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ON THE SOLUTION OF A NONLINEAR FRACTIONAL FUNCTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. This paper investigates the existence of a positive solution to a general scalar delayed population model of fractional order that is a nonlinear fractional functional differential equation by applying a nonlinear alternative of Leray-Schauder type in a cone and the generalization of Gronwall's lemma for singular kernels, improving previously known results. Further, we show the continuous dependence of the solution on the order and the initial condition of nonlinear fractional functional differential equations and obtain an Mittag-Leffler functional estimate of the solution by virtue of the generalized Gronwall inequality.

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1. Introduction

In 2006 Ye et al. [3] have addressed the question of the existence of positive solutions for a general scalar delayed population model of fractional order

$$\begin{cases} D^\alpha[x(t)-x(0)] = x(t)f(t,x_t) & , \quad t \in I, \\ x(t) = \phi(t) \geq 0 & , \quad t \in [-r,0], \end{cases} \tag{1.1}$$

where $I = [0,T]$, $0 < \alpha < 1$, D^α is the standard Riemann-Liouville fractional derivative, $\phi \in C$ and $f : I \times C \to R^+$ is continuous. As usual, $C = C([-r,0]; R^+)$ is the space of continuous function from $[-r,0]$ to $R^+, r > 0$, equipped with the sup norm $\|\phi\| = \max_{-r\leq\theta\leq 0} |\phi(\theta)|, R^+ = [0,+\infty)$ and x_t denotes the function in C defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$. By using the sub- and super-solution method, they obtained some sufficient conditions for its existence of positive solutions. However, it is an essential condition that $f(t,\cdot)$ be nondecreasing in [3] (also in [8,10,11])

Therefore, the first aim in this paper is concerned with the existence of a positive solution to Eq (1.1). By applying a nonlinear alternative of Leray-Schauder type in a cone and the generalization of Gronwall's lemma for singular kernels [9], we obtain

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the the existence of its positive solution, removing the condition that the function $f(t, \cdot)$ be nondecreasing, which improves the previously known results [3].

Then, this paper investigates the dependence of the solution on the order and the initial condition for fractional functional differential equations and an Mittag-Leffler functional estimate of the solution by virtue of the generalized Gronwall inequality [6].

2. Integral inequalities and definitions

In this section, we introduce integral inequalities which can be used in nonlinear fractional functional differential equations, theorems and definitions of fractional integral and derivative [1,4].

Let us recall the standard Gronwall inequality which can be found in [5, p. 14].

Theorem 2.1. *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)ds, \quad t \in [t_0, T)$$

where all the functions involved are continuous on $[t_0, T), T \leq +\infty$*, and* $k(t) \geq 0$*, then* $x(t)$ *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)\exp[\int_{s}^{t} k(u)du]ds, \quad t \in [t_0, T).$$

If, in addition, $h(t)$ *is nondecreasing, then*

$$x(t) \leq h(t)\exp(\int_{t_0}^{t} k(s)ds), \quad t \in [t_0, T).$$

Theorem 2.2 ([6, p. 188])**.** *Suppose* $b \geq 0, \alpha > 0$ *and* $a(t)$ *is a nonnegative function locally integrable on* $0 \leq t < T$ *(some* $T \leq +\infty$*), and suppose* $u(t)$ *is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$u(t) \leq a(t) + b\int_{0}^{t} (t-s)^{\alpha-1}u(s)ds$$

on this interval; then

$$u(t) \leq a(t) + \int_{0}^{t} \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right] ds, \qquad 0 \leq t < T.$$

Corollary 2.3. *Under the hypothesis of Theorem 2.2, let* $a(t) \equiv a$ *on* $[0, T)$ *and* $b \equiv \frac{1}{\Gamma(\alpha)}$*, then*

$$u(t) \leq aE_{\alpha}(t^{\alpha}),$$

where E_{α} *is the Mittag-Leffler function defined by* $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}$*.*

Proof. The hypotheses imply

$$\begin{aligned} u(t) &\leq a[1+\int_0^t \sum_{n=1}^{\infty} \frac{1}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}ds] \\ &= a\sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= aE_\alpha(t^\alpha). \end{aligned}$$

The proof is complete. □

Theorem 2.4. [7] *Let E be a Banach space with $C \subseteq E$ closed and convex. Assume U is a relatively open subset of C with $0 \in U$ and $F : \overline{U} \to C$ is a continuous, compact map. Then either*

(1) F has a fixed point in $\overline{U}$; or

(2) there exists $u \in \partial U$ and $\lambda \in (0,1)$ with $u = \lambda F u$.

