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EXTENSION OF STOCHASTIC INTEGRAL IN BANACH SPACE

FARIBORZ ASADIAN

Department of Mathematics and Computer Science, Fort Valley State University,
Fort Valley, Georgia 31030 USA

ABSTRACT. The space of distributions on an abstract Wiener space (H, B) is constructed through the second quantification of a self-adjoint unbounded operator on the Cameron-Martin space H. This construction is used to enlarge the domain of the adjoint of the stochastic derivative, thereby generalizing stochastic integration of Hilbert valued processes with respect to a Wiener process in B. We apply this generalization to the study of an abstract stochastic linear system driven by a cylindrical Wiener process.

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1. INTRODUCTION

In the late 1980's, Korezlioglu and Ustunel, using the methods of white noise analysis developed earlier by Kontrate'v, Kubo, Yokoi, and Tekenaka, constructed spaces of distributions on an abstract Wiener space in the framework of Malliavin calculus (Korezlioglu & Ustunel, 1990). These spaces are larger than the widely used space of Watanabe distributions (Shigekawa 2004). This development on an abstract Wiener space, of which the classical Wiener space is an important example, provided an appropriate setting in which to consider problems of interest in quantum field theory and stochastic partial differential equations. Later, Nualart and Rozovskii used this construction to study the advection-diffusion equation with random potential driven by white noise on $L^2([0, T] \times R^d)$. In their paper (Nualart & Rozovskii, 1997) it was shown that the solution to this equation has stochastic support in an appropriately weighted Gaussian space of square integrable random elements.

The main purpose of this paper is to use the distribution theory developed by Korezlioglu and Ustunel to extend the definition of Ito stochastic integral in an abstract Wiener space (H, B). This generalization is accomplished by considering stochastic integration as the adjoint of the stochastic derivative and by using the duality relationships between $\mathcal{L}^2$-classes of test functions and distribution processes. This

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development is then applied to the study of the abstract linear stochastic system

$$\begin{cases} \frac{du}{dt} & = \mathcal{A}\, u(t) + u(t)\dot{W}(t) \\ u(0) & = u_0, \end{cases} \tag{1.1}$$

where $\dot{W}$ is a space-time white noise based on $L^2([0,T], L^2(R^d))$ and $\mathcal{A}$ is the infinitesimal generator associated with a time-homogeneous Markov process in R^d. One concrete example is the system in which $\mathcal{A}$ is a uniformly elliptic differential operator

$$\mathcal{A} = \frac{1}{2}\sum_{i,j=1}^{d} g_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d}\sigma_i(x)\frac{\partial}{\partial x_i}.$$

It is well known that if the coefficients $G(x) = (g_{i,j}(x))$ and $\sigma(x) = (\sigma_i(x))$ possess some regularity properties, then this operator is the infinitesimal generator associated with a time-homogeneous Markov process satisfying the Feller property. In fact, if $B(t), t \geq 0$, is a Brownian motion in R^d, and if the coefficients satisfy some smoothness properties, then the strong solution of the stochastic differential equation

$$dX(t) = \sqrt{G(X(t))}dB(t) + \sigma(X(t))dt \tag{1.2}$$

is a time-homogeneous Markov process, and for each t and $x \in R^d$, the transition probability $p(x,t,dy)$ associated with this Markov process has a density that is square Lebesgue integrable (Friedman 1975 - Chapter 6). It follows that $\mathcal{A}$ is the infinitesimal generator of the C_0-semigroup of operators on $L^2(R^d)$ defined by $T_t\phi(x) = \int \phi(y)p(x,t,dy)$.

The investigation in this paper is carried out in the context of abstract Wiener space. The approach taken here to study system (1.1) is based on (Nualart & Rozovskii, 1997) in that we use a deterministic equation associated with the abstract stochastic system (1.1) to find the coefficients of the chaos expansion of the solution of the system. In their paper, Nualart and Rozovskii use the Feynman Kac formula to obtain a stochastic representation of the Fourier coefficients of the solution. In this paper, however, we use perturbation of the infinitesimal generator $\mathcal{A}$ by bounded operators to prove that the solution belongs to the space of distributions.

2. A REVIEW OF GAUSSIAN ANALYSIS IN BANACH SPACE

2.1. **Preliminaries and Notation.** Let B be a separable Banach space with norm $\|\cdot\|$, and let μ be a nondegenerate Gaussian measure in B. Then there is a separable Hilbert subspace H of B with norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ in which μ is the cylindrical (finitely additive) Gauss measure. The pair (H, B) is called an abstract Wiener space, and H is known as the Reproducing Kernel Hilbert Space of μ (Kuo, 1975). Upon identification of H^* and H via Riesz Representation Theorem, we obtain the inclusions

$$B^* \subseteq H^* \equiv H \subseteq B.$$

In this way the dual space B^* of B is identified with a dense subset of H.

In all that follows K will be a separable Hilbert space. We denote the Hilbert space of square-μ integrable K-valued functions by $(\mathcal{L}^2; K)$, and if $K = R$, by $(\mathcal{L}^2)$. We denote the Hilbert space of Hilbert Schmidt linear functions from H to K by $L_2(H, K)$. This space is identified with the tensor product $H \otimes K$. For every positive integer n, we denote the tensor product of n copies of H by $\otimes^n H$. We define the symmetrization operator $S_n : \otimes^n H \to \otimes^n H$ by

$$S_n (h_1 \otimes \cdots \otimes h_n) = \frac{1}{n!} \sum_{\sigma} h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes \cdots \otimes h_{\sigma(n)},$$

where the summation is over all permutations σ of $\{1, 2, \cdots, n\}$. We denote the symmetrization of $h_1 \otimes \cdots \otimes h_n$ by $h_1 \cdots h_n$ and the tensor product of n factors of h by h^n. The range of S_n in $\otimes^n H$ will be denoted by $\otimes_s^n H$. The operator S_n is a projection, and hence $\otimes_s^n H$ is a closed subspace of $\otimes^n H$.

Let $h \in H$, and let $\{x_n^*; n = 1, 2, \cdots\}$ be a sequence in B^* that converges in the norm of H to h. For each n, the everywhere defined function $\phi_n : x \mapsto x_n^*(x)$ is normally distributed with mean 0 and variance $|x_n^*|^2$. The sequence ϕ_n is Cauchy in $(\mathcal{L}^2)$ and converges to a μ-almost everywhere defined variable δh that is normally distributed with mean 0 and variance $|h|^2$. For each natural number n and vectors $h_1, \cdots, h_n \in H$ and $k \in K$, we define

$$\delta^n (h_1 \cdots h_n \otimes k) = \delta h_1 \, \delta h_2 \, \cdots \, \delta h_n \, k.$$

It is easy to verify that for every $h \in H$ and natural number n, $\int (\delta h)^{2n} d\mu = \frac{(2n)!}{2^n n!} |h|^{2n}$. Using this and invoking Hölder's inequality we infer the existence of a constant C_n such that

$$\int |\delta^n (h_1 \cdots h_n \otimes k)|_K^2 d\mu \le C_n |h_1 \cdots h_n \otimes k|^2_{L_2(\otimes^n H, K)}$$

for every $h_1, \cdots, h_n \in H$ and $k \in K$. We extend the definition of δ^n by linearity to the dense subspace of $L_2 (\otimes_s^n H, K)$ consisting of all finite linear combinations of elements of the type $h_1 \cdots h_n \otimes k$.

