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EXISTENCE RESULTS FOR FRACTIONAL ORDER INTEGRAL EQUATIONS OF MIXED TYPE IN FRÉCHET SPACES

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ABSTRACT. In this paper we discuss the existence of solutions for an integral equation of mixed type. We rely on a generalization on Fréchet spaces of a Krasnosel'skii type fixed point theorem due to Avramescu and on a nonlinear alternative of Leray-Schauder type for contraction maps in Fréchet spaces due to Frigon and Granas.

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1. INTRODUCTION

This paper is concerned with the existence of solutions for the following integral equation,

$$y(t) = f(t) + \int_0^t \frac{g(t,s,y(s))}{(t-s)^{1-\alpha}}ds + \int_0^\infty \frac{h(t,s,y(s))}{(t-s)^{1-\alpha}}ds,\ t \in [0,+\infty), \tag{1.1}$$

where $f : \mathbb{R}_+ \to \mathbb{R}^d$, $g : D \times \mathbb{R}^d \to \mathbb{R}^d$, $h : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ are continuous functions, and $D := \{(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+,\ s \le t\}$, $d > 1$ and $\alpha \in (0,1)$. Integral equations arise naturally in many applications in describing numerous real world problems; see for instance the books by Agarwal *et al.* [2], Agarwal and O'Regan [3], Corduneanu [11], Deimling [12], and O'Regan and Meehan [20] and the references therein. Also quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, kinetic theory of gases, in the theory of neutron transport and in traffic theory. Especially, the so-called quadratic integral equation of Chandrasekher type can be very often encountered in many applications; see for instance the book by Chandrasekher [10] and the research papers by Banas *et al.* [4, 5], Benchohra and Darwish [6], Burton and Zhang [8], Darwish [13], Hu *et al.* [15], Kelley [17], Leggett [18], and Stuart [21] and the references therein.

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2. PRELIMINARIES

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1 ([16, 19])**.** The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \leq 0$, and $\varphi_\alpha \to \delta(t)$ as $\alpha \to 0$, where δ is the delta function.

Consider the functional space

$$C_c := \{y : \mathbb{R}_+ \to \mathbb{R}^d |\ y \text{ is continuous}\},$$

equipped with the numerable families of seminorms

$$\|y\|_n := \sup_{t\in[0,n]} \|y(t)\|.$$

This family of semi norms determines on C_c a structure of a Fréchet space (i.e., a linear, metrisable, and complete space), its topology being the one of the uniform convergence on compact subsets of $\mathbb{R}_+$. We also mention that a family $\mathcal{A} \subset C_c$ is relatively compact if and only if for each $n \geq 1$, the restrictions to $[0,n]$ of all functions from $\mathcal{A}$ form an equicontinuous and uniformly bounded set.

Definition 2.2 ([1]))**.** The operator $H : C_c \to C_c$ is called a contraction if there is a sequence $L_n \in [0,1)$, such that

$$\|Hy - Hx\|_n \leq L_n \|y - x\|_n,\ \forall y, x \in C_c,\ \forall n \geq 1.$$

Proposition 2.3 ([9])**.** *Every contraction admits a unique fixed point.*

Theorem 2.4 (Nonlinear Alternative of Avramescu, [1])**.** *Let X be a Fréchet space and let $A, B : X \to X$ be two operators satisfying the following hypotheses:*

(i) *A is contraction;*
(ii) *B is compact operator.*

Then either one of the following statements holds:

(S_1) *The operator $A + B$ has a fixed point;*
(S_2) *the set $\{x \in X |\ x = \lambda A\left(\dfrac{x}{\lambda}\right) + \lambda B(x),\ \lambda \in (0,1)\}$ is unbounded.*

Theorem 2.5 ([14])**.** *Let Ω be a closed subset of a Fréchet space X such that $0 \in \Omega$ and let $F : \Omega \to X$ be a contraction such that $F(\Omega)$ bounded. Then either:*

(C1) *F has a unique fixed point; or*

(C2) *there exist $\lambda \in (0,1)$, $n \in \mathbb{N}$ and $u \in \partial\Omega^n$ such that $\|u - \lambda F(u)\|_n = 0$.*