Lemma 2.5 (The generalization of Gronwall's lemma for singular kernels [9])**.** *Let $\nu : [0,T] \to [0,\infty)$ be a real functional and $\omega(t)$ be a nonnegetive, locally integrable function on $[0,T]$, and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$\nu(t) \leq \omega(t) + a\int_0^t \frac{\nu(s)}{(t-s)^\alpha}ds, t \in [0,T]$$

Then there $\exists K = K(\alpha)$, such that

$$\nu(t) \leq \omega(t) + Ka\int_0^t \frac{\omega(s)}{(t-s)^\alpha}ds, t \in [0,T]$$

for every $t \in [0,T]$.

Definition 2.6. Let $f : [a,b] \to R$, and $f \in L^1[a,b]$. The left-sided Riemann-Liouville fractional integral [1,4] of f of order α is defined as

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt,$$

where $\alpha > 0, a < x < b$.

Definition 2.7. The left-sided Riemann-Liouville fractional derivative [1,4] of a function $f : [a,b] \to R$ is defined as

$$D_a^\alpha f(x) = D^m I_a^{m-\alpha} f(x),$$

where $m = [\alpha] + 1, D^m = \frac{d^m}{dt^m}, a < x < b$.

We denote D_0^α by D^α and I_0^α by I^α. If the fractional derivative $D_a^\alpha f(x)$ is integrable, then [1, p. 71]

$$I_a^\alpha(D_a^\beta f(x)) = I_a^{\alpha-\beta}f(x) - [I_a^{1-\beta}f(x)]_{x=a}\frac{x^{\alpha-1}}{\Gamma(\alpha)}, \quad 0 < \beta \leq \alpha < 1. \tag{2.1}$$

If f is continuous on $[a,b]$, then $[I_a^{1-\beta}f(x)]_{x=a}=0$ and Eq (3) reduces to

$$I_a^{\alpha}(D_a^{\beta}f(x)) = I_a^{\alpha-\beta}f(x), \quad 0<\beta\le\alpha<1. \tag{2.2}$$

3. Existence of a positive solution

In this section, we discuss the existence of a positive solution to Eq (1.1)

$$\begin{cases} D^{\alpha}[x(t)-x(0)] = x(t)f(t,x_t) & , \quad t\in I, \\ x(t)=\phi(t)\ge 0 & , \quad t\in[-r,0], \end{cases}$$

where $I=[0,T]$, $0<\alpha<1$, D^{α} is the standard Riemann-Liouville fractional derivative, $\phi\in C$ and $f: I\times C\to R^+$ is continuous. As usual, $C=C([-r,0];R^+)$ is the space of continuous function from $[-r,0]$ to R^+, $r>0$, equipped with the sup norm $\|\phi\|=\max_{-r\le\theta\le 0}|\phi(\theta)|$, $R^+=[0,+\infty)$ and x_t denotes the function in C defined by $x_t(\theta)=x(t+\theta)$, $-r\le\theta\le 0$.