Theorem 2.1. *The collection consisting of finite sums of random variables $e^{\delta h} k$, where $h \in H$ and $k \in K$, is a dense subset of $(\mathcal{L}^2; K)$.*

For each $h \in H$, the norm of $e^{\delta h}$ in $(\mathcal{L}^2)$ is $e^{|h|^2/2}$. We adopt the notation widely used in white noise analysis and denote the renormalization $e^{\delta h - |h|^2/2}$ of $e^{\delta h}$ by $: e^{\delta h} :$.

2.2. **Homogeneous Chaos.** Let $\hat{\mathcal{H}}_0(K) = K$, and for $n \ge 1$, define $\hat{\mathcal{H}}_n(K)$ to be the closure of the linear span, in $(\mathcal{L}^2; K)$, of the constant vectors (in K) and the random elements of the form $\delta^m(h_1 \otimes \cdots \otimes h_m)k, k \in K, h_1, \cdots, h_m \in H$ and $1 \le m \le n$. Then

$$\hat{\mathcal{H}}_0(K) \subset \hat{\mathcal{H}}_1(K) \subset \cdots \subset \hat{\mathcal{H}}_n(K) \subset \cdots.$$

Definition 2.2. We let $\mathcal{H}_0(K) = K$, and for every $n \geq 1$, we define $\mathcal{H}_n(K) = \hat{\mathcal{H}}_n(K) \ominus \hat{\mathcal{H}}_{n-1}(K)$; i.e., $\mathcal{H}_n(K)$ is the orthogonal complement of $\hat{\mathcal{H}}_{n-1}(K)$ in $\hat{\mathcal{H}}_n(K)$. The space $\mathcal{H}_n(R)$ will be denoted $\mathcal{H}_n$.

Theorem 2.3. $(\mathcal{L}^2; K) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n(K)$. *(This sum is an orthogonal sum.)*

It can be shown that $\mathcal{H}_n(K)$ is the closed linear subspace of $(\mathcal{L}^2; K)$ generated by finite linear combinations of random elements $H_n(\delta h)k$, where $h \in H, |h| = 1$, $k \in K$, and H_n is the n^{th} Hermite polynomial. Recall that $H_n(x) = \frac{d^n}{dt^n}\left[e^{tx-t^2/2}\right]\Big|_{t=0}$.

Definition 2.4. Let n be a nonnegative integer, and let $f \in L_2(\otimes_s^n H, K)$ be a linear combination of elements of the type $h_1 \cdots h_n \otimes k$, where $h_1, \cdots, h_n \in H$ and $k \in K$. We define the n^{th} multiple Wiener integral, $\delta^{(n)}(f)$, of f by $\delta^{(n)}(f) = J_n \delta^n(f)$, where J_n is the orthogonal projection of $(\mathcal{L}^2; K)$ onto $\mathcal{H}_n(K)$.

Remark: It is clear that if $h \in H$, $|h| = 1$, and $k \in K$, then

$$\delta^{(n)}\left(h^n \otimes k\right) = H_n(\delta h)k.$$

The densely defined linear function $\frac{1}{\sqrt{n!}}\delta^{(n)} : L_2(\otimes_s^n H, K) \to \mathcal{H}_n(K)$ is unitary. Therefore $\delta^{(n)}$ can be extended to a bounded linear map defined on the entire space $L_2(\otimes_s^n H, K)$. Consequently, for each n, the space $\mathcal{H}_n(K)$ can be identified with $L_2(\otimes_s^n H, K)$, and for each $F \in (\mathcal{L}^2; K)$ there is a sequence $\{f_m\, ;\, m = 0, 1, 2, \cdots\}$ such that $f_m \in L_2(\otimes_s^m H, K)$ for each m and $F = \sum_{m=0}^{\infty} \delta^{(m)}(f_m)$. Furthermore the norm of F in $(\mathcal{L}^2\, ;\, K)$ is $\sqrt{\sum_{m=0}^{\infty} m! |f_m|^2_{L_2(\otimes_s^m H, K)}}$.

3. STOCHASTIC INTEGRATION IN ABSTRACT WIENER SPACE

3.1. **Construction of Test and Distribution spaces.** Let (H, B) be a fixed abstract Wiener space, and let Q : Domain $Q(\subset H) \to H$ be an operator that satisfies the following properties:

1. There is an orthonormal basis $\{e_i\, ;\, i = 1, 2, \ldots\}$ of H and a sequence of numbers $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots$ such that $Q\left(e_i\right) = \lambda_i\, e_i$ for every positive integer i.

2. There is a positive number α such that $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{\alpha}} < \infty$.

3. The set $H_\infty := \cap_{n=1}^{\infty}\text{Domain}\left(Q^n\right)$ is dense in H.

Note that these conditions imply that the operator Q has a bounded inverse and that $A^{-\alpha/2}$ is a Hilbert Schmidt operator with Hilbert Schmidt norm $\sum_{j=1}^{\infty} \frac{1}{\lambda_j^{\alpha}}$.

For each real number β, we denote by H_β the Hilbert space obtained by completing H_∞ (in H) with respect to the norm

$$|h|_\beta = \sqrt{\langle Q^\beta h, h\rangle} = \sqrt{\sum_{j=1}^{\infty} \lambda_j^{\beta} \langle h, e_j\rangle^2}.$$

We denote the space $\cup_{\beta>0} H_{-\beta}$ by $H_{-\infty}$. For each $\beta > 0$, we can identify the dual space of H_β with $H_{-\beta}$. Thus we obtain the following inclusions:

$$H_\infty \subset H_\beta \subset H \subset H_{-\beta} \subset H_{-\infty}.$$

For each $\beta \in R$, the set $\{\lambda_j^{-\beta/2} e_j\,;\, j = 1, 2, \dots\}$ is an orthonormal basis of H_β. Therefore, if $\beta \geq 0$ and $h \in H_{-\beta}$, then $|h|^2_{-\beta} = \sum_{j=1}^\infty \frac{1}{\lambda_j^\beta} \langle h, e_j\rangle^2$.

