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NEW STOCHASTIC INTEGRALS, OSCILLATION THEOREMS AND ENERGY IDENTITIES

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ABSTRACT. This paper is divided into three parts on diverse aspects of stochastic analysis, namely

1) newly defined stochastic integrals such as the stochastic Simpson and stochastic quadrature integrals and their relation to the recently introduced stochastic α-integral by the author (DSA, Vol. 15 (2), 2006),

2) oscillation theorems for second order stochastic differential equations (SDEs) which show the almost sure oscillation property of all linear undamped oscillators perturbed by additive, non-degenerate martingale-type noise for all measurable random initial data (this generalizes results from X. Mao (1997), Markus and Weerasinghe (1988)),

3) expected energy formulas for linear stochastic oscillators with additive noise under adequate discretization by midpoint-type methods (the latter generalizes independent results from Hong, Scherer and Wang (NPSC, Vol. 14 (1), 2006) and the author (2004 for the beam problem, 2005 for stochastic wave equation)). Energy-exact stochastic-numerical methods (called improved midpoint methods) for linear second order SDEs are constructed and verified along non-equidistant partitions.

These results can be applied to quadrature methods such as Newton-Cotes formulas for stochastic integrals, to analysis of the oscillatory and energy behavior of stochastically perturbed Schrödinger equations, stochastic oscillators, beam models and stochastic wave equations for randomly vibrating strings.

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1. INTRODUCTION

This paper focuses on 3 dynamic aspects of stochastic analysis, stochastic integration and 2nd order stochastic differential equations (SDEs). In Section 2, we report on newly defined stochastic integrals such as the stochastic Simpson integral and its relation to the recently introduced stochastic alpha-integral by the author [10]. There it is shown that the Simpson integral coincides with the well-known Stratonovich integral under appropriate conditions. Besides, we introduce the more general stochastic quadrature integral which indeed gives new stochastic integrals which do not coincide neither with the famous Itô nor with the Stratonovich integrals in general. Section 3

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discusses oscillation theorems for second order stochastic differential equations which show the almost sure oscillation property of all linear undamped oscillators perturbed by additive, non-degenerate martingale-type noise for all measurable random initial data (this generalizes results from X. Mao [6], Markus and Weerasinghe [5]). Eventually, Section 4 presents expected energy formulas for linear stochastic oscillators with additive noise under discretization by midpoint-type methods (the latter generalizes independent results from Hong, Scherer and Wang [2] and the author (preprints in 2004, 2005)). There, a new numerical method (called improved midpoint method) is shown to be energy-exact along any nonrandom time-partition and with any integrable, adapted initial data. An appendix (Section 5) states the law of iterated logarithms (LIL) for stochastic processes with independent increments (PII) which is needed to prove oscillation theorems for SDEs.

The presented results can be applied to quadrature methods such as Newton-Cotes formulas for stochastic integrals, to analysis of the oscillatory and energy behavior of stochastically perturbed Schrödinger equations, stochastic oscillators, beam models and stochastic wave equations for randomly vibrating strings, and its adequate discretization in a dynamically consistent fashion. In passing, for introductory discussions on Itô-Riemann stochastic quadratures, see Allen [1] and Schurz [11].

2. NEW STOCHASTIC INTEGRALS: STOCHASTIC SIMPSON AND QUADRATURE INTEGRALS

Let $\|X(t)\|_p = (\mathbb{E}\,[\|X(t)\|_d^p])^{1/p}$ for $\mathbb{R}^d$-valued random variables $X(t)$ on the complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, $\|.\|_d$ a vector norm on $\mathbb{R}^d$ with Euclidean scalar product $< .,. >_d$. Furthermore, $< X >_0^t$ denotes the quadratic variation of $\mathcal{F}_t$-adapted d-dimensional real-valued stochastic process $X = (X(t))_{t\geq 0}$ and $< X, Y >_0^t$ the quadratic covariation for processes X and Y. Set $\Delta X_n = X(t_{n+1}) - X(t_n)$ and $\Delta t_n = |t_{n+1} - t_n|$. Throughout this paper, we only deal with processes X with $< X >_a^b < +\infty$. For essentials on stochastic analysis, see [1], [3], [4], [7], [8], [9], [15].

Definition 2.1. For a given Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$ and d-dimensional stochastic process, the (stochastic) **Simpson integral** from a to $b > a$ of $f(X)$ along process X is defined by

$$\mathop{\$\!\!\!\int}_a^b f(X(t))dX(t) := \tag{2.1}$$

$$\lim_{N\to+\infty} \frac{1}{6} \sum_{n=0}^{N-1} \left\langle f(X(t_n)) + 4f(\frac{X(t_{n+1}) + X(t_n)}{2}) + f(X(t_{n+1})), \Delta X_n \right\rangle_d$$

if the limit exists, where this limit is understood in the sense $L^2(\Omega, \mathcal{F}, \mathbb{P})$ along nonrandom partitions

$$a = t_0 < t_1 < \cdots < t_n < t_{n+1} < \cdots < t_N = b. \tag{2.2}$$

Theorem 2.1. *Suppose that* $f \in C^1$ *on the range of* $X(t)$ *for* $a \le t \le b$, $X = (X(t))_{t\ge 0}$ *is a continuous d-dimensional stochastic process on the filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ *with* $\mathbb{E}[\langle X\rangle_a^b] < +\infty$, *and*

$$\mathbb{E}[\langle X\rangle_a^b] < +\infty, \quad \mathbb{E}\Big[\int_a^b (\|f(X(t))\|_d^2 + \|\nabla f(X(t))\|_{d\times d}^2) d\langle X\rangle_a^t\Big] < +\infty. \tag{2.3}$$

Then

$$\begin{aligned}\mathbb{S}\!\!\int_a^b f(X(t))dX(t) &= \int_a^b \langle f(X(t)), dX(t)\rangle_d + \frac{1}{2}\langle f(X), X\rangle_a^b \\ &= \alpha\!\!\int_a^b \langle f(X(t)), dX(t)\rangle_d\end{aligned} \tag{2.4}$$

where $\alpha = \frac{1}{2}$.