Using Eq (2.1) and Eq (2.2), Eq (1.1) is equivalent to the integral equation

$$x(t)=\begin{cases} x(0)+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}x(s)f(s,x_s)ds & , \quad t\in I, \\ \phi(t)\ge 0 & , \quad t\in[-r,0]. \end{cases} \tag{3.1}$$

Let $y(\cdot):[-r,T]\to[0,+\infty)$ be the function defined by

$$y(t)=\begin{cases} \phi(0) & , \quad t\in I, \\ \phi(t)\ge 0 & , \quad t\in[-r,0], \end{cases}$$

Then $y_0=\phi$. For each $z\in C(I,R)$ with $z(0)=0$, we denote by $\bar{z}$ the function define by

$$\bar{z}(t)=\begin{cases} z(t) & , \quad t\in I, \\ 0 & , \quad t\in[-r,0]. \end{cases}$$

We can decompose $x(\cdot)$ as $x(t)=\bar{z}(t)+y(t)$, $t\in[-r,T]$, which implies $x_t=\bar{z}_t+y_t$, for $t\in I$. Therefore, Eq (3.1) is equivalent to the integral equation

$$z(t)=I^{\alpha}[z(t)+\phi(0)]f(t,\bar{z}_t+y_t), t\in I. \tag{3.2}$$

Let set

$$A_0=\{z\in C(I,R): \overline{z}_0=0\},$$

and let $\|z\|_C$ be the seminorm in A_0 defined by

$$\|z\|_C=\|\overline{z}_0\|+\|z\|=\|z\| =: \sup\{|z(t)|: t\in I\}, \quad z\in A_0.$$

So A_0 is a Banach space with norm $\|\cdot\|_C$. Let K be a cone of A_0

$$K=\{z\in A_0; z(t)\ge 0, t\in I\},$$

and let

$$K^*=\{x(t)\in C([-r,T],R^+); x(t)=\phi(t)\ge 0, t\in[-r,0]\}.$$

Define the operator $F: K \to K$ by

$$Fz(t) = I^\alpha[z(t) + \phi(0)]f(t, \bar{z}_t + y_t), t \in I.$$

Theorem 3.1. *Let $f: I \times C \to R^+$ be a given continuous function, and $\exists M > 0$ such that $f(t, x_t) \leq M$, then Eq (1.1) has at least a positive solution $x^* \in K^*$.*

Proof. Obviously, the operator $F: K \to K$ is continuous and completely continuous by virtue of Lemma 3.1 in [3].

Next, we will show there exists an open set $U \subseteq K$ with $z \neq \lambda F(z)$ for $\lambda \in (0,1)$ and $z \in \partial U$.

Let $z \in K$ be any solution of

$$z = \lambda F z, \lambda \in (0,1). \tag{3.3}$$

Since $F: K \to K$ is continuous and completely continuous, we have

$$\begin{aligned} |z(t)| = |\lambda F z(t)| &\leq I^\alpha[z(t) + \phi(0)]f(t, \bar{z}_t + y_t) \\ &\leq M I^\alpha[z(t) + \phi(0)] \\ &\leq \frac{M\|\phi\|T^\alpha}{\Gamma(\alpha+1)} + M\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s)|ds. \end{aligned}$$

By virtue of Lemma 2.5, there exists a constant $K = K(\alpha)$, we get

$$\begin{aligned} |z(t)| &\leq R + \frac{MK}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}R ds \\ &\leq R + \frac{MKRT^\alpha}{\Gamma(\alpha+1)} =: \overline{R}, \quad t \in I, \end{aligned}$$

where

$$R = \frac{M\|\phi\|T^\alpha}{\Gamma(\alpha+1)}$$

so

$$\|z\| \leq \overline{R}.$$

Now, using the above inequality, we know any solution z of Eq (3.3) satisfies $\|z\| \neq \overline{R}$, let

$$U = \{z \in K; \|z\| < \overline{R} + 1\}.$$

Therefore, Theorem 2.4 guarantees that Eq (3.2) has at least a positive solution $z \in \overline{U}$. Hence, Eq (1.1) has at least a positive solution $x^* \in K^*$, satisfying $\|x^*\| \leq \max\{\|\phi\|, \overline{R} + 1\}$, and the proof is complete. ∎

Example 3.2. Consider the fractional functional differential equation

$$\begin{cases} D^\alpha x(t) = (t-1)^2 x \sin^2 x(t-r) & , \quad t \in I, \\ x(t) = \phi(t) \geq 0 & , \quad t \in [-r, 0], \end{cases} \tag{3.4}$$

where $0 < \alpha < 1$, $0 < T \leq 2$.