3. MAIN RESULTS

In this section we present two results for Equation (1.1). The first one relies on the nonlinear alternative due to Avramescu (Theorem 2.4) and the second one the nonlinear alternative for contraction maps due to Frigon and Granas (Theorem 2.5). We will admit the following hypotheses:

(h_1) there exist $p \in L^1_{loc}(\mathbb{R}_+, \mathbb{R}_+)$, with $\bar{p}_n = \sup_{t\in[0,n]} \int_0^\infty \frac{p(s)}{(t-s)^{1-\alpha}} ds < \infty$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|h(t,s,y)\| \leq p(s)\psi(\|y\|) \text{ for a.e. } (t,s) \in \mathbb{R}^+ \times \mathbb{R}^+ \text{ and each } y \in \mathbb{R}^d;$$

(h_2) there exists a continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$\|g(t,s,y) - g(t,s,x)\| \leq \eta(s)\|y - x\| \text{ for all } (t,s) \in D \text{ and each } x, y \in \mathbb{R}^d;$$

(h_3) For each $n \in \mathbb{N}$, there exists a constant $R_n > 0$ such that

$$\limsup_{R_n \longrightarrow +\infty} \frac{\left(1 - \|\eta\|_n \frac{n^\alpha}{\alpha}\right) R_n}{\frac{n^\alpha}{\alpha} R_n + \|f\|_n + \|\xi\|_n + \psi(R_n)\bar{p}_n} > 1,$$

with $\frac{n^\alpha}{\alpha}\|\eta\|_n < 1$.

Our first main result reads as follows.

Theorem 3.1. *Assume that hypotheses (h_1)–(h_3) hold. Then equation (1.1) admits a unique solution in C_c.*

Proof. Transform Equation (1.1) into a fixed point problem. Consider the operator $H : C_c \to C_c$ defined by

$$(Hy)(t) := f(t) + \int_0^t \frac{g(t,s,y(s))}{(t-s)^{1-\alpha}} ds + \int_0^\infty \frac{h(t,s,y(s))}{(t-s)^{1-\alpha}} ds, \ t \in \mathbb{R}_+.$$

Clearly the fixed points of H are solutions of the Equation (1.1). For the proof, we will apply Theorem 2.4. To this end, let us set $y(t) = x(t) + \xi(t)$. Then we write (1.1) as

$$x(t) = (Ax)(t) + (Bx)(t), \tag{3.1}$$

where

$$(Ax)(t) := f(t) + \int_0^t \frac{g(t,s,x(s)+\xi(s))}{(t-s)^{1-\alpha}} ds - \xi(t),$$

$$(Bx)(t) := \int_0^\infty \frac{h(t,s,x(s)+\xi(s))}{(t-s)^{1-\alpha}} ds.$$

The proof will be given in several steps.

Step 1: *A is a contraction mapping.*

Let $y,\, \bar{y} \in C_c$; then using (h_2), for each $t \in [0,n],\ n \in \mathbb{N}$,

$$\begin{aligned}\|A(y)(t)-A(\bar{y})(t)\| &\leq \int_0^t (t-s)^{\alpha-1}\|g(t,s,y(s)+\xi(s))-g(t,s,\bar{y}(s)+\xi(s))\|ds\\ &\leq \int_0^t (t-s)^{\alpha-1}\eta(s)\|y(s)-\bar{y}(s)\|ds\\ &\leq \|y-\bar{y}\|_n\int_0^t (t-s)^{\alpha-1}\eta(s)ds\\ &\leq \|y-\bar{y}\|_n\frac{n^\alpha}{\alpha}\|\eta\|_n.\end{aligned}$$

So A is a contraction mapping.

Step 2: *We show that B is continuous and completely continuous.*

Claim 1: B is continuous.

Let $(y_m)_m$ be a sequence $(y_m)_m \subset C_c$, $y_m \longrightarrow y$ in C_c, that is, $\forall \varepsilon > 0$, $\forall n \geq 1$, $\exists N = N(\varepsilon, n)$, $\forall m \geq N$, $\|y_m - y\|_n < \varepsilon$.