Remark 2.2. That means that the stochastic Simpson integral coincides with the Stratonovich integral (or α-integral with $\alpha = 1/2$) under appropriate assumptions. Therefore, this concept will not lead to essentially new types of stochastic integrals under given smooth assumptions of Theorem 2.1.

Proof. For the sake of abbreviation of notation, suppose $d = 1$. Define the discrete stochastic Simpson integral

$$S_a^b(N) \;:=\; \frac{1}{6}\sum_{n=0}^{N-1}\Big[f(X(t_n)) + 4f(\frac{X(t_{n+1}) + X(t_n)}{2}) + f(X(t_{n+1}))\Big]\Delta X_n \tag{2.5}$$

along partitions (2.2). Recall the definition of discrete outer stochastic α-integral given by

$$Q_a^b(\alpha)(N) \;:=\; \sum_{n=0}^{N-1}\Big[\alpha f(X(t_{n+1}) + (I_d - \alpha)f(X(t_n)\Big]\Delta X_n$$

along partitions (2.2), where I_d is the unit $d \times d$ matrix and α $d \times d$ real-valued matrix as introduced originally by Schurz [10]. Obviously, under assumptions (2.3), we can establish the convergence of the discrete Simpson integral (2.5) to the continuous Simpson integral (2.1) as $\max_{n=1,2,\dots,N} |t_n - t_{n-1}| \to 0$ as $N \to +\infty$. Similarly, convergence for the α-integrals holds. Now, for fixed $N \in \mathbb{N}$, decompose the discrete Simpson integral into

$$\begin{aligned}S_a^b(N) &= \frac{1}{6}\sum_{n=0}^{N-1} f(X(t_n))\Delta X_n + \frac{4}{6}\sum_{n=0}^{N-1} f(\frac{X(t_{n+1})+X(t_n)}{2})\Delta X_n + \frac{1}{6}\sum_{n=0}^{N-1} f(X(t_{n+1}))\Delta X_n \\ &= \frac{1}{6}Q_a^b(0)(N) + \frac{2}{3}Q_a^b(\frac{1}{2})(N) + \frac{1}{6}Q_a^b(1)(N) + \frac{2}{3}R_a^b(N) \\ &= \frac{1}{6}\text{ Itô } + \frac{2}{3}\text{ Stratonovich } + \frac{1}{6}\text{ Fisk } + \frac{2}{3}R_a^b(N)\end{aligned}$$

along partitions (2.2) with remainder term

$$R_a^b(N) = \sum_{n=0}^{N-1} \Big[f(\frac{X(t_{n+1})+X(t_n)}{2}) - \frac{f(X(t_{n+1}))+f(X(t_n))}{2} \Big] \Delta X_n. \tag{2.6}$$

Using twice Theorem 2.5 (identity (2.14)) from [10], we find that

$$\begin{aligned} S_a^b(N) &= \frac{1}{6}Q_a^b(0)(N) + \frac{2}{3}(Q_a^b(0)(N) + \frac{1}{2}\langle f(X), X\rangle_a^b) + \frac{1}{6}(Q_a^b(0)(N) \\ &\quad + \langle f(X), X\rangle_a^b) + \frac{2}{3}R_a^b(N) \\ &= Q_a^b(0)(N) + [\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{6}]\langle f(X), X\rangle_a^b + \frac{2}{3}R_a^b(N) \\ &= Q_a^b(\alpha)(N) + \frac{2}{3}R_a^b(N) \end{aligned}$$

with $\alpha = 1/2$. It remains to take the limit $N \to +\infty$. Note that the remainder term $R_a^b(N) \to 0$ in L^2 as $N \to +\infty$ under assumption (2.3) (cf. Schurz [10], arguing as in proof of Theorem 4.1, p. 250–251). Thus, as $N \to +\infty$ in above equation array, we verify relation (2.4). (Note that $\max_{n=1,2,\ldots,N} |t_n - t_{n-1}| \to 0$ as $N \to +\infty$.) □

Let us generalize the previous concept of integration. Suppose that $q_i \in \mathbb{R}^1$ are normalized nonrandom weights with $\sum_{i=0}^k q_i = 1$ (the latter normalization is important for consistency of integrals below) where $k \in \mathbb{N} \setminus \{0\}$ is fixed. Recall $\Delta X_n = X(t_{n+1}) - X(t_n)$ and define differences

$$\Delta_0 X_n = X(t_{n+1}) - X(t_n), \quad \forall k > 0 \ : \ \Delta_k X_n = X(t_{n+1}) - X(t_{n+1-k}).$$

Definition 2.3. For a given Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$ and d-dimensional stochastic process, the k**-stage quadrature integral of the first kind** from a to $b > a$ of $f(X)$ along process X is defined by

$$\oint_a^{b} {}^{1}_{k} f(X(t))dX(t) := \lim_{N\to+\infty} \sum_{n=0}^{N-1} \sum_{i=0}^{k} \langle q_i f(X(t_{n,i})), \Delta X_n\rangle_d \tag{2.7}$$

if the limit exists, where this limit is understood in the sense $L^2(\Omega, \mathcal{F}, \mathbb{P})$ along nonrandom partitions (2.2) with sub-partitions

$$t_n = t_{n,0} < t_{n,2} < \cdots < t_{n,i-1} < t_{n,i} < \cdots < t_{n,k} \le t_{n+1}. \tag{2.8}$$