Using Theorem 3.1, $f(t, x_t) \le M = 1$, Eq (3.4) has at least a positive solution $x^* \in K^*$, satisfying $\|x^*\| \le \max\{\|\phi\|, \overline{R} + 1\}$.

4. Dependence of solution on parameters and Mittag-Leffler functional estimation of solution

In recent years, more authors have attention to the dependence of the solution on the order and the initial condition of fractional ordinary differential equations [1,2]. However, no contribution exists, as far as we know, concerning the dependence of the solution on the order and the initial condition of nonlinear fractional functional differential equations. Therefore, this section is concerned with the continuous dependence of the solution on the order and the initial condition to nonlinear fractional functional differential equations in terms of the Riemann-Liouville fractional derivatives. We will consider the solution of initial value problems with neighboring orders and neighboring initial values and finally obtain a law which also follows in the solution of a retarded functional differential equation of integer order. Lastly, We obtain an Mittag-Leffler functional estimate of the solution by the generalized Gronwall inequality.

Theorem 4.1. *Let $\alpha, \beta \in (0,1)$ and $f : I \times C \to R^+$ be continuous, Further assume*

$$|uf(t,u_t) - vf(t,v_t)| \le \|f\||u - v| + L\|u_t - v_t\|, \forall u_t, v_t \in C, t \in I,$$

where L is a constant. Let u and v are the continuous solutions of Eq (1.1) and

$$\begin{cases} D^\beta[y(t) - y(0)] = y(t)f(t, y_t) & , \quad t \in I, \\ y(t) = \psi(t) & , \quad t \in [-r, 0]. \end{cases} \tag{4.1}$$

respectively, then for $t \in I$, the following holds:

$$\|u(t) - v(t)\| \le A(t) + \int_0^t [\sum_{n=1}^{\infty} (\|f\| + L)^n \frac{(t-s)^{n\alpha - 1}}{\Gamma(n\alpha)} A(s)]ds,$$

where

$$A(t) = \|\phi - \psi\| + \frac{\|f\|L_1}{\Gamma(\alpha)}|\frac{t^\alpha}{\alpha} - \frac{t^\beta}{\beta}| + \|f\|L_1\frac{t^\beta}{\beta}|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\beta)}|, L_1 = \max_{-r \le t \le T} |y(t)|,$$

and

$$\|f\| = \max\{f(t, x_t) | t \in I, x_t \in C\}, \quad \|u(t) - v(t)\| = \max_{-r \le s \le t} |u(s) - v(s)|.$$

Proof. The solutions of Eq (1.1) and Eq (4.1) are given by

$$u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha - 1} u(s) f(s, u_s) ds, t \in I, \quad \text{and} \quad u(t) = \phi(t), t \in [-r, 0].$$

and

$$v(t) = v(0) + \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta - 1} v(s) f(s, v_s) ds, t \in I, \quad \text{and} \quad v(t) = \psi(t), t \in [-r, 0].$$

respectively. It follows that, for $t \in I$,

$$
\begin{aligned}
&|u(t)-v(t)| \\
&\le |u(0)-v(0)| + |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)f(s,u_s)ds \\
&\qquad -\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)f(s,v_s)ds| \\
&\le \|\phi-\psi\| + |\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)f(s,u_s)ds \\
&\qquad -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)f(s,v_s)ds| \\
&+|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)f(s,v_s)ds - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\beta-1}v(s)f(s,v_s)ds| \\
&+|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\beta-1}v(s)f(s,v_s)ds - \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}v(s)f(s,v_s)ds| \\
&\le \|\phi-\psi\| + \frac{\|f\|L_1}{\Gamma(\alpha)}|\frac{t^\alpha}{\alpha}-\frac{t^\beta}{\beta}| + \|f\|L_1\frac{t^\beta}{\beta}|\frac{1}{\Gamma(\alpha)}-\frac{1}{\Gamma(\beta)}| \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[\|f\||u(s)-v(s)| + L\|u_s-v_s\|]ds \\
&\le A(t) + \frac{\|f\|+L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds.
\end{aligned}
$$