It follows from properties (2) and (3) that the space H_∞ endowed with the topology generated by the norms $|\cdot|_\beta$ is a nuclear space that is continuously and densely embedded in every H_β. Therefore the symmetric product, $\otimes_s^n H_\infty$, of n copies of H_∞ is densely and continuously embedded in $\otimes_s^n H_\beta$. This implies that the collection of all finite linear combinations of vectors of the form $h^n \otimes k$, where $h \in H_\infty$ and $k \in K$, is dense in $L_2(\otimes_s^i H_\beta; K)$ for every β.

For every positive integer n, real number β, and vectors $h_1, \dots, h_n \in H_\infty$ we define

$$\begin{aligned}(Q^\beta)^{\otimes n}(h_1 \cdots h_n) &= (Qh_1)\cdots(Qh_n)\\ &= \sum_{i_1,\dots,i_n=1}^{\infty} \lambda_{i_1}^\beta \cdots \lambda_{i_n}^\beta \langle h_1, e_{i_1}\rangle \cdots \langle h_n, e_{i_n}\rangle e_{i_1}\cdots e_{i_n}.\end{aligned}$$

The construction of Korezlioglu and Ustunel utilizes the dense subspace of $(\mathcal{L}^2; K)$ consisting of all random elements $F = \sum_{n=0}^\infty \delta^{(n)}(f_n)$ such that each f_n is of the form $f_n = \sum_{i=1}^j h_i^n \otimes k_i$, $h_i \in H_\infty, k_i \in K$, $i = 1, \dots, j$, for some j. For each $\beta \in R$, the norm $\|\cdot\|_\beta$ is defined in this subspace by $\| F \|_\beta = \sum_{n=0}^\infty n! |(Q^{\beta/2})^{\otimes n} f_n)|^2_{L_2(\otimes_s^n H, K)}$. We denote the completion of $(\mathcal{L}^2; K)$ with respect to this norm by $(\mathcal{L}^{2,\beta}; K)$. In this way, for every $\beta \geq 0$ we obtain the following inclusions

$$(\mathcal{L}^{2,\beta}; K) \subseteq (\mathcal{L}^2; K) \subseteq (\mathcal{L}^{2,-\beta}; K).$$

An approach equivalent to this will be taken in the next subsection to construct the spaces of test functions and distributions on the classical Wiener space.

3.2. **Classical Wiener Space.** Let the Hilbert spaces $H_\beta, \beta \in R$, be those defined above. For each $t \in (0, T]$ and $y \in B$ we let $\mu_t(y, dx) = \mu\left(\frac{dx - y}{\sqrt{t}}\right)$. Let $C = C([0, T], B)$ be the Banach space of continuous paths in B all starting from 0. In C we define the measure $\mathcal{P}$ on cylindrical subsets of B in the following way:

$$\begin{aligned}&\mathcal{P}\{\omega; w(t_0) \in A_0, \dots, \omega(t_n) \in A_n\}\\ &\qquad = \int_{A_n} \cdots \int_{A_1} \int_{A_0} \mu_{t_n - t_{n-1}}(y_{n-1}, dy_n) \cdots \mu_{t_1}(y_0, dy_1)\delta_0(dy_0),\end{aligned}$$

where $0 = t_0 < t_1 \cdots < t_n \leq T$ and $A_0, A_1, \dots A_n$ are Borel subsets of B. It can be shown that this measure has σ-additive extension to the Borel field of C. We will denote the expectation of a random element F defined on C with respect to this measure by $E\,F$.

The Reproducing Kernel Hilbert Space associated with $\mathcal{P}$ is the subcollection C' of C consisting of absolutely continuous H-valued paths. This space equipped with the inner product $\langle \phi, \psi \rangle_{C'} = \int_0^T \langle \phi'(s), \psi'(s) \rangle_H ds$ is a Hilbert space, and the pair (C', C) itself is an abstract Wiener space. Henceforth, the constructions and notations used in section 2 will be applied in this setting.

The space $L^2([0,T], H)$ of square-Lebesgue integrable H-valued paths is identified with C' through the unitary map $\phi \mapsto \int_0^{\cdot} \phi$. Using this map we can also identify $L^2([0,T]^n, L_2(\otimes_s^n H, K))$ with $L_2(\otimes_s^n C', K)$. For a function f that belongs to the space $L^2([0,T]^n, L_2(\otimes_s^n H, K))$ we define $\delta^{(n)}(f)$ to be the value of the n^{th} multiple Wiener integral of the element in C' that is identified with f. Each $F \in (\mathcal{L}^2; K)$ has chaos expansion $F = \sum_{n=0}^{\infty} \delta^{(n)}(f_n)$ in which each function f_n belongs to $L^2([0,T]^n, L_2(\otimes_s^n H, K))$.

For the remainder of this and the next subsection, we let β be a fixed non-negative number. We say that a function F in $(\mathcal{L}^2; K)$ with chaos decomposition $\sum_{n=0}^{\infty} \delta^{(n)}(f_n)$ belongs to $(\mathcal{L}^{2,\beta}; K)$ if f_n belongs to $L^2([0,T]^n, L_2(\otimes_s^n H_\beta, K))$ for every n and $\sum_{n=0}^{\infty} n! |f_n|^2_{L^2([0,T]^n, L_2(\otimes_s^n H_\beta, K))} < \infty$.

For $F = \sum_{n=0}^{\infty} \delta^{(n)}(f_n) \in (\mathcal{L}^2; K)$ we define $\| F \|^2_{-\beta} = \sum_{n=0}^{\infty} n! |f_n|^2_{L_2(\otimes_s^n H_{-\beta}, K)}$. Note that this is a weaker norm than the norm in $(\mathcal{L}^2; K)$. The completion of $(\mathcal{L}^2; K)$ with respect to this norm is denoted by $(\mathcal{L}^{2,-\beta}; K)$. This space is identified with the dual space of $(\mathcal{L}^{2,\beta}; K)$, and thus we obtain the inclusions

$$(\mathcal{L}^{2,\beta}; K) \subseteq (\mathcal{L}^2; K) \subseteq (\mathcal{L}^{2,-\beta}; K).$$

Let $f \in L^2([0,T]^n, L_2(\otimes_s^n H_\beta, K))$. We define the (stochastic) derivative of $\delta^{(n)} f$ by $\mathcal{D}_\beta[\delta^{(n)}(f)] = n\delta^{(n-1)}(f)$. Note that the random element obtained belongs to $\mathcal{H}_{n-1}(L^2([0,T], H_\beta \otimes K))$. A function F in $(\mathcal{L}^{2,\beta}; K)$ is said to be (stochastically) differentiable if applying the derivative to each term of its chaos expansion produces a convergent series in $(\mathcal{L}^{2,\beta}; L^2([0,T], H_\beta \otimes K))$. Therefore, a function F in $(\mathcal{L}^{2,\beta}; K)$ with chaos decomposition $F = \sum_{n=0}^{\infty} \delta^{(n)}(f_n)$ is stochastically differentiable if $\sum_{n=0}^{\infty} n\, n! |f_n|^2_{L^2([0,T]^n, L_2(\otimes_s^n H_\beta, K))} < \infty$. The space consisting of such functions is denoted by $(\mathcal{D}^{2,\beta}; K)$.