Definition 2.4. For a given Borel-measurable function $f : \mathbb{R}^d \to \mathbb{R}^d$ and d-dimensional stochastic process, the k**-stage quadrature integral of the second kind** from a to $b > a$ of $f(X)$ along process X is defined by

$$\oint_a^{b} {}^{2}_{k} f(X(t))dX(t) := \lim_{N\to+\infty} \sum_{n=k}^{N-1} \sum_{i=0}^{k} \langle q_i f(X(t_{n+1-k+i})), \Delta_k X_n\rangle_d \tag{2.9}$$

if the limit exists, where this limit is understood in the sense $L^2(\Omega, \mathcal{F}, \mathbb{P})$ along nonrandom partitions (2.2).

Remark 2.5. Weights q_i above could be replaced even by matrices $q_i \in \mathbb{R}^{d\times d}$. However, such a concept we leave to the further interest of the reader. A corresponding investigation can be conducted for this more general case in a similar manner. We shall focus on the case of one-dimensional weights and the case $k = 2$ for simplicity below. If $k = 0$ then the quadrature integral of first kind coincides with the Itô-integral and the quadrature integral of second kind with the backward integral. If $k > 1$ all integrals with non-smooth f might differ.

Theorem 2.2. *Suppose that $f \in C^1$ on the range of $X(t)$ for $a \le t \le b$, $X = (X(t))_{t\ge 0}$ is a continuous d-dimensional stochastic process on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$, and hypothesis (2.3) is satisfied. Then*

$$\oint_a^{b}{}_{2}^{1} f(X(t))dX(t) = \int_a^b \langle f(X(t)), dX(t)\rangle_d + \alpha\langle f(X), X\rangle_a^b$$
$$= \oint_a^b{}_{\alpha} \langle f(X(t)), dX(t)\rangle_d \qquad (2.10)$$

where $\alpha = q_1/2 + q_2$ and $q_0 + q_1 + q_2 = 1$.

Remark 2.6. That means that the stochastic quadrature integral coincides with the α-integral with $\alpha = q_1/2 + q_2$ under appropriate assumptions. Therefore, this concept may lead to new types of stochastic integrals (neither Itô nor Stratonovich integrals) which are strongly related to the concept of α-integrals under given smooth assumptions of Theorem 2.2. However, if $q_1 = -2q_2$ then we receive the well-known Itô integral. If $q_1 + 2q_2 = 1$ then we obtain the well-known Stratonovich integral (both facts due to the consistency condition and relation to α-integrals, see [10]).

Proof. For the sake of abbreviation of notation, suppose $d = 1$. Define the discrete stochastic Quadrature integral of the first kind by

$$Q_a^b(N) := \sum_{n=0}^{N-1}\Big[q_0 f(X(t_n)) + q_1 f(\frac{X(t_{n+1}) + X(t_n)}{2}) + q_2 f(X(t_{n+1}))\Big]\Delta X_n \qquad (2.11)$$

along partitions (2.2). Now, for fixed $N \in \mathbb{N}$, decompose the discrete quadrature integral (2.11) into

$$\begin{aligned} Q_a^b(N) &= q_0 \sum_{n=0}^{N-1} f(X(t_n))\Delta X_n + q_1 \sum_{n=0}^{N-1} f(\frac{X(t_{n+1}) + X(t_n)}{2})\Delta X_n \\ &\quad + q_2 \sum_{n=0}^{N-1} f(X(t_{n+1}))\Delta X_n \\ &= q_0 Q_a^b(0)(N) + q_1 Q_a^b(\frac{1}{2})(N) + q_2 Q_a^b(1)(N) + q_1 R_a^b(N) \\ &= q_0 \text{Itô} + q_1 \text{Stratonovich} + q_2 \text{Fisk} + q_1 R_a^b(N) \end{aligned}$$

along partitions (2.2) with remainder term $R_a^b(N)$ defined by (2.6). Using twice Theorem 2.5 (identity (2.14)) from [10], we find that

$$\begin{aligned} Q_a^b(N) &= q_0 Q_a^b(0)(N) + q_1(Q_a^b(0)(N) + \frac{1}{2}\langle f(X), X\rangle_a^b) + q_2(Q_a^b(0)(N) \\ &\quad + \langle f(X), X\rangle_a^b) + q_1 R_a^b(N) \\ &= Q_a^b(0)(N) + \left[\frac{q_1}{2} + q_2\right]\langle f(X), X\rangle_a^b + q_1 R_a^b(N) = Q_a^b(\alpha)(N) + q_1 R_a^b(N) \end{aligned}$$

with $\alpha = q_1/2 + q_2$ (recall that consistency condition $q_0 + q_1 + q_2 = 1$ holds). It remains to take the limit $N \to +\infty$. Note that $R_a^b(N) \to 0$ in L^2 under (2.3) and $\max_{n=1,2,\ldots,N} |t_n - t_{n-1}| \to 0$ as $N \to +\infty$. Thus, as N tends to infinity in above equation, we can confirm the validity of (2.10). Hence, Theorem 2.2 is verified. □