Therefore, for $t \in I$,

$$\|u(t)-v(t)\| \le A(t) + \frac{\|f\|+L}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u(s)-v(s)\|ds.$$

An application of Theorem 2.2 yields:

$$\|u(t)-v(t)\| \le A(t) + \int_0^t [\sum_{n=1}^{\infty}(\|f\|+L)^n\frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}A(s)]ds.$$

and Theorem is proved. □

Corollary 4.2. *Under the hypothesis of Theorem* 4.1*, if* $\alpha = \beta \in (0,1)$*, then for* $t \in I$*,*

$$\|u(t)-v(t)\| \le \|\phi-\psi\|E_\alpha((\|f\|+L)t^\alpha).$$

Proof. If $\alpha = \beta$, then for $t \in I$,

$$A(t) = \|\phi-\psi\|.$$

By Theorem 4.1,we obtain

$$\begin{aligned}\|u(t)-v(t)\| &\le \|\phi-\psi\|+\|\phi-\psi\|\int_0^t[\sum_{n=1}^{\infty}(\|f\|+L)^n\frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}]ds\\ &\le \|\phi-\psi\|\sum_{n=0}^{\infty}\frac{(\|f\|+L)^n t^{n\alpha}}{\Gamma(n\alpha+1)}\le\|\phi-\psi\|E_\alpha((\|f\|+L)t^\alpha).\end{aligned}$$

The proof is complete. □

When $\beta=1$ in Eq (4.1), Eq (4.1) becomes a general scalar delayed population model of the form

$$\begin{cases}\dot{y}(t)=y(t)f(t,y_t) & , \quad t\in I,\\ y(t)=\psi(t) & , \quad t\in[-r,0].\end{cases}\tag{4.2}$$

where $f: I\times C\to R$ is a continuous function. Eq (4.2) is usually considered in population dynamics, where $y(t)$ denotes the density of a single population species at time t, r stands for the maturation period of the species and $f(t,x_t)$ is the growth function. Therefore, we obtain a theorem:

Theorem 4.3. *Under the hypothesis of Theorem* 4.1, *suppose* $\beta=1$, *Eq* (4.1) *becomes Eq* (4.2). *Let* u *and* v *are the continuous solutions of Eq* (1.1) *and Eq* (4.2) *respectively, then for* $t\in I$,

$$\|u(t)-v(t)\|\le A(t)+\int_0^t[\sum_{n=1}^{\infty}(\|f\|+L)^n\frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}A(s)]ds.$$

where

$$A(t)=\|\phi-\psi\|+\|f\|L_1|\frac{t^\alpha}{\Gamma(\alpha+1)}-t|,\quad L_1=\max_{-r\le t\le T}|y(t)|,$$

and

$$\|f\|=\max\{f(t,x_t)|t\in I,x_t\in C\},\quad \|u(t)-v(t)\|=\max_{-r\le s\le t}|u(s)-v(s)|.$$

Proof. The solutions of Eq (1.1) and Eq (4.2) are given by

$$u(t)=u(0)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}u(s)f(s,u_s)ds, t\in I,\quad and\quad u(t)=\phi(t), t\in[-r,0].$$

and

$$v(t)=v(0)+\int_0^t v(s)f(s,v_s)ds, t\in I,\quad and\quad v(t)=\psi(t), t\in[-r,0].$$

respectively. It follows that for $t \in I$,

$$
\begin{aligned}
&|u(t) - v(t)| \\
&\le |u(0) - v(0)| + |\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) f(s, u_s) ds - \int_0^t v(s) f(s, v_s) ds| \\
&\le \|\phi - \psi\| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |u(s) f(s, u_s) - v(s) f(s, v_s)| ds \\
&+|\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) f(s, v_s) ds - \int_0^t v(s) f(s, v_s) ds| \\
&\le A(t) + \frac{\|f\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s) - v(s)\| ds.
\end{aligned}
$$