A function $F \in (\mathcal{L}^{2,-\beta}; L^2([0,T], H \otimes K))$ belongs to the domain of the adjoint, $\mathcal{D}^*_{-\beta}$, of $\mathcal{D}_\beta$ if for every $G \in (\mathcal{D}^{2,\beta}; K)$ we have $E\langle F, \mathcal{D}_\beta G \rangle_{L_2([0,T], H\otimes K)} \leq C|G|_{(\mathcal{L}^{2,\beta}; K)}$, where C is a constant depending on F. In this case an element $\mathcal{D}^*_{-\beta} F$ exists in $(\mathcal{L}^{2,-\beta}; K)$ such that

$$E\langle F, \mathcal{D}_\beta G \rangle_{L^2([0,T], H\otimes K)} = E\langle \mathcal{D}^*_{-\beta} F, G \rangle_K.$$

It is clear from this definition that the domain of $\mathcal{D}^*_{-\beta}$ is a closed subspace of $(\mathcal{L}^{2,-\beta}, L^2([0,T], H\otimes K))$.

Members of the domain of $\mathcal{D}^*_{-\beta}$ can be characterized in terms of their chaos expansions. Let $F = \sum_{n=0}^{\infty} \delta^{(n)}(f_n)$ be in $(\mathcal{L}^{2,-\beta}; L_2([0,T], H\otimes K))$. For each n, let $\tilde{f}_n \in L_2([0,T]^{n+1}, L_2(\otimes_s^{n+1} H_{-\beta}, K))$ be the symmetrization of f_n. Then F belongs to the domain of $\mathcal{D}^*_{-\beta}$ if the following condition is satisfied

$$\sum_{n=0}^{\infty} (n+1)! |\tilde{f}_n|^2_{L^2([0,T]^{n+1}, L_2(\otimes_s^{n+1} H_{-\beta}, K))} < \infty.$$

3.3. **Extension of Ito Integral.** Let $\mathcal{F}$ be the completion of the Borel field of C with respect to the measure $\mathcal{P}$, and for each $t > 0$, let $\mathcal{F}_t$ be the σ-algebra generated by the null sets of $\mathcal{F}$ and all random variables of the type $\delta(1_{[0,s)} \times h)$, where $0 < s \le t$ and $h \in H$. Using a density argument we can easily prove that if $f \in L^2([0,T]^n, L_2(\otimes_s^n H_\beta, K))$, then

$$E(\delta^{(n)}(f) \mid \mathcal{F}_t) = \delta^{(n)}(f\, 1_{[0,t)}^{\otimes n}).$$

From this fact we get the following results.

Lemma 3.1. *Let $F = \sum_n \delta^{(n)}(f_n)$ be a random element in $(\mathcal{L}^{2,\beta}; K)$ that is $\mathcal{F}_t$-measurable for some $t \ge 0$. Then, for every n, $f_n(s_1, \dots, s_n) = 0$ if $s_i > t$ for at least one $i = 1, \dots, n$.*

Corollary 3.2. *If $F \in (\mathcal{D}^{2,\beta}; K)$ is $\mathcal{F}_t$-measurable for some $t \ge 0$, then $(\mathcal{D}_\beta F)(s) = 0$ for every $s > t$.*

Lemma 3.3. *Let $\xi \in (\mathcal{L}^{2,-\beta})$ be $\mathcal{F}_t$-measurable for some $t \ge 0$, and let $h \in H$ and $k \in K$. For every $t' > t$, the element $\xi h \otimes k 1_{[t,t')}(s)$ belongs to the domain of $\mathcal{D}^*_{-\beta}$ and*

$$\mathcal{D}^*_{-\beta}(\xi 1_{[t,t')} h \otimes k) = \xi \mathcal{D}^*_{-\beta}(h \otimes k 1_{[t,t')}).$$

Proof. Let $\{\xi_n\,;\, n = 1, 2, \dots\}$ be a sequence of $\mathcal{F}_t$-measurable elements in $(\mathcal{D}^{2,\beta})$ that converges to ξ in $(\mathcal{L}^{2,-\beta})$. Then the sequence $\xi_n \langle h, W(t') - W(t)\rangle k$ converges to $\xi \langle h, W(t') - W(t)\rangle k$ in $(\mathcal{L}^{2,-\beta}; K)$.

Now, let G be a polynomial function in $(\mathcal{L}^{2,\beta})$. Then

$$\begin{aligned} E\, G \xi_n \mathcal{D}^*_{-\beta}(h 1_{[t,t')}) &= E \langle \mathcal{D}_\beta(G\xi_n), h 1_{[t,t')}\rangle_{L^2([0,T],H)} \\ &= E\, \left[\langle \mathcal{D}_\beta G, \xi_n h 1_{[t,t')}\rangle_{L^2([0,T],H)} + G \langle \mathcal{D}_\beta \xi_n, h 1_{[t,t')}\rangle_{L^2([0,T],H)} \right] \\ &= E\, \langle \mathcal{D}_\beta G, \xi_n h 1_{[t,t')}\rangle_{L^2([0,T],H)} \\ &= E\, G \mathcal{D}^*_{-\beta}(\xi_n h 1_{[t,t')}), \end{aligned}$$

using the fact $\langle \mathcal{D}_\beta \xi_n, h 1_{[t,t')}\rangle_{L^2([0,T],H)} = 0$. Now invoking the density of polynomial functions in $(\mathcal{L}^{2,\beta})$ and the fact that $\mathcal{D}^*_{-\beta}$ is closed we infer that $\xi(h \otimes k 1_{[t,t')})$ is in the domain of $\mathcal{D}^*_{-\beta}$ and $\mathcal{D}^*_{-\beta}(\xi 1_{[t,t')} h \otimes k) = \xi \mathcal{D}^*_{-\beta}(h \otimes k 1_{[t,t')})$. □

We call a process $F \in (\mathcal{L}^{2,-\beta}; L^2([0,T], H\otimes K))$ nonanticipating if for every $t \geq 0$, $h \in H$ and $k \in K$, the random variable $\langle F(t), h\otimes k\rangle_{H\otimes K}$ is $\mathcal{F}_t$-measurable. We denote by $(\mathcal{L}_a^{2,-\beta}; L^2([0,T], H \otimes K))$ the subcollection of $(\mathcal{L}^{2-\beta}; L^2([0,T], H \otimes K)$ consisting of processes that are nonanticipating.