Example. Consider integrals

$$\int_0^T W(t)dW(t)$$

where W is a standard Wiener process and $T > 0$ nonrandom (hence $f(x) = x$ and $X = W$ here). Then, for Itô, Stratonovich, α, Simpson and Quadrature integrals we obviously have

$$\begin{aligned} \int_0^T W(t)dW(t) &= \frac{(W(T))^2 - T}{2}, \ \int_0^T W(t) \circ dW(t) = \oint\!\!\!\!\!\!S_0^T W(t)dW(t) = \frac{(W(T))^2}{2} \\ \oint\!\!\!\!\!\!\alpha_0^T W(t)dW(t) &= \frac{(W(T))^2 - (1-2\alpha)T}{2}, \ \oint\!\!\!\!\!\!{}^{1}_{2}{}_0^T W(t)dW(t) = \frac{(W(T))^2 + (q_2 - q_0)T}{2}. \end{aligned}$$

One can clearly recognize that the quadrature integral Q_k^1 differs from the other integrals. Moreover, it depends on the presence of asymmetric terms in its definition (expressed by the relation of its q_2- and q_0-weights). Hence, the Simpson integral must coincide with the symmetric integral of Stratonovich type (for $\int W dW$ at least).

Remark 2.7. More general, for integrals $\int_a^b f(X(t))dX(t)$, we may arrive at the open conjecture that

$$\oint\!\!\!\!\!\!{}^{1}_{2}{}_a^b f(X(t))dX(t) = F(X(b)) - F(X(a)) + \frac{(q_2 - q_0)}{2}\int_a^b \|f'(X(t))\|_1^2 d\langle X\rangle_0^t$$

for sufficiently smooth functions $f : \mathbb{R}^1 \to \mathbb{R}^1$ (e.g. locally linearizable such as convex ones), continuous processes X with finite quadratic variation and nonrandom limits a and b, where F is a antiderivative of f. Is this really true?

Remark 2.8. Note that, for same weights q_i and smooth f, X, we have

$$\begin{aligned} &\oint\!\!\!\!\!\!{}^{1}_{0}{}_a^b f(X(t))dX(t) = \text{Itô integral}, \quad \oint\!\!\!\!\!\!{}^{2}_{0}{}_a^b f(X(t))dX(t) = \text{ Backward integral} \\ &\oint\!\!\!\!\!\!{}^{2}_{1}{}_a^b f(X(t))dX(t) = \oint\!\!\!\!\!\!\alpha_a^b f(X(t))dX(t) \quad \text{where } \alpha = q_0 = (1 - q_1). \end{aligned}$$

3. OSCILLATION THEOREMS FOR LINEAR 2ND ORDER SDEs

Let $\mathcal{B}(S)$ be the Borel σ-algebra of inscribed set S and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ a complete probability basis. Consider stochastic differential equations (stochastic oscillator with additive Gaussian noise)

$$\ddot{x} + \omega^2 x = \sigma\xi(t) \tag{3.1}$$

driven by white noise ξ, with real eigenfrequency $\omega \geq 0$ and real noise intensity $\sigma \neq 0$. This equation for a stochastic oscillator can be rewritten to the equivalent two-dimensional system of SDEs

$$dX(t) = Y(t)dt \tag{3.2}$$

$$dY(t) = -\omega^2 X(t)dt + \sigma dW(t) \tag{3.3}$$

driven by the standard Wiener process W (i.e. $W(t) = \int_0^t \xi(s)ds$) and started at $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^2))$-measurable initial data $X(0) = X_0 = x_0$, $Y(0) = Y_0 = y_0$.

Theorem 3.1 (Explicit Representation)**.** *Assume that both $X(0) = X_0$ and $Y(0) = Y_0$ are $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^1))$-measurable initial data, and $\omega > 0$. Then the (strong) solution $(X(t), Y(t))$ of oscillator* (3.1) *satisfies ($\mathbb{P}$-a.s.)*

$$X(t) = R_1 \cos(\omega t - \delta_1) + \frac{\sigma}{\omega}\int_0^t \sin(\omega(t-s))dW(s) \tag{3.4}$$

$$Y(t) = R_2 \cos(\omega t - \delta_2) + \sigma\int_0^t \cos(\omega(t-s))dW(s) \tag{3.5}$$

for all times $t \geq 0$, where

$$R_1 = \sqrt{[X(0)]^2 + [Y(0)/\omega]^2}, \quad \delta_1 = \arctan(Y(0)/[\omega X(0)]),$$
$$R_2 = \sqrt{[\omega X(0)]^2 + [Y(0)]^2}, \quad \delta_2 = \arctan(-\omega X(0)/Y(0))$$

(if $Y(0) = 0$ or $X(0) = 0$ then above we take the limit $Y(0) \to 0+$ or $X(0) \to 0+$, respectively; if both $X(0) = 0$ and $Y(0) = 0$ then we set $R_1 = 0$ and $R_2 = 0$).

Remark. The proof uses the well-known variation of parameters formula and trigonometric identities in a standard manner. So we omit it here. For more details, see [12].