Therefore, for $t \in I$,

$$\|u(t) - v(t)\| \le A(t) + \frac{\|f\| + L}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|u(s) - v(s)\| ds.$$

An application of Theorem 2.2 yields:

$$\|u(t) - v(t)\| \le A(t) + \int_0^t [\sum_{n=1}^{\infty} (\|f\| + L)^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} A(s)] ds.$$

and Theorem 4.3 is proved. □

Remark 4.4. It follows from Corollary 4.2 and Theorem 4.3 that for $t \in I$,

$$\|u(t) - v(t)\| \le \|\phi - \psi\| \exp((\|f\| + L)t),$$

if $\alpha = \beta = 1$.

Next we will show the dependence of the solution on the order and the initial condition to the general nonlinear fractional delay differential equation by the same method.

Theorem 4.5. *Let $\alpha, \beta \in (0, 1)$ and $f : I \times C \to R^+$ be continuous, Further assume*

$$|f(t, u_t) - f(t, v_t)| \le L\|u_t - v_t\|, \forall u_t, v_t \in C, t \in I,$$

where L is a constant. Let u and v are the continuous solutions of

$$\begin{cases} D^{\alpha}[x(t) - x(0)] = f(t, x_t) & , \quad t \in I, \\ x(t) = \phi(t) & , \quad t \in [-r, 0]. \end{cases} \tag{4.3}$$

and

$$\begin{cases} D^{\beta}[y(t) - y(0)] = f(t, y_t) & , \quad t \in I, \\ y(t) = \psi(t) & , \quad t \in [-r, 0]. \end{cases} \tag{4.4}$$

respectively, then for $t \in I$, the following holds:

$$\|u(t) - v(t)\| \le A(t) + \int_0^t [\sum_{n=1}^{\infty} L^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)} A(s)] ds,$$

where

$$A(t) = \|\phi - \psi\| + \frac{\|f\|}{\Gamma(\alpha)}\left|\frac{t^\alpha}{\alpha} - \frac{t^\beta}{\beta}\right| + \|f\|\frac{t^\beta}{\beta}\left|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\beta)}\right|,$$

and

$$\|f\| = \max\{f(t, x_t)|t \in I, x_t \in C\}, \quad \|u(t) - v(t)\| = \max_{-r\leq s\leq t} |u(s) - v(s)|.$$

Corollary 4.6. *Under the hypothesis of Theorem* 4.5*, if* $\alpha = \beta \in (0, 1)$*, then for* $t \in I$*,*

$$\|u(t) - v(t)\| \leq \|\phi - \psi\| E_\alpha(Lt^\alpha),$$

where E_α *is the Mittag-Leffler function defined by* $E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)} (\alpha > 0)$*.*

Theorem 4.7. *Under the hypothesis of Theorem* 4.5*, suppose* $\beta = 1$*, Eq* (4.4) *becomes*

$$\begin{cases} \dot{y}(t) = f(t, y_t) & , \quad t \in I, \\ y(t) = \psi(t) & , \quad t \in [-r, 0]. \end{cases} \tag{4.5}$$

Let u *and* v *are the continuous solutions of Eq* (4.3) *and Eq* (4.5) *respectively, then for* $t \in I$*,*

$$\|u(t) - v(t)\| \leq A(t) + \int_0^t [\sum_{n=1}^{\infty} L^n \frac{(t - s)^{n\alpha-1}}{\Gamma(n\alpha)} A(s)] ds.$$

where

$$A(t) = \|\phi - \psi\| + \|f\|\left|\frac{t^\alpha}{\Gamma(\alpha + 1)} - t\right|,$$

and

$$\|f\| = \max\{f(t, x_t)|t \in I, x_t \in C\}, \quad \|u(t) - v(t)\| = \max_{-r\leq s\leq t} |u(s) - v(s)|.$$

Remark 4.8. It follows from Corollary 4.6 and Theorem 4.7 that for $t \in I$,

$$\|u(t) - v(t)\| \leq \|\phi - \psi\| \exp(Lt),$$

if $\alpha = \beta = 1$.