Let us fix $n \geq 1$. From the convergence of (y_m) and the continuity of ξ, there is $r \geq 0$ such that $\|y_m+\xi\|_n \leq r$, $\|y+\xi\|_n \leq r$, $\forall m$. Consider $\varepsilon > 0$. By hypothesis (h_1), there is $t_0 > 0$, such that

$$\int_{t_0}^\infty \frac{p(s)}{(t-s)^{1-\alpha}}ds < \frac{\varepsilon}{4\psi(r)}. \tag{3.2}$$

Since h is uniformly continuous on the set $([0,n]\times[0,t_0])\times\overline{B(r)}$ (here $\overline{B(r)} := \{x \in \mathbb{R}^d : \|x\| \leq r\}$), it follows that for all $t \in [0,n]$, $s \in [0,t_0]$, and $m \geq N$,

$$\|h(t,s,y_m(s)+\xi(s))-h(t,s,y(s)+\xi(s))\| < \frac{\varepsilon}{2\displaystyle\int_0^{t_0}\frac{1}{(t-s)^{1-\alpha}}ds}.$$

Therefore, for every $t \in [0,n]$ and $m \geq N$, we have

$$\begin{aligned}&\|(By_m)(t)-(By)(t)\|\\ &\qquad\leq \int_0^{t_0}\left\|\frac{1}{(t-s)^{1-\alpha}}(h(t,s,y_m(s)+\xi(s))-h(t,s,y(s)+\xi(s)))\right\|ds\\ &\qquad\quad+\int_{t_0}^\infty\left\|\frac{1}{(t-s)^{1-\alpha}}(h(t,s,y_m(s)+\xi(s))-h(t,s,y(s)+\xi(s)))\right\|ds\\ &\qquad\leq \frac{\varepsilon}{2}+2\psi(r)\int_{t_0}^\infty\frac{p(s)}{(t-s)^{1-\alpha}}ds\\ &\qquad\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Hence,

$$\|By_m - By\|_n \leq \varepsilon,\ \forall m \geq N$$

and the continuity of B is proved.

Claim 2: B maps bounded sets into bounded sets in C_c.

Indeed, it is enough to show that there exists a positive constant k such that for each for each $y \in Q = \{y \in C_c : \exists q_n > 0,\ \|y\|_n \leq q_n,\ n \geq 1\}$, one has $\|By\|_n \leq k$. By (h_1) for all $t \in [0,n]$ and $y \in Q$, we have

$$\|(By)(t)\| \leq \psi(q_n + \xi_n) \int_0^{\infty} \frac{p(s)}{(t-s)^{1-\alpha}} ds := l,$$

where

$$\xi_n = \sup_{t \in [0,n]} \{\|\xi(t)\|\}.$$

Thus $\|B(y)\|_n \leq l$. So, $\{By|_{[0,n]} :\ y \in Q\}$ is bounded.

Claim 3: B maps bounded sets into equicontinuous sets in C_c.

Let $\varepsilon > 0$ be arbitrarily fixed and $t_0 > 0$ be given by (3.2). By hypothesis (h_2), it follows that $h(t,s,x)$ is uniformly continuous on the set

$$\tilde{D} := [0,n] \times [0,t_0] \times \overline{B(\rho_n)},$$

where

$$\overline{B(\rho_n)} = \{y \in \mathbb{R}^d :\ \|y\| \leq \rho_n\},$$

with

$$\rho_n := q_{[t_0]+1} + \xi_n.$$

Hence, there is a $\delta > 0$ such that for all $t_1, t_2 \in [0,n]$ with $|t_1 - t_2| < \delta$ and all $y \in \overline{B(\rho_n)}$,

$$\|h(t_1, s, y(s) + \xi(s)) - h(t_2, s, y(s) + \xi(s)\| < \frac{\alpha\varepsilon}{4n^{\alpha}}.$$