Theorem 3.2 (A.S. Persistence of Oscillations About 0)**.** *Assume that $\omega > 0$, $\sigma \neq 0$, and $X(0) = x_0$, $Y(0) = \dot{x}_0$ are $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}^1))$-measurable. Then both components of solution vector $(X(t),Y(t))_{t\geq 0}$ of stochastic oscillator* (3.1) *have infinitely many zero's ($\mathbb{P}$-a.s.), i.e. both $X(t)$ and $Y(t)$ oscillates around zero infinitely often ($\mathbb{P}$ almost surely) as time t advances.*

Proof. Suppose that $\omega > 0$ and $\sigma \neq 0$ throughout the entire proof. **First**, let $X(0) = Y(0) = 0$. Then we have the components X and Y are continuous time, Gaussian

$\mathcal{N}(0, Var(t))$-distributed $\mathcal{F}_t$-martingales with finite variance $Var(t)$ (which can even be explicitly calculated). Moreover, the representations of X and Y satisfy

$$\begin{aligned} X(t) &= \frac{\sigma}{\omega}\int_0^t \sin(\omega(t-s))dW(s) \\ Y(t) &= \sigma\int_0^t \cos(\omega(t-s))dW(s) \end{aligned}$$

for all times $t \geq 0$. Note that both X and Y as stochastic integrals with non-anticipating integrands are processes with independent increments. Hence, using the law of iterated logarithm (**LIL**, see Shiryaev [15], Stout [16] and Wang [17] (as summarized by Theorem 5.1 in the appendix, Section 5) confirms that both X and Y must oscillate around zero (a.s., as time-shifted Wiener processes).

Second, let $|X(0)| > 0$. Recall general representations (3.4) and (3.5) from Theorem 3.1. It suffices to consider the X-component since the the proof for Y-component is very similar to that of X-component. Now consider $X(t)$ at discrete instants

$$t_k := \frac{\pi/2 + 2k\pi + \arctan(Y(0)/[\omega X(0)])}{\omega} = \frac{\pi/2 + 2k\pi + \delta_1}{\omega}$$

where $k \in \mathbb{N}$ (if $X(0) = 0$ then one sets $\arctan() = -\pi/2$ in above definition of t_k). Note that $t_k - t_{k-1} = \Delta_k = \frac{2\pi}{\omega}$. Define the sequence $(U_n)_{m\in\mathbb{N}}$ by $U_n = X(t_n), n \in \mathbb{N}$. Thereby $U_n = \frac{\sigma}{\omega}\int_0^{t_n} \sin(\omega(t_n - s))dW(s)$. We can decompose U_n by $U_n = \sum_{k=1}^n Z_k$ where $(Z_k)_{k\in\mathbb{N}}$ is a sequence of independent Gaussian distributed random variables satisfying

$$Z_k = \frac{\sigma}{\omega}\int_{t_{k-1}}^{t_k} \sin(\omega(t_n - s))dW(s)$$

with mean zero and variance

$$\begin{aligned} \mathbb{E}\,[Z_k]^2 &= \frac{\sigma^2}{\omega^2}\int_{t_{k-1}}^{t_k} \sin^2(\omega(t_n - s))ds = \frac{\sigma^2}{2\omega^2}\int_{t_{k-1}}^{t_k} (1 - \cos(2\omega(t_n - s)))ds \\ &= \frac{\sigma^2}{2\omega^2}[s + \frac{\sin(2\omega(t_n - s))}{2\omega}]_{t_{k-1}}^{t_k} \\ &= \frac{\sigma^2}{2\omega^2}[t_k - t_{k-1} + \frac{\sin(2\omega(t_n - t_k))}{2\omega} - \frac{\sin(2\omega(t_n - t_{k-1}))}{2\omega}] \\ &= \frac{\sigma^2\pi}{\omega^3} \end{aligned}$$

where $\delta_1 = \arctan(Y(0)/[\omega X(0)])$. Here, we have used the fact that

$$\begin{aligned} \sin(2\omega(t_n - t_k)) &= \sin(2\omega(t_n - t_{k-1} - \Delta_k)) = \sin(2\omega(t_n - t_{k-1} - \frac{2\pi}{\omega})) \\ &= \sin(2\omega(t_n - t_{k-1}) - 4\pi) = \sin(2\omega(t_n - t_{k-1})). \end{aligned}$$

Consequently, all random variables Z_k of U_n are identically distributed. Now, standard limit theorems on family of sums of i.i.d. random variables (such as laws of iterated logarithms (**LIL**), see Shiryaev [15], Stout [16] and Wang [17] (see Theorem 5.1 in the appendix, Section 5) say that infinitely many oscillations of the sequence $(U_n)_{n\in\mathbb{N}}$ around zero must happen (almost surely) as n tends to infinity (by infinite number of sign changes of $(U_n)_{n\in\mathbb{N}}$). (Recall that X has always a continuous time modification.)

Third, suppose that $X(0) = 0$ and $Y(0) \neq 0$. Then

$$X(t) = Y_0 \frac{\sin(\omega t)}{\omega} + \frac{\sigma}{\omega}\int_0^t \sin(\omega(t-s))dW(s).$$

Now, consider $X(t)$ at discrete instants $t_k := k\dfrac{2\pi}{\omega}$, hence $t_k - t_{k-1} = 2\pi/\omega$. Define the sequence $U_n = X(t_n), n \in \mathbb{N}$ as before. Note that $U_n = \frac{\sigma}{\omega}\int_0^{t_n} \sin(\omega(t_n - s))dW(s)$ can be decomposed in terms of $U_n = \sum_{k=1}^n Z_k$ where $(Z_k)_{k\in\mathbb{N}}$ is a sequence of independent identically Gaussian distributed random variables $Z_k \in \mathcal{N}(0, \sigma^2\pi/\omega^3)$. Therefore, the **LIL** (i.e. Theorem 5.1 in Section 5) gives the presence of infinite number of oscillations of $(U_n)_{n\in\mathbb{N}}$ around 0, hence of X too.