Example 4.9.

$$\begin{cases} D^\alpha[x(t) - x(0)] = \frac{x(t)\cdot x^2(t-r)}{1+x^2(t-r)} & , \quad t \in I, \\ x(t) = ae^t, a \geq 1 & , \quad t \in [-r, 0]. \end{cases} \tag{4.6}$$

and

$$\begin{cases} D^{\alpha+\delta}[y(t) - y(0)] = \frac{y(t)\cdot y^2(t-r)}{1+y^2(t-r)} & , \quad t \in I, \\ y(t) = e^t & , \quad t \in [-r, 0]. \end{cases} \tag{4.7}$$

where $\alpha \in (0, 1]$, δ is a small constant, such that $0 < \alpha + \delta \leq 1$, we have

$$x(t) = a + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} \cdot \frac{x(s) \cdot x^2(s - r)}{1 + x^2(s - r)} ds, \quad t \in I,$$

and

$$y(t) = 1 + \frac{1}{\Gamma(\alpha + \delta)} \int_0^t (t - s)^{\alpha+\delta-1} \cdot \frac{y(s) \cdot y^2(s - r)}{1 + y^2(s - r)} ds, \quad t \in I,$$

When $t \in I$,

$$
\begin{aligned}
|\frac{x(t)\cdot x^2(t-r)}{1+x^2(t-r)} - \frac{y(t)\cdot y^2(t-r)}{1+y^2(t-r)}| &\leq |x(t)-y(t)| + (L_1^2 + L_1L_2)\|x_t - y_t\| \\
&\leq (1+L_1^2+L_1L_2)\|x(t)-y(t)\|.
\end{aligned}
$$

where

$$L_1 = \max_{-r\leq t\leq T}|y(t)|,\ L_2 = \max_{-r\leq t\leq T}|x(t)|.$$

Obviously, by virtue of Theorem 4.1, when $t \in I$, we get

$$\|x(t)-y(t)\| \leq A(t) + \int_0^t [\sum_{n=1}^{\infty}(1+L_1^2+L_1L_2)^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}A(s)]ds, \quad t\in I,$$

where

$$A(t) = a-1+\frac{L_1}{\Gamma(\alpha)}|\frac{t^\alpha}{\alpha} - \frac{t^\beta}{\beta}| + \frac{L_1 t^\beta}{\beta}|\frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\beta)}|,$$
$$\|x(t)-y(t)\| = \max_{-r\leq s\leq t}|x(s)-y(s)|.$$

Then for $t \in I$, the following hold:

(i) if $\delta \to 0$, $a \to 1$, then $\|x(t)-y(t)\| \to 0$.

(ii) if $\delta = 0$, then $\|x(t)-y(t)\| \leq (a-1)E_\alpha((1+L_1^2+L_1L_2)t^\alpha)$.

(iii) if $\alpha+\delta = 1$, using Theorem 4.3, we obtain

$$\|x(t)-y(t)\| \leq A(t) + \int_0^t [\sum_{n=1}^{\infty}(1+L_1^2+L_1L_2)^n \frac{(t-s)^{n\alpha-1}}{\Gamma(n\alpha)}A(s)]ds, \quad t\in I,$$

where

$$A(t) = a-1+L_1|\frac{t^\alpha}{\Gamma(\alpha+1)} - t|.$$

when $\alpha = 1$, $\delta = 0$, then $\|x(t)-y(t)\| \leq (a-1)e^{(1+L_1^2+L_1L_2)t}$.

Example 4.10. Consider the equation

$$\begin{cases} D^{1-\theta}[x(t)-x(0)] = x(t) & , \quad 0 < t \leq T, \\ x(t) = e^t, & , \quad t \in [-r, 0]. \end{cases} \tag{4.8}$$

where $\theta \in (0,1)$ is a small parameter. Next, we discuss its approximate solution.