Now

$$\begin{aligned}
\|(By)(t_1) - (By)(t_2)\| \leq\ & \int_0^{t_0} \left\| \frac{1}{(t_1-s)^{1-\alpha}} h(t_1, s, y(s) + \xi(s)) \right. \\
& \left. - \frac{1}{(t_2-s)^{1-\alpha}} h(t_2, s, y(s) + \xi(s)) \right\| ds \\
& + \psi(\rho_n) \int_{t_0}^{\infty} \frac{p(s)}{(t_1-s)^{1-\alpha}} ds + \psi(\rho_n) \int_{t_0}^{\infty} \frac{p(s)}{(t_2-s)^{1-\alpha}} ds \\
\leq\ & \int_0^{t_0} \left| \frac{1}{(t_1-s)^{1-\alpha}} \|h(t_1, s, y(s) + \xi(s)) - h(t_2, s, y(s) + \xi(s))\| \right. \\
& \left. - \left\| \left(\frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}} \right) h(t_2, s, y(s) + \xi(s)) \right\| \right| ds \\
& + \psi(\rho_n) \int_{t_0}^{\infty} \frac{p(s)}{(t_1-s)^{1-\alpha}} ds + \psi(\rho_n) \int_{t_0}^{\infty} \frac{p(s)}{(t_2-s)^{1-\alpha}} ds \\
\leq\ & \int_0^{t_0} \frac{1}{(t_1-s)^{1-\alpha}} \|h(t_1, s, y(s) + \xi(s)) - h(t_2, s, y(s) + \xi(s))\| ds \\
& - \int_0^{t_0} \left\| \left(\frac{1}{(t_2-s)^{1-\alpha}} - \frac{1}{(t_1-s)^{1-\alpha}} \right) h(t_2, s, y(s) + \xi(s)) \right\| ds
\end{aligned}$$

$$
\begin{aligned}
& +\psi(\rho_n)\int_{t_0}^{\infty}\frac{p(s)}{(t_1-s)^{1-\alpha}}ds+\psi(\rho_n)\int_{t_0}^{\infty}\frac{p(s)}{(t_2-s)^{1-\alpha}}ds \\
\leq\ & \frac{\alpha\varepsilon}{4n^{\alpha}}\int_0^{t_0}\frac{1}{(t_1-s)^{1-\alpha}}ds \\
& -\psi(\rho_n)\int_0^{t_0}\left|\frac{1}{(t_2-s)^{1-\alpha}}-\frac{1}{(t_1-s)^{1-\alpha}}\right|p(s))ds \\
& +\psi(\rho_n)\int_{t_0}^{\infty}\frac{p(s)}{(t_1-s)^{1-\alpha}}ds+\psi(\rho_n)\int_{t_0}^{\infty}\frac{p(s)}{(t_2-s)^{1-\alpha}}ds \\
\leq\ & \frac{\alpha\varepsilon}{4n^{\alpha}}\frac{2n^{\alpha}}{\alpha}-\psi(\rho_n)\frac{\varepsilon}{4\psi(\rho_n)}+\psi(\rho_n)\frac{\varepsilon}{4\psi(\rho_n)}+\psi(\rho_n)\frac{\varepsilon}{2\psi(\rho_n)} \\
\leq\ & \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

Hence the set $\{By|_{[0,n]},\ y\in D\}$ is equicontinuous.

Step 3: *A priori bounds on solutions.*

To apply Theorem 2.4, we must check S_2; i.e., it remains to show that the set

$$\mathcal{E}=\{y\in C_c : y=\lambda A\left(\frac{y}{\lambda}\right)+\lambda B(y),\ \text{for some } 0<\lambda<1\}$$

is bounded. Let $y\in\mathcal{E}$; by (h_1) and (h_2), we have for each $t\in[0,n]$