Fourth, we may proceed similarly with the analysis of Y-component with phase angle δ_2 in representation (3.5) in order to confirm the conclusion of Theorem 3.2. However, for this procedure, one needs to distinguish between the cases $Y(0) = 0$ and $Y(0) \neq 0$. □

4. ENERGY IDENTITIES FOR LINEAR STOCHASTIC OSCILLATORS

Consider **midpoint methods** applied to oscillator (3.1)

$$X_{n+1} = X_n + \bar{Y}_n h_n, \quad Y_{n+1} = Y_n - \omega^2 \bar{X}_n h_n + \sigma \Delta W_n \tag{4.1}$$

where $h_n = t_{n+1} - t_n$ as current step size, $\Delta W_n = W(t_{n+1}) - W(t_n) \in \mathcal{N}(0, h_n)$ as independent Gaussian noise increments, using current arithmetic means

$$\bar{X}_n = \frac{X_{n+1} + X_n}{2} \quad \text{and} \quad \bar{Y}_n = \frac{Y_{n+1} + Y_n}{2}.$$

In what follows, we shall generalize results from Hong, Scherer and Wang [2] who considered the energy functional of this method for the special case $\omega = 1$, $X_0 = 1$ and $Y_0 = 0$ only along equidistant partitions $(t_n)_{n\in\mathbb{N}}$ of intervals $[0, T]$. In fact, we allow to have random initial data in $L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ (i.e. the decisive condition to guarantee the finiteness of expected energy) and non-equidistant partitions of intervals $[0, T]$. Let $0 = t_0 \leq t_1 \leq \cdots \leq t_n \leq \cdots \leq t_{n_T} = T$ form a nonrandom partition of $[0, T]$ and

$$\mathcal{E}(t) = \frac{\omega^2[X(t)]^2 + [Y(t)]^2}{2}$$

be the energy functional related to (3.1). Then the following more general result is found.

Theorem 4.1 (Energy Identity for Midpoint Methods)**.** *Assume that*

(i) W *(i.e.* $W(t)=\int_0^t \xi(s)ds$*) is a standard Wiener process and*
(ii) *initial data* $X(0)=X_0$, $Y(0)=Y_0$ *are* $L^2(\Omega,\mathcal{F}_0,\mathbb{P})$*-integrable.*

Then, $\forall t_n \geq 0$*, the mean energy* $\mathbb{E}\,[\mathcal{E}(t_n)]$ *is finite and grows linearly in* t*. More precisely, we have* $\forall \omega,\sigma \in \mathbb{R}^1\ \forall X_0,Y_0 \in L^2(\Omega,\mathcal{F}_0,\mathbb{P})$

$$
\begin{aligned}
0 \leq \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2} t_n \min_{k=0,1,\ldots,n-1} \frac{1}{1+\omega^2 h_k^2/4} \leq \mathbb{E}\,[\mathcal{E}(t_n)] \ &= \ \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2} \sum_{k=0}^{n-1} \frac{h_k}{1+\omega^2 h_k^2/4} \\
\leq \ \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2} t_n \max_{k=0,1,\ldots,n-1} \frac{1}{1+\omega^2 h_k^2/4} \ &\leq \ \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2} t_n < +\infty
\end{aligned}
$$

which renders to be an equality for equidistant partitions.

Proof. More general, consider the **drift-implicit θ-methods**

$$
\begin{aligned}
X_{n+1} &= X_n + (\theta_n Y_{n+1} + (1-\theta_n) Y_n) h_n \\
Y_{n+1} &= Y_n - \omega^2 (\theta_n X_{n+1} + (1-\theta_n) X_n) h_n + \sigma \Delta W_n
\end{aligned}
\tag{4.2}
$$

with nonrandom parameter-sequence $(\theta_n)_{n\in\mathbb{N}}$, where

$$
\theta_n \in \mathbb{R}^1, h_n = t_{n+1} - t_n, \Delta W_n = W(t_{n+1}) - W(t_n) \in \mathcal{N}(0,h_n).
$$

First, rewrite this system of equations for (X,Y) to as

$$
\begin{aligned}
X_{n+1} &= X_n + (2\theta_n \bar{Y}_n + (1-2\theta_n) Y_n) h_n \\
Y_{n+1} &= Y_n - \omega^2 (2\theta_n \bar{X}_n + (1-2\theta_n) X_n) h_n + \sigma \Delta W_n.
\end{aligned}
$$

Second, multiply the components of these equations by $\omega^2 \bar{X}_n$ and $\bar{Y}_n$ (resp.) to arrive at

$$
\begin{aligned}
&\frac{\omega^2}{2}(X_{n+1}^2 - X_n^2) = (2\theta_n \omega^2 \bar{Y}_n \bar{X}_n + (1-2\theta_n)\omega^2 Y_n \bar{X}_n) h_n \\
&\frac{1}{2}(Y_{n+1}^2 - Y_n^2) = -\omega^2 (2\theta_n \bar{Y}_n \bar{X}_n + (1-2\theta_n) \bar{Y}_n X_n) h_n + \sigma \bar{Y}_n \Delta W_n.
\end{aligned}
$$

Third, adding both equations leads to

$$
\mathcal{E}_{n+1} = \mathcal{E}_n - \omega^2 (1-2\theta_n)[Y_n \bar{X}_n - \bar{Y}_n X_n] h_n + \sigma \bar{Y}_n \Delta W_n. \tag{4.3}
$$