Theorem 3.4. *Let $F \in (\mathcal{L}_a^{2,-\beta}; L^2([0,T], H \otimes K))$. Then F belongs to the domain of $\mathcal{D}^*_{-\beta}$.*

Proof. Let $\xi \in (\mathcal{L}_a^{2,-\beta}; L^2([0,T], H \otimes K))$ be a bounded simple process with jumps at $0 = t_0 \leq t_1 < \cdots < t_l = T$. Then for each $j = 0, \ldots, l$ there is a bounded $H \otimes K$-valued random variable ξ_j in $(\mathcal{L}^{2,-\beta}; H \otimes K)$ that is $\mathcal{F}_{t_j}$-measurable such that $\xi(t) = \xi_j$ if $t_j \leq t < t_{j+1}$. If $\{e_m\,;\, m \in N\}$ is an ONB of H and $\{k_n\,;\, n \in N\}$ is an ONB of K, then we have

$$\xi(s) = \sum_{m,n=1}^{\infty} \sum_{j=0}^{l} \langle \xi_j, e_m \otimes k_n\rangle_{H\otimes K} e_m \otimes k_n 1_{[t_j,t_{j+1})}(s).$$

Using the above lemma and the fact that $\mathcal{D}^*_{-\beta}$ is closed we get

$$\mathcal{D}^*_{-\beta}\xi = \sum_{m,n=1}^{\infty} \sum_{j=0}^{l} \langle \xi_j, e_m \otimes k_n\rangle_{H_{-\beta}\otimes K} \mathcal{D}^*_{-\beta}(e_m \otimes 1_{[t_j,t_{j+1})}) k_n.$$

We clearly see that

$$|\mathcal{D}^*_{-\beta}\xi|^2_{(\mathcal{L}^{2,-\beta};K)} = |\xi|^2_{(\mathcal{L}^{2,-\beta};L^2([0,T],H\otimes K))}.$$

The assertion of the theorem follows since the subcollection of $(\mathcal{L}_a^{2,-\beta}; L^2([0,T], H \otimes K))$ consisting of bounded simple processes of the type considered in this proof is dense in this space. □

The (canonical) B-valued Wiener process is the map $W : [0,T] \times C \to B$ defined by $W(t,\omega) = \omega(t)$. Clearly this process is adapted to the filtration $\mathcal{F}_t, t \in [0,T]$. Note that if $h \in H$, then the random variable $\langle h\,,\, W(t) - W(s)\rangle$ (as an element in $(\mathcal{L}^{2,-\beta})$) and the random variable $\mathcal{D}^*_{-\beta}(h\, 1_{(s,t]})$ are identically distributed. Clearly the definition of the random element $\mathcal{D}^*_{-\beta}F$ given above for every $F \in (\mathcal{L}_a^{2,-\beta}; L^2([0,T], H \otimes K))$ reduces to the definition of the ordinary Ito integral of F driven by the process $W(t)$ if $\beta = 0$. Therefore, $\mathcal{D}^*_{-\beta}$, $\beta > 0$ provides an extension of Ito integral. This extension which preserves the essential properties of Ito integral is useful in the study of stochastic partial differential equations.

3.4. **Example.** In this subsection, we let H be the Hilbert space, $L^2(R^d)$, of square Lebesgue integrable functions on R^d, where d is some positive integer. If $L^2(R^d)$ is completed with respect to a norm $\|\cdot\| = |T \cdot|_{L^2(R^d)}$, where $T : L^2(R^d) \to L^2(R^d)$ is any positive Hilbert Schmidt operator, then we obtain a Hilbert space B such that the pair (H, B) is an abstract Wiener space. One example of such an operator T is the square of the inverse of the differential operator $\widetilde{Q}$ introduced in the next paragraph. The operator T can be used to carry out the construction of Hilbert spaces $L^{2,\beta}(R^d), \beta \in R$ as described in subsection 3.1. Recall that the space $L^2([0,T], L^2(R^d))$ is identified with the Reproducing Kernel Hilbert Space associated with the probability measure $\mathcal{P}$ constructed on the space of continuous paths $C([0,T], B)$ as explained in subsection 3.2. We can now use the spaces $L^{2,\beta}(R^d)$ to construct the spaces of test functions and distributions on $(\mathcal{L}^2; K)$ for any Hilbert space K. (See subsection 3.2.) This construction can be utilized to investigate stochastic differential equations driven by a *cylindrical* Wiener process in $L^2(R^d)$.

In the example that we will consider in this subsection we choose the differential operator $\widetilde{Q} = \prod_{i=1}^d (-\frac{\partial^2}{\partial x_i^2} + x_i^2 + 1)/2$ to construct the spaces of test functions and distributions. It is well known that the orthonormal eigenvalues of this operator are the tensor products $e_{i_1} \otimes \cdots \otimes e_{i_d}$ of Hermite functions. Recall that the nth Hermite function is $e_n(x) = (-1)^n (2^n n! \sqrt{\pi})^{-1/2} e^{x^2/2} \frac{d^n}{dx^n} \left[e^{-x^2} \right]$. The eigenvalue of $\widetilde{Q}$ corresponding to the product $e_{i_1} \otimes \cdots \otimes e_{i_d}$ is $(i_1+1)\cdots(i_d+1)$. It is also known that there exists a constant C such that $|e_n|_\infty < \frac{C}{\sqrt[12]{n}}$ for every n.

Let $\{\psi_j; j = 1, 2, \dots\}$ be an orthonormal basis of $L^2([0,T])$ consisting of bounded and smooth functions. We let Γ be the collection of all $(d+1)$-tuples of positive integers. For each $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \Gamma$ we let

$$\Phi_\alpha(s, x_1, \dots, x_d) = \psi_{\alpha_0}(s) e_{\alpha_1}(x_1) \cdots e_{\alpha_d}(x_d).$$

The collection $\{ \Phi_\alpha ; \alpha \in \Gamma \}$ is an orthonormal basis of $L^2([0,T], L^2(R^d))$.

We let Λ be the collection consisting of all sequences of nonnegative integers that are indexed by Γ and contain only a finite number of nonzero terms. If $\tau = (\tau_\alpha ; \alpha \in \Gamma) \in \Lambda$, then we set $|\tau| = \sum_{\alpha\in\Gamma} \tau_\alpha$ and $\tau! = \prod_{\alpha\in\Gamma} \tau_\alpha!$.

