\le q$, on the order $m\le q$ and with coefficients depending only $j$ and $q$ such that

$$V_n\left((t-x)^q;x\right) = \sum_{j=0}^{[q/2]}\frac{P_{j,q}(x)}{n^{q-j}}, \qquad x\in R_0, \quad n\in N,$$

where $[q/2]$ denotes the integral part of $q/2$. Moreover $V_n\left(t^q;0\right)=0$ for all $n\in N$ and $q\in N$ (see [9, p. 125]).

It is known [8, Eq. (13)] that:

**Theorem 1.1.** *Suppose that $f\in C_{p,q}$, $p,q\in N_0$. Then there exists a positive constant $M_1(p,q)$ such that for all $(x,y)\in R_0^2$*

$$w_{p,q}(x,y)\,|V_{m,n}(f;x,y)-f(x,y)| \le M_1(p,q)\,\omega_1\left(f,C_{p,q};\sqrt{\frac{x(x+1)}{m}},\sqrt{\frac{y(y+1)}{n}}\right), \tag{7}$$

*$m,n\in N$, where*

$$\omega_1(f,C_{p,q};t,s) := \sup_{0\le h\le t,\,0\le\delta\le s}\|\Delta_{h,\delta}f(\cdot,\cdot)\|_{p,q}, \qquad t,s\ge 0, \tag{8}$$

*$\Delta_{h,\delta}f(x,y) := f(x+h,y+\delta)-f(x,y)$, $(x+h,y+\delta)\in R_0^2$ is the modulus of continuity of $f\in C_{p,q}$.*

From (8) it follows that

$$\lim_{t,s\to 0+}\omega_1(f,C_{p,q};t,s)=0 \tag{9}$$

for every $f \in C_{p,q}$, $p, q \in N_0$. Moreover $\omega(f, C_{0,0}; s, t)$ is nondecreasing function of variables $s$, $t$ and

$$\omega(f, C_{0,0}; c_1 s, c_2 t) \leq (2 + c_1 + c_2)\omega(f, C_{0,0}; s, t).$$

It is easy to verify that operators $A_{m,n}$ give the same rate of convergence as operators $V_{m,n}$.

In this paper we propose a new family of linear operators. The result is a form convenient for applications. Thus these operators, may play an important role in the applications to actual approximation schemes. We shall show that these operators have better approximation proporties then classical Baskakov operators.

Let $C^p_{0,0}$, $p \in N_0$, be the set of all $f \in C_{0,0}$ with all partial derivatives $\frac{\partial^k f}{\partial x^{k-i} \partial y^i}$, $0 \leq i \leq k \leq p$ belonging also to $C$ with the norm (4) ($C^0_{0,0} \equiv C_{0,0}$).

In this paper by $M_i(\alpha), i = 1, 2, ...$, we shall denote suitable positive constants depending only on indicated parameter $\alpha$.

## 2. **Main results**

We introduce the following class of operators in $C^p_{0,0}$, $p \in N_0$.

**Definition 2.1.** We define the class of operators $D_n$ by the formula

$$D_n(f; p; x, y) := \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{n-1+j}{j} x^j (1+x)^{-n-j} \times \binom{n-1+k}{k} y^k (1+y)^{-n-k} f\left(\frac{j}{n}, \frac{k}{n}\right) \sum_{i=0}^{p} \frac{d^i f\left(\frac{j}{n}, \frac{k}{n}\right)}{i!}, \tag{10}$$

$f \in C^p_{0,0}$, $(x, y) \in R^2_0$, $n \in N$, where $d^i f(s, t)$ is the $i$-th differential

$$d^i f(s,t) = \sum_{k=0}^{i} \binom{i}{k} \frac{\partial^i f(s,t)}{\partial x^{i-k} \partial y^k} (x-s)^{i-k} (y-t)^k.$$

In this section we shall state some estimate of the rate of convergence of $D_p$. We shall use the modulus of continuity defined by (8).