For this equation, it is very difficult for us to obtain its analytic solution. However, we can obtain its approximate solution by virtue of Theorem 4.3. In fact, we can introduce the delay differential equation

$$\begin{cases} Dy = y & , \quad t \in I, \\ y(t) = e^t & , \quad t \in [-r, 0]. \end{cases} \tag{4.9}$$

Since Eq (4.8) and Eq (4.9) have the same initial condition, we get the corresponding $A(t)$

$$A(t) = L_1|\frac{t^{1-\theta}}{\Gamma(1-\theta+1)} - t|, \quad L_1 = \max_{-r\leq t\leq T}|y(t)|.$$

So, using Theorem 4.3, we have

$$\|x(t)-e^t\| \leq A(t) + \int_0^t [\sum_{n=1}^{\infty} \frac{(t-s)^{n(1-\theta)-1}}{\Gamma(n(1-\theta))} A(s)]ds, \quad t \in I.$$

Obviously, when $\theta \to 0$, then

$$\|x(t)-e^t\| \to 0 \Rightarrow x(t) \to e^t.$$

Therefore, when θ is a small parameter, we use the solution $(t) = e^t$ close to Eq (4.8).

Theorem 4.11 (Mittag-Leffler functional estimate of solution)**.** *Suppose x is the continuous solution of Eq* (1.1)*,then*

$$|x(t)| \leq \|\phi\| E_\alpha(\|f\| t^\alpha).$$

where $\|f\| = \max\{f(t,x_t)|t \in I, x_t \in C\}$.

Proof. Suppose x is the continuous solution of Eq (1.1) ,then

$$x(t) = x(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) f(s,x_s) ds, t \in I, \quad and \quad x(t) = \phi(t), t \in [-r,0].$$

therefore , for $t \in I$,

$$|x(t)| \leq \|\phi\| + \frac{\|f\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| ds.$$

Applying Corollary 2.3. we get

$$|x(t)| \leq \|\phi\| E_\alpha(\|f\| t^\alpha).$$

and Theorem 4.11 is proved. □

Remark 4.12. By virtue of Lemma 2.5, $\exists K = K(\alpha)$, we have another estimate of solution to Eq (1.1):

$$\begin{aligned} |x(t)| &\leq \|\phi\| + \frac{\|f\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |x(s)| ds \\ &\leq \|\phi\| + \frac{K\|f\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|\phi\| ds \\ &\leq \|\phi\| + \frac{K\|f\|\|\phi\| t^\alpha}{\Gamma(\alpha+1)}, \quad t \in I. \end{aligned}$$

For α =1, Eq (1.1) becomes a retarded functional differential equation

$$\begin{cases} Dx(t) = x(t) f(t,x_t) & , \quad t \in I, \\ x(t) = \phi(t) & , \quad t \in [-r,0]. \end{cases} \tag{4.10}$$

It is equivalent to the integral equation

$$x(t) = x(0) + \int_0^t x(s) f(s,x_s) ds, t \in I, \quad and \quad x(t) = \phi(t), t \in [-r,0].$$

Corollary 4.13. *Suppose x is the continuous solution of Eq* (4.10)*, then*

$$|x(t)| \leq \|\phi\| \exp(\|f\|t).$$

Proof. Suppose x is the continuous solution of Eq (4.10), then

$$x(t) = x(0) + \int_0^t x(s)f(s, x_s)ds, \quad t \in I, \quad and \quad x(t) = \phi(t), \quad t \in [-r, 0].$$

therefore , for $t \in I$, applying Theorem 2.1, we get

$$|x(t)| \leq \|\phi\| + \|f\| \cdot \int_0^t |x(s)|ds \leq \|\phi\| \exp(\|f\|t).$$

□

Therefore, we also get an exponential estimate [5, p. 16] on how the solution of Eq (4.10) depends on ϕ and f.

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