For each $\tau \in \Lambda$ we let $I_\tau = \{\alpha \in \Gamma ; \tau_\alpha \neq 0\}$. We let x_τ be the sequence $(x_\alpha ; \alpha \in I_\tau)$ of variables that has as many variables in it as there are nonzero terms in τ. If $I_\tau = \{\alpha_1, \dots, \alpha_i\}$, then the differential operator $\frac{\partial^{|\tau|}}{\partial x_{\alpha_i}^{\tau_{\alpha_i}} \cdots \partial x_{\alpha_1}^{\tau_{\alpha_1}}}$ will be denoted by $\frac{\partial^{|\tau|}}{\partial x_\tau^\tau}$.

For each $\tau = (\tau_\alpha) \in \Lambda$ we set

$$\Psi_\tau = \frac{1}{\sqrt{\tau!}} \prod_{\alpha \in I_\tau} H_{\tau_\alpha}(\delta(\Phi_\alpha)) = \frac{1}{\sqrt{\tau!}} \frac{\partial^{|\tau|}}{\partial x_\tau^\tau} \left[: e^{\sum_{\alpha\in I_\tau} x_\alpha \delta(\Phi_\alpha)} : \right] \Big|_{x_\tau = (0,\dots,0)}.$$

The collection $\{\Psi_\tau ; \tau \in \Lambda\}$ is an orthonormal basis of $(\mathcal{L}^2)$.

For each $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d) \in \Gamma$ we denote the product $(\alpha_1+1)\cdots(\alpha_d+1)$ by λ_α. Let Q be the unbounded operator that is defined on $L^2([0,T], L^2(R^d))$ by

$$Q(\Phi_\alpha)(s,x) = \psi_{\alpha_0}(s) \widetilde{Q}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_d})(x) = \lambda_\alpha \Phi_\alpha(s,x), \qquad \alpha \in \Gamma,$$

where $\widetilde{Q}$ is the differential operator on $L^2(R^d)$ defined above. We use this operator and the construction method explained in section 3.1 to obtain a collection of Hilbert spaces $L^2([0,T],L^{2,b}(R^d))$ satisfying the following inclusions

$$L^2([0,T],L^{2,b}(R^d)) \subseteq L^2([0,T],L^2(R^d)) \subseteq L^2([0,T],L^{2,-b}(R^d)),$$

for every $b \geq 0$. We get the corresponding spaces

$$(\mathcal{L}^{2,b};K) \subseteq (\mathcal{L}^2;K) \subseteq (\mathcal{L}^{2,-b};K)$$

as described in section 3.2.

For every $\tau \in \Lambda$ and $b \in R$, we denote the product $\prod_{\alpha\in\Gamma} \lambda_\alpha^{b\tau_\alpha}$ by $\lambda^{b\tau}$. Then for every $b > 0$ and orthonormal basis $\{k_i\,;\, i = 1,2,\dots\}$ of K, the collection $\{\lambda^{b\tau/2}\Psi_\tau k_i\,;\,\tau\in\Lambda\,, i = 1,2,\dots\}$ is an orthonormal basis of $(\mathcal{L}^{2,-b};K)$. Let u be an element of the space $(\mathcal{L}^{2,-b};\, L^2([0,T],L^2(R^d)))$. Then for every $(t,x)\in[0,T]\times R^d$ we have

$$\begin{aligned}\| \, u(t,x)\, \|_{-b}^2 &= \sum_{\tau\in\Lambda} \frac{1}{\lambda^{b\tau}} (E\,[u(t,x)\Psi_\tau])^2 \\ &= \sum_{\tau\in\Lambda} \frac{1}{\lambda^{b\tau}\tau!}\frac{\partial^{|\tau|}}{\partial x_\tau^\tau} E\,\left[u(t,x)\;:e^{\sum_{\alpha\in I_\tau} x_\alpha\delta(\Phi_\alpha)}:\right]\Big|_{x_\tau=(0,\dots,0)}.\end{aligned}$$

Let $\{p(t,x,dy)\,;\, t\geq 0,\ x\in R^d\}$ be a collection of transition probabilities in R^d that possess square integrable densities with respect to the Lebesgue measure in R^d. Let $T_t, t\geq 0$, be the C_0-semigroup of operators on $L^2(R^d)$ associated with these probabilities; i.e., $T_t(\phi)(x) = \int \phi(y)p(t,x,dy)$ for every $\phi\in L^2(R^d)$. Furthermore, we assume that the map $x\mapsto T_t(\phi)(x)$ is continuous. We denote the infinitesimal generator of the semigroup $T_t, t\geq 0$ by $\mathcal{A}$. We assume that the domain of the adjoint of $\mathcal{A}$ contains the collection of test functions as a dense subspace. See the introduction for an example of an unbounded operator satisfying these properties.

It is well known that a (time-homogeneous Markov) process $X(t), t\geq 0$ exists in R^d defined on a probability space $(\Omega,\mathcal{B})$ and adapted to a filtration $\mathcal{B}_t\subset\mathcal{B}, t\geq 0$ such that for every $x\in R^d$, there is a probability measure P^x defined on $(\Omega,\mathcal{B})$ satisfying $P^x(X(0)=x)=1$ and $P^x(X(t+h)\in dy\,|\,\mathcal{B}_t) = p(t,X(t),dy)$ a.s. for every $h\geq 0$ and $t\geq 0$. We will denote expectation with respect to this measure by E^x.

We are now ready to consider the stochastic system

$$\begin{cases} \frac{du}{dt} &= \mathcal{A}\,u(t) + u(t)\dot{W}(t) \\ u(0) &= u_0, \end{cases} \tag{3.1}$$

where W is a white noise based on $L^2([0,T],L^2(R^d))$, and u_0 is a bounded continuous function on R^d.

We use the definition in (Mikulevicius and Rozovskii, 1994) and (Nualart and Rozovskii, 1997) and call a nonanticipating random field $u(t,x)$ a solution of the above system if there is $b>0$ for which the following conditions are satisfied:

1. $u(t,x) \in (\mathcal{L}^{2,-b})$ for every $(t,x) \in [0,T] \times R^d$, and $\int_0^T \int_K \| u(t,x) \|^2_{(\mathcal{L}^{2,-b})} \, dxdt < \infty$ for every bounded neighborhood K of the origin in R^d.