**Theorem 2.2.** *Suppose that $p \in N_0$. Then for every $f \in C^p_{0,0}$ we have*

$$D_n(f; p; x, y) - f(x, y) = o(n^{-p/2}), \qquad (x, y) \in R^2_0, \ n \to \infty. \tag{11}$$

*Proof.* We first suppose that $f \in C^p_{0,0}$, $p \in N$. We shall need the folowing modified Taylor formula

$$f(x,y) = \sum_{i=0}^{p} \frac{d^i f\left(\frac{j}{n},\frac{k}{n}\right)}{i!} + \frac{1}{(p-1)!} \times \int_0^1 (1-t)^{p-1} \Big\{ d^p f\left(\frac{j}{n} + t\left(x - \frac{j}{n}\right), \frac{k}{n} + t\left(y - \frac{k}{n}\right)\right) - d^p f\left(\frac{j}{n},\frac{k}{n}\right)\Big\} dt.$$

This implies that

$$\begin{aligned} &|D_n(f;p;x,y) - f(x,y)| \\ &\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{n-1+j}{j} x^j (1+x)^{-n-j} \binom{n-1+k}{k} y^k (1+y)^{-n-k} \\ &\quad \times \left| \sum_{i=0}^{p} \frac{d^i f\left(\frac{j}{n},\frac{k}{n}\right)}{i!} - f(x,y)\right| \\ &\quad \le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{n-1+j}{j} x^j (1+x)^{-n-j} \binom{n-1+k}{k} y^k (1+y)^{-n-k} \frac{1}{(p-1)!} \\ &\quad \times \int_0^1 (1-t)^{p-1} \left| d^p f\left(\frac{j}{n} + t\left(x - \frac{j}{n}\right), \frac{k}{n} + t\left(y - \frac{k}{n}\right)\right) \right. \\ &\quad \left. - d^p f\left(\frac{j}{n},\frac{k}{n}\right)\right| dt. \end{aligned}$$

Observe that

$$\begin{aligned} &\left| d^p f\left(\frac{j}{n} + t\left(x - \frac{j}{n}\right), \frac{k}{n} + t\left(y - \frac{k}{n}\right)\right) - d^p f\left(\frac{j}{n},\frac{k}{n}\right)\right| \\ &\quad \le \sum_{i=0}^{p} \binom{p}{i} \omega\left(\frac{\partial^p f}{\partial x^{p-i}\partial y^i}; t\left|x - \frac{j}{n}\right|, t\left|y - \frac{k}{n}\right|\right) \times \left|x - \frac{j}{n}\right|^{p-i} \left|y - \frac{k}{n}\right|^i. \end{aligned}$$

Therefore

$$\begin{aligned} |D_n(f;p;x,y) - f(x,y)| &\le \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{n-1+j}{j} x^j (1+x)^{-n-j} \binom{n-1+k}{k} y^k (1+y)^{-n-k} \\ &\times M_2(p) \sum_{i=0}^{p} \omega\left(\frac{\partial^p f}{\partial x^{p-i}\partial y^i}; n^{-1/2}, n^{-1/2}\right) \\ &\times \left(2 + \left|x - \frac{j}{n}\right| n^{1/2} + \left|y - \frac{k}{n}\right| n^{1/2}\right) \left|x - \frac{j}{n}\right|^{p-i} \left|y - \frac{k}{n}\right|^i, \end{aligned}$$

where $M_2(p)$ is a positive constant depending only on $p$. Applying the above inequality and (6), we obtain

$$\begin{aligned}|D_p(f;p;x,y)-f(x,y)| \le M_1(p)\sum_{i=0}^{p}\omega\left(\frac{\partial^p f}{\partial x^{p-i}\partial y^i};n^{-1/2},n^{-1/2}\right)\\ \times\Big\{2V_n\left(|x-s|^{p-i};x\right)V_n\left(|y-t|^i;y\right)\\ +n^{1/2}V_n\left(|x-s|^{p+1-i};x\right)V_n\left(|y-t|^i;y\right)\\ +n^{1/2}V_n\left(|x-s|^{p-i};x\right)V_n\left(|x-t|^{i+1};y\right)\Big\}.\end{aligned}$$

Using the Hölder inequality and the properties A and B, we get

$$\begin{aligned}V_n\left(|x-s|^{p-i};x\right) &\le \left(V_n\left((x-s)^{2(p-i)};x\right)V_n(1;x)\right)^{1/2}\\ &=\left(\sum_{j=0}^{[q/2]}\frac{P_{j,q}(x)}{n^{q-j}}\right)^{1/2}=O(n^{-(p-i)/2}),\quad n\to\infty.\end{aligned}$$

Analogously we obtain

$$\begin{aligned}V_n\left(|y-t|^i;y\right)&=O(n^{-i/2}),\quad n\to\infty,\\ V_n\left(|x-s|^{p+1-i};x\right)&=O(n^{-(p+1-i)/2}),\quad n\to\infty,\\ V_n\left(|y-t|^{i+1};y\right)&=O(n^{-(i+1)/2}),\quad n\to\infty.\end{aligned}$$

This implies that

$$|D_p(f;p;x,y)-f(x,y)|=o(n^{-p/2}),\quad p\in N,\quad n\to\infty.$$

□

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