Fourth, note that

$$
\begin{aligned}
[Y_n \bar{X}_n - \bar{Y}_n X_n] &= \frac{1}{2}[Y_n X_{n+1} - Y_{n+1} X_n], \\
\bar{Y}_n &= \frac{2Y_n - \omega^2 (X_n + \theta_n (1-2\theta_n) Y_n h_n) h_n + \sigma \Delta W_n}{2(1+\omega^2 \theta_n^2 h_n^2)}, \\
\mathbb{E}\,[\sigma \bar{Y}_n \Delta W_n] &= \frac{\sigma^2}{2(1+\omega^2 \theta_n^2 h_n^2)} h_n.
\end{aligned}
$$

Fifth, pulling over expectations and summing over n in equation (4.3) for the pathwise evolution of related energy yield that

$$\mathbb{E}\,[\mathcal{E}(t_n)] \quad = \quad \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2}\sum_{k=0}^{n-1}\frac{h_k}{1+\omega^2\theta_k^2 h_k^2} - \frac{\omega^2}{2}\sum_{k=0}^{n-1}(1-2\theta_k)\mathbb{E}\,[Y_k X_{k+1} - Y_{k+1}X_k]h_k.$$

Now, it remains to set $\theta_n = 0.5$ for all $n \in \mathbb{N}$. Thus, one obtains

$$\mathbb{E}\,[\mathcal{E}(t_n)] \quad = \quad \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2}\sum_{k=0}^{n-1}\frac{h_k}{1+\omega^2 h_k^2/4} < +\infty$$

since the expected initial energy is finite under (*ii*). Consequently, by estimating the series with minimum and maximum in a standard fashion, the conclusion of Theorem 4.1 is confirmed. □

Remark. Theorem 4.1 remains true if all ΔW_n are independent quantities with $\mathbb{E}\,[\Delta W_n] = 0$ and $\mathbb{E}\,[(\Delta W_n)^2] = h_n$. (So Gaussian property is not essential for its validity.) Theorem 4.1 says also that midpoint methods underestimate the exact mean energy (the formula below is also called **trace formula** in [14])

$$\forall t \geq 0 \;:\; \mathbb{E}\,[\mathcal{E}(t)] = \mathbb{E}\,[\mathcal{E}(0)] + \frac{1}{2}\sigma^2 t \tag{4.4}$$

of underlying continuous SDE (3.1) (however they are consistent as maximum step size tends to zero). The proof of Theorem 4.1 also shows that the situation of inadequate replication of expected energy is not improving with the use of more general drift-implicit θ-methods (including forward Euler and backward Euler methods as well).

Extracting results from the previous proof of Theorem 4.1 gives the following immediate consequence.

Corollary 4.1 (Expected Energy Identity for θ-Methods (4.2))**.** *Under the same assumptions* (i)–(ii) *as in Theorem* 4.1*, we have the expected energy identity*

$$\mathbb{E}\,[\mathcal{E}(t_n)] = \mathbb{E}\,[\mathcal{E}(0)] + \frac{\sigma^2}{2}\sum_{k=0}^{n-1}\frac{h_k}{1+\omega^2\theta_k^2 h_k^2} - \frac{\omega^2}{2}\sum_{k=0}^{n-1}(1-2\theta_k)\mathbb{E}\,[Y_k X_{k+1} - Y_{k+1}X_k]h_k \tag{4.5}$$

along any nonrandom partition $(t_n)_{n\in\mathbb{N}}$ *for the drift-implicit* θ*-methods* (4.2) *with any nonrandom parameters* $\theta_k \in \mathbb{R}^1$*, any nonrandom constants* $\omega, \sigma \in \mathbb{R}^1$ *and any random initial data* $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$*.*

However, the observed bias in the energy-evolution under discretization can be even removed by the energy-exact **improved midpoint methods**

$$X_{n+1} = X_n + \bar{Y}_n h_n, \quad Y_{n+1} = Y_n - \omega^2 \bar{X}_n h_n + \sigma\sqrt{1+\omega^2 h_n^2/4}\,\Delta W_n \tag{4.6}$$

where the involved quantities are defined as for midpoint methods (4.1). In passing, we note that this numerical method is new to the best of our knowledge. Moreover, it is a consistent one with an exact replication of the temporal evolution of underlying continuous time energy.

Theorem 4.2 (Exact Energy Identity for Improved Midpoint Methods)**.** *Under the same assumptions* (i)–(ii) *as in Theorem* 4.1*, we have the exact energy identity (called* **trace formula** *in a more general context, see* [14]*)*

$$\mathbb{E}\left[\mathcal{E}(t_n)\right] = \mathbb{E}\left[\mathcal{E}(0)\right] + \frac{1}{2}\sigma^2 t_n \tag{4.7}$$

along any nonrandom partition $(t_n)_{n\in\mathbb{N}}$ *for the methods* (4.6) *with any nonrandom constants* $\omega, \sigma \in \mathbb{R}^1$ *and any random initial data* $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$*.*

Proof. More general, consider the **fully implicit θ-methods**

$$\begin{aligned} X_{n+1} &= X_n + (\theta_n Y_{n+1} + (1-\theta_n)Y_n)h_n \\ Y_{n+1} &= Y_n - \omega^2(\theta_n X_{n+1} + (1-\theta_n)X_n)h_n + \sigma_n \Delta W_n \end{aligned} \tag{4.8}$$

with nonrandom implicitness-parameters θ_n, where

$$\sigma_n = \sigma\sqrt{1+\omega^2\theta_n^2 h_n^2}, \theta_n \in \mathbb{R}^1, h_n = t_{n+1} - t_n, \Delta W_n = W(t_{n+1}) - W(t_n) \in \mathcal{N}(0, h_n).$$