2. For every sequence $\tau \in \Lambda$ and specific values assigned to the terms of x_τ, the function u_τ defined by $u_\tau(t,x) = Eu(t,x) : e^{\sum_{\alpha \in I_\tau} x_\alpha \delta(\Phi_\alpha)} :$ satisfies the following deterministic system in the weak sense:

$$\begin{cases} \frac{du_\tau}{dt}(t) & = \mathcal{A}\, u_\tau(t) + \sum_{\alpha \in I_\tau} x_\alpha \Phi_\alpha u_\tau(t) \\ u_\tau(0) & = u_0. \end{cases} \tag{3.2}$$

It follows from condition (1) that for every test function ϕ, the function $u(t,x)\phi(x)$ belongs to $(\mathcal{L}^{2,-b}_a; L^2([0,T], L^2(R^d))$, and hence to the domain of $\mathcal{D}^*_{-b}$ (Theorem 3.4). The solution of the stochastic system is therefore considered in the weak sense, and stochastic integration is $\mathcal{D}^*_{-b}$ for some $b>0$. To see how systems (3.1) and (3.2) are related, assume that u_τ, for a given $\tau \in \Lambda$, is a solution of (3.2) in the weak sense. Then for each test function $\phi \in \mathrm{Dom}(\mathcal{A}^*)$ we have

$$\begin{aligned} \langle u_\tau(t,\cdot), \phi \rangle_{L^2(R^d)} = \langle u_0, \phi \rangle_{L^2(R^d)} &+ \int_0^t \langle u_\tau(s,\cdot), \mathcal{A}^* \phi(\cdot) \rangle_{L^2(R^d)} ds \\ &+ \int_0^T \sum_{\alpha \in I_\tau} x_\alpha \langle \Phi_\alpha(s,\cdot) u_\tau(s,\cdot) 1_{[0,t)}(s), \phi(\cdot) \rangle_{L^2(R^d)} ds. \end{aligned}$$

If u satisfies conditions 1 and 2 above, then we have

$$\begin{aligned} &\int_0^T \sum_{\alpha \in I_\tau} x_\alpha \langle \Phi_\alpha(s,\cdot)\ ,\ u_\tau(s,\cdot) 1_{[0,t)}(s) \phi(\cdot) \rangle_{L^2(R^d)} ds \\ &\quad = \left\langle \sum_{\alpha \in I_\tau} x_\alpha \Phi_\alpha(\cdot,\cdot)\ ,\ u_\tau(\cdot,\cdot) 1_{[0,t)}(\cdot)\phi(\cdot) \right\rangle_{L^2([0,T],L^2(R^d))} \\ &\quad = E \ll \sum_{\alpha \in I_\tau} x_\alpha \Phi_\alpha(\cdot,\cdot) : e^{\sum_{\alpha \in I_\tau} x_\alpha \delta(\Phi_\alpha)} :,\ u(\cdot,\cdot) 1_{[0,t)}(\cdot)\phi(\cdot) \gg \\ &\quad = E \ll \mathcal{D}_b : e^{\sum_{\alpha \in I_\tau} x_\alpha \delta(\Phi_\alpha)} :, u(\cdot,\cdot) 1_{[0,t)}(\cdot)\phi(\cdot) \gg \\ &\quad = E\ : e^{\sum_{\alpha \in I_\tau} x_\alpha \delta(\Phi_\alpha)} : \mathcal{D}^*_{-b}[u(\cdot,\cdot) 1_{[0,t)}(\cdot)\phi(\cdot)]. \end{aligned}$$

Here we have used the symbolism $\ll \cdot,\cdot \gg$ to denote the duality relationship between $L^2([0,T], L^{2,b}(R^d))$ and $L^2([0,T], L^{2,-b}(R^d))$. Since the collection containing the random elements $: e^{\sum_{\alpha \in I_\tau} x_\alpha \delta(\Phi_\alpha)} :$, $\tau \in \Lambda$ and $x_\alpha \in R$ is dense in $(\mathcal{L}^{2,b})$, we infer that u

satisfies the equation

$$\langle u(t,\cdot),\phi(\cdot)\rangle_{L^2(R^d)} = \langle u_0(\cdot),\phi(\cdot)\rangle_{L^2(R^d)} + \int_0^t \langle u(s,\cdot),\mathcal{A}^*\phi(\cdot)\rangle_{L^2(R^d)} ds + \mathcal{D}^*_{-b}[u(s,\cdot)1_{[0,t)}(s)\phi(\cdot)].$$

Note that the stochastic integral on the right is the extension of Ito integral defined above.

Theorem 3.5. *There is a $b > 0$ and a solution u of system* (3.1) *that satisfies conditions* 1 *and* 2.

Proof. For each $t \in [0,T]$ and $\tau \in \Lambda$, and for each set of values assigned to $x_\alpha, \alpha \in I_\tau$, multiplication by the function $\sum_{\alpha\in I_\tau} x_\alpha \Phi_\alpha(t,\cdot)$ defines a bounded operator on $L^2(R^d)$. System (3.2) is a linear system in which the unbounded operator $\mathcal{A}$ is perturbed by the bounded operators $\sum_{\alpha\in I_\tau} x_\alpha \Phi_\alpha(t,\cdot)$. We know from the theory of semigroups (Pazy, 1983) that the solution of this system is given by $u_\tau = \sum_{n=0}^\infty S_{n,\tau}(t)u_0$, where

$$S_{0,x_\tau}(t)u_0 = T(t)u_0, \qquad \text{and for } n > 1,$$
$$S_{n,x_\tau}(t)u_0 = \sum_{\alpha\in I_\tau} x_\alpha \int_0^t T(t-s)\left[\Phi_\alpha(s,\cdot)S_{n-1,x_\tau}(s)(u_0)\right] ds.$$

Let $\tau \in \Lambda$ be a sequence that has nonzero entries $\tau_{\alpha_1},\ldots,\tau_{\alpha_m}$. Then the τ^{th} Fourier coefficient of $u(t,x)$ is

$$E\,u(t,x)\Psi_\tau = \frac{1}{\sqrt{\tau!}}\frac{\partial^{|\tau|}}{\partial x_{\alpha_1}^{\tau_{\alpha_1}}\cdots\partial x_{\alpha_m}^{\tau_{\alpha_m}}} E\left[u(t,x) : e^{\sum_{\alpha\in I_\tau} x_\alpha \delta(\Phi_\alpha)} :\right]\Bigg|_{x_{\alpha_1}=\cdots=x_{\alpha_m}=0}. \tag{3.3}$$

Clearly, if $\tau' \neq \tau$, then $\frac{\partial^{|\tau|}}{\partial x_{\alpha_1}^{\tau_{\alpha_1}}\cdots\partial x_{\alpha_m}^{\tau_{\alpha_m}}}\left[S_{i,x_{\tau'}}(t)u_0\right]\Big|_{x_{\alpha_1}=\cdots=x_{\alpha_m}=0} = 0$ for every i. Therefore, if $|\tau| = n$, then we only need to consider $S_{n,x_\tau}(t)u_0$ to find the τ^{th} Fourier coefficient of $u(t,x)$. Note that

$$S_{n,x_\tau}(t)u_0 = \sum_{i_1,\ldots,i_n=1}^{m} x_{\alpha_{i_1}}\cdots x_{\alpha_{i_n}} \Upsilon_{\alpha_{i_1},\ldots,\alpha_{i_n}}(t)u_0,$$

where

$$\Upsilon_{\alpha_{i_1},\ldots,\alpha_{i_n}}u_0 = \int_0^t T(t-s_n)\Big[\Phi_{\alpha_{i_n}}(s_n,\cdot)\int_0^{s_n} T(s_n-s_{n-1})\Big[\Phi_{\alpha_{i_{n-1}}}(s_{n-1},\cdot)\int_0^{s_{n-1}} T(s_{n-1}-s_{n-2}) \cdots\Big[\Phi_{\alpha_{i_2}}(s_2,\cdot)\int_0^{s_2} T(s_2-s_1)\Big[\Phi_{\alpha_{i_1}}(s_1,\cdot)(T(s_1)u_0)ds_1\Big]ds_2\Big]\cdots ds_{n-1}\Big]ds_n\Big].$$