$$
\begin{aligned}
|y(t)| \leq\ & \lambda\left|A\left(\frac{y}{\lambda}\right)-A(0)\right|+\lambda|A(0)|+|By(t)| \\
\leq\ & \int_0^t\eta(s)\|y\|_n(t-s)^{\alpha-1}ds+\int_0^t(t-s)^{\alpha-1}g(t,s,\xi(s))ds+|f(t)|+|\xi(t)| \\
& +\int_0^{\infty}p(s)\psi(\|y\|_n)(t-s)^{\alpha-1}ds \\
\leq\ & \|y\|_n\int_0^t\eta(s)(t-s)^{\alpha-1}ds+\int_0^t(t-s)^{\alpha-1}g(t,s,\xi(s))ds \\
& +|f(t)|+|\xi(t)|+\psi(\|y\|_n)\int_0^{\infty}p(s)(t-s)^{\alpha-1}ds \\
\leq\ & \|y\|_n\|\eta\|_n\frac{n^{\alpha}}{\alpha}+\frac{n^{\alpha}}{\alpha}\|g\|_n+\|f\|_n+\|\xi\|_n+\psi(\|y\|_n)\bar{p}_n.
\end{aligned}
$$

Thus,

$$\|y\|_n\left[1-\|\eta\|_n\frac{n^{\alpha}}{\alpha}\right]\leq\frac{n^{\alpha}}{\alpha}\|g\|_n+\|f\|_n+\|\xi\|_n+\psi(\|y\|_n)\bar{p}_n.$$

$$\frac{\left[1-\|\eta\|_n\frac{n^{\alpha}}{\alpha}\right]\|y\|_n}{\frac{n^{\alpha}}{\alpha}\|g\|_n+\|f\|_n+\|\xi\|_n+\psi(\|y\|_n)\bar{p}_n}\leq 1. \tag{3.3}$$

From (h_3), there exists $R_n>0$ such that for each $y\in\mathcal{E}$ with $\|y\|_n>R_n$, the condition (3.3) is violated. Therefore,

$$\|y\|_n\leq R_n.$$

Set

$$\Upsilon=\{y\in C_c : \sup\{\|y(t)\|,\ 0\leq t\leq n\}\leq R_n+1 \text{ for all } n\in\mathbb{N}\}.$$

Clearly Υ is a closed subset of C_c. From the choice of Υ there is no $y \in \partial\Upsilon$, such that $y = \lambda B(y)$ for some $\lambda \in (0,1)$. Then the statement (S_2) in Theorem 2.4 does not hold. The nonlinear alternative of Avramescu implies that (S_1) holds, so the operator $A+B$ has a fixed point y^*. Then $x^*(t) = y^*(t) + \xi(t)$, $t \in (0,+\infty)$ is a fixed point of the operator H, which is the solution of the problem (1.1). $\square$

Now we shall present our second result which will be based upon Theorem 2.5. Before, we introduce the following assumptions:

(k_1) $f : [0,+\infty) \to \mathbb{R}^d$ is a continuous function;

(k_2) For all $Q > 0$, there exists $l_Q \in C(\mathbb{R}, \mathbb{R}_+)$, such that

$$|g(t,s,y) - g(t,s,x)| \le l_Q(t)\|y-x\|_n, \text{ for each } y,\ x \in C_c, \text{ with } \|y\|_n \le Q,\ \|x\|_n \le Q;$$

(k_3) There exists $L \in L^1(\mathbb{R}_+, \mathbb{R}_+)$ with $\bar{L}_n = \sup\limits_{t\in[0,n]} \int_0^\infty (t-s)^{\alpha-1} L(s)ds < \infty$, such that:

$$|h(t,s,y) - h(t,s,x)| \le L(t)\|y-x\|_n, \text{ for each } y,\ x \in C_c;$$

(k_4) There exists a function $q \in C(\mathbb{R}, \mathbb{R}_+)$ and a continuous nondecreasing function $\varphi : \mathbb{R}_+ \to (0,\infty)$ such that :

$$|g(t,s,y)| \le q(s)\ \varphi(|y|), \text{ for each } y \in C_c, \text{ and } \ s,\ t \in (0,\infty);$$

(k_5) For each $n \in \mathbb{N}$, there exists a constant $M_n > 0$, such that

$$\frac{M_n}{\|f\|_n + \varphi(\|y\|_n)\|q\|_n \frac{n^\alpha}{\alpha} + \psi(\|y\|_n)\overline{p}_n} > 1. \tag{3.4}$$

Theorem 3.2. *Let the assumptions (h_1), and (k_1)–(k_5) be satisfied. If, in addition,*

$$(\|l_{M_n}\|_n \frac{n^\alpha}{\alpha} + \bar{L}_n) < 1, \tag{3.5}$$

then the equation (1.1) has a unique solution.