Recall that each $\alpha \in \Gamma$ is a $(d+1)$-tuple with its first coordinate used for the time parameter t and the rest used for the space variable. Let $\alpha = (\alpha_0,\ldots,\alpha_d) \in \Gamma$.

We denote the d-tuple obtained by deleting the first component of α by $\tilde{\alpha}$, and we denote the function $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_d}$ by $e_{\tilde{\alpha}}$. If $\tau \in \Lambda$ is a sequence that has nonzero entries $\tau_{\alpha_1}, \ldots, \tau_{\alpha_m}$ and if $|\tau| = n$, then we have

$$\begin{aligned}
[S_{n,x_\tau}(t)u_0](x) &= \sum_{\alpha_1,\ldots,\alpha_n \in I_\tau} x_{\alpha_1} \cdots x_{\alpha_n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \psi_{\alpha_1,0}(s_1) \cdots \psi_{\alpha_n,0}(s_n) \\
&\quad \times \int_{(R^d)^{n+1}} e_{\tilde{\alpha}_1}(y_1) \cdots e_{\tilde{\alpha}_n}(y_n) u_0(y_{n+1}) p(s_n, y_n, dy_{n+1}) p(s_{n-1} - s_n, y_{n-1}, dy_n) \\
&\qquad \cdots p(t - s_1, x, dy_1) ds_n \cdots ds_1 \\
&= \sum_{\alpha_1,\ldots,\alpha_n \in I_\tau} x_{\alpha_1} \cdots x_{\alpha_n} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \psi_{\alpha_1,0}(s_1) \cdots \psi_{\alpha_n,0}(s_n) \\
&\qquad \times E^x u_0(X_t) e_{\tilde{\alpha}_n}(X_{t-s_n}) \cdots e_{\tilde{\alpha}_1}(X_{t-s_1}) ds_n \cdots ds_1 \\
&= \frac{x_{\alpha_1}^{\tau_{\alpha_1}} \cdots x_{\alpha_m}^{\tau_{\alpha_m}}}{\tau_{\alpha_1}! \cdots \tau_{\alpha_m}!} E^x \Big[u_0(X_t) \left(\int_0^t \Phi_{\alpha_1}(r, X(t-r)) dr \right)^{\tau_{\alpha_1}} \\
&\qquad \cdots \left(\int_0^t \Phi_{\alpha_m}(r, X(t-r)) dr \right)^{\tau_{\alpha_m}} \Big].
\end{aligned}$$

It follows from (3.3) that if $|\tau| = n$, then the τ^{th} Fourier coefficient of $u(t,x)$ is

$$\begin{aligned}
E\, u(t,x) \Psi_\tau &= \frac{1}{\sqrt{\tau!}} \frac{\partial^{|\tau|}}{\partial x_\tau^\tau} [S_{n,x_\tau}(t)(u_0)(x)] \Big|_{x_\tau = (0,\ldots,0)} \\
&= \frac{1}{\sqrt{\tau_{\alpha_1}! \cdots \tau_{\alpha_m}!}} E^x \Big[u_0(X_t) \left(\int_0^t \Phi_{\alpha_1}(r, X(t-r)) dr \right)^{\tau_{\alpha_1}} \\
&\qquad \cdots \left(\int_0^t \Phi_{\alpha_m}(r, X(t-r)) dr \right)^{\tau_{\alpha_m}} \Big].
\end{aligned}$$

So for every $b > 0$ we have

$$\begin{aligned}
&E \parallel u(t,x) \parallel^2_{(\mathcal{L}^{2,-b})} \\
&\quad = \sum_{n=0}^\infty \sum_{\tau \in \Lambda, |\tau| = n} \frac{1}{\tau! \lambda^{b\tau}} \left[E^x \left(u_0(X_t) \prod_{\alpha \in I_\tau} \left(\int_0^t \Phi_\alpha(r, X(t-r)) dr \right)^{\tau_\alpha} \right) \right]^2 \\
&\quad \le \sum_{n=0}^\infty E^x \left[(u_0(X_t))^2 \left(\sum_{\tau \in \Lambda, |\tau| = n} \frac{1}{\tau!} \prod_{\alpha \in I_\tau} \left(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t \Phi_\alpha(r, X(t-r)) dr \right)^{2\tau_\alpha} \right) \right] \\
&\quad = \sum_{n=0}^\infty \frac{1}{n!} E^x \left[(u_0(X_t))^2 \left(\sum_{\alpha \in \Gamma} \left(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t \Phi_\alpha(r, X(t-r)) dr \right)^2 \right)^n \right]. \qquad (3.4)
\end{aligned}$$

Recall that for $\alpha = (\alpha_0, \alpha_1 \ldots, \alpha_d) \in \Gamma$, λ_α denotes the product $(\alpha_1 + 1) \cdots (\alpha_d + 1)$. Note that

$$\sum_{\alpha\in\Gamma} \Big(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t \Phi_\alpha(r, X(t-r))dr\Big)^2$$

$$= \sum_{\alpha_1,\ldots,\alpha_d=1}^{\infty} \sum_{\alpha_0=1}^{\infty} \left(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t \psi_{\alpha_0}(r)(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_d})(X(t-r))dr\right)^2$$

$$= \sum_{\alpha_1,\ldots,\alpha_d=1}^{\infty} \left(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t (e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_d})(X(t-r))dr\right)^2,$$

since $\{\psi_i\,;\, i = 1, 2, \ldots\}$ is an orthonormal basis of $L^2([0, T])$. Therefore, using the estimate $|e_n|_\infty < \frac{C}{n^{1/12}}$ we see that

$$\sum_{\alpha\in\Gamma} \left(\frac{1}{\lambda_\alpha^{b/2}} \int_0^t \Phi_\alpha(r, X(t-r))dr\right)^2 < \left(\sum_{i=1}^{\infty} \frac{C^2}{(i+1)^b i^{1/6}}\right)^d t^2.$$

The assertion of the theorem follows from this estimate and estimate (3.4). □

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