Proof. Transform Equation (1.1) into a fixed-point problem. Consider the operator $\mathcal{F} : C_c \to C_c$ defined by :

$$(\mathcal{F}y)(t) = f(t) + \int_0^t (t-s)^{\alpha-1} g(t,s,y(s))ds + \int_0^\infty (t-s)^{\alpha-1} h(t,s,y(s))ds, \ \ t \in (0,\infty].$$

Let y be a possible solution Equation (1.1). Given $n \in \mathbb{N}$ and $t \le n$, then in view of $(k_1), (k_3)$ and (k_4), we have

$$\begin{aligned}
|y(t)| &\le |f(t)| + \int_0^t (t-s)^{\alpha-1}|g(t,s,y(s))|ds + \int_0^\infty (t-s)^{\alpha-1}|h(t,s,y(s))|ds \\
&\le |f(t)| + \int_0^t (t-s)^{\alpha-1}q(s)\varphi(\|y\|_n)ds + \int_0^\infty (t-s)^{\alpha-1}p(s)\psi(\|y\|_n)ds \\
&\le \|f\|_n + \varphi(\|y\|_n)\|q\|_n \frac{n^\alpha}{\alpha} + \psi(\|y\|_n)\overline{p}_n.
\end{aligned}$$

Using the nondecreasing character of φ and ψ, we obtain

$$\frac{\|y\|_n}{\|f\|_n+\varphi(\|y\|_n)\|q\|_n\frac{n^\alpha}{\alpha}+\psi(\|y\|_n)\overline{p}_n}\le 1.$$

From (3.4) it follows that for each $n\in\mathbb{N}$

$$\|y\|_n\neq M_n.$$

Now, set

$$\Omega=\{y\in C_c:\|y\|_n\le M_n,\ \text{for each } n\in\mathbb{N}\}.$$

Clearly, Ω is a closed subset of C_c. We shall show that $\mathcal{F}:\Omega\to C_c$ is a contraction operator. Indeed, consider $y,\overline{y}\in\Omega$; for each $t\in[0,n]$ and $n\in\mathbb{N}$, from $(k_2)-(k_3)$ we have

$$\begin{aligned}
|(\mathcal{F}y)(t)-(\mathcal{F}\overline{y})(t)| &\le \int_0^t(t-s)^{\alpha-1}|g(t,s,y(s))-g(t,s,\bar{y}(s))|ds\\
&\quad+\int_0^\infty(t-s)^{\alpha-1}|h(t,s,y(s))-h(t,s,\bar{y}(s))|ds\\
&\le \int_0^t l_{M_n}(s)(t-s)^{\alpha-1}|y(s)-\bar{y}(s)|ds\\
&\quad+\int_0^\infty L(s)(t-s)^{\alpha-1}|y(s)-\bar{y}(s)|ds\\
&\le \|y-\bar{y}\|_n\|l_{M_n}\|_n\frac{n^\alpha}{\alpha}+\|y-\bar{y}\|_n\bar{L}_n\\
&\le \|y-\bar{y}\|_n[\|l_{M_n}\|_n\frac{n^\alpha}{\alpha}+\bar{L}_n].
\end{aligned}$$

Therefore,

$$\|\mathcal{F}y-\mathcal{F}\overline{y}\|_n\le[\|l_{M_n}\|_n\frac{n^\alpha}{\alpha}+\bar{L}_n]\|y-\bar{y}\|_n.$$

So by (3.5) the operator $\mathcal{F}$ is a contraction for all $n\in\mathbb{N}$. From the choice of Ω there is no $y\in\partial\Omega$ such that $y=\lambda\ \mathcal{F}(y)$ for some $\lambda\in(0,1)$. Then the statement $(C2)$ in Theorem 2.3 does not hold. The nonlinear alternative of Leray-Schauder type [14] implies that $(C1)$ holds, so that the operator $\mathcal{F}$ has a unique fixed-point y in Ω which is a solution to Equation (1.1). This completes the proof. □

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