First, rewrite this system of equations for (X, Y) to as

$$\begin{aligned} X_{n+1} &= X_n + (2\theta_n \bar{Y}_n + (1-2\theta_n)Y_n)h_n \\ Y_{n+1} &= Y_n - \omega^2(2\theta_n \bar{X}_n + (1-2\theta_n)X_n)h_n + \sigma_n \Delta W_n. \end{aligned}$$

Second, multiply the components of these equations by $\omega^2\bar{X}_n$ and $\bar{Y}_n$ (resp.) to get

$$\begin{aligned} &\frac{\omega^2}{2}(X_{n+1}^2 - X_n^2) = (2\theta_n\omega^2\bar{Y}_n\bar{X}_n + (1-2\theta_n)\omega^2 Y_n\bar{X}_n)h_n \\ &\frac{1}{2}(Y_{n+1}^2 - Y_n^2) = -\omega^2(2\theta_n\bar{Y}_n\bar{X}_n + (1-2\theta_n)\bar{Y}_n X_n)h_n + \sigma_n\bar{Y}_n\Delta W_n. \end{aligned}$$

Third, adding both equations leads to

$$\mathcal{E}_{n+1} = \mathcal{E}_n - \omega^2(1-2\theta_n)[Y_n\bar{X}_n - \bar{Y}_n X_n]h_n + \sigma_n\bar{Y}_n\Delta W_n.$$

Fourth, note that

$$\begin{aligned} &[Y_n\bar{X}_n - \bar{Y}_n X_n] = \frac{1}{2}[Y_n X_{n+1} - Y_{n+1}X_n], \\ &\bar{Y}_n = \frac{2Y_n - \omega^2(X_n + \theta_n(1-2\theta_n)Y_n h_n)h_n + \sigma_n\Delta W_n}{2(1+\omega^2\theta_n^2 h_n^2)}, \\ &\mathbb{E}\left[\sigma_n\bar{Y}_n\Delta W_n\right] = \frac{\sigma^2}{2}h_n. \end{aligned}$$

Fifth, pulling over expectations and summing over n yield that

$$\mathbb{E}\left[\mathcal{E}(t_n)\right] = \mathbb{E}\left[\mathcal{E}(0)\right] + \frac{\sigma^2}{2}t_n - \frac{\omega^2}{2}\sum_{k=0}^{n-1}(1-2\theta_k)\mathbb{E}\left[Y_k X_{k+1} - Y_{k+1}X_k\right]h_k < +\infty.$$

Recall that all step sizes h_n, parameters ω, σ and θ_n are supposed to be nonrandom. It remains to set $\theta_n = 0.5$ to verify the energy-identity (4.7). □

Extracting results from the previous proof of Theorem 4.2 yields the following.

Corollary 4.2 (Expected Energy Identity for θ-Methods (4.8))**.** *Under the same assumptions* (i)–(ii) *as in Theorem* 4.1*, we have the expected energy identity*

$$\mathbb{E}\left[\mathcal{E}(t_n)\right] = \mathbb{E}\left[\mathcal{E}(0)\right] + \frac{\sigma^2}{2}t_n - \frac{\omega^2}{2}\sum_{k=0}^{n-1}(1-2\theta_k)\mathbb{E}\left[Y_k X_{k+1} - Y_{k+1}X_k\right]h_k \qquad (4.9)$$

along any nonrandom partition $(t_n)_{n\in\mathbb{N}}$ *for the improved implicit* θ*-methods* (4.8) *with any nonrandom parameters* $\theta_k \in \mathbb{R}^1$*, any nonrandom constants* $\omega, \sigma \in \mathbb{R}^1$ *and any random initial data* $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$*.*

Remark. For more results on energy identities and oscillations, see forthcoming papers of author. Notice that relations (4.4) for continuous energy of SDE (3.1) and (4.7) for discrete energy of numerical methods (4.6) are indeed identical at the partition-instants t_n for all parameters ω, σ and initial values $X_0, Y_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P})$! Thus we answer the question that such numerical methods indeed exist (which are consistent too). The conclusions of Theorems 4.1, 4.2 and Corollaries 4.1, 4.2 are still valid if all ΔW_n are independent random variables with $\mathbb{E}\left[\Delta W_n\right] = 0$ and $\mathbb{E}\left[(\Delta W_n)^2\right] = h_n$.

5. APPENDIX: LIL FOR PII

The following theorem represents a summary of major results found by Stout [16] (discrete case) and Wang [17] (continuous case). It is also known as the **law of iterated logarithms** (LIL) for processes with independent increments (PII). This result extends classical results of Kolmogorov, Lévy and Strassen.

Theorem 5.1 (Stout-Wang's LIL for PII)**.** *Let* $X = (X(t))_{t\geq 0}$ *be a stochastic process with independent increments (PII) along a continuous or discrete time scale* $t \in \mathbb{T}$ *with* $\mathbb{T} = [0, +\infty)$ *or* $\mathbb{T} = \mathbb{N}$*. Assume that*

(i) $\forall t \in \mathbb{T} : \mathbb{E}\left[X(t)\right] = 0, \quad V(t) := \mathbb{E}\left[(X(t))^2\right] < +\infty, \quad \lim_{t\to+\infty} V(t) = +\infty,$

(ii) $\mathbb{E}\left[\sup_{t>0}\left(\frac{\Delta X_t}{t}\right)^2\right] < +\infty$ *where* $\Delta X_t = X(t) - X(t-)$ *for* $\mathbb{T} = [0, +\infty)$ *and* $\Delta X_t = X(t) - X(t-1)$ *for* $\mathbb{T} = \mathbb{N}$*.*

Then

$$\mathbb{P}\left(\left\{\limsup_{t\to+\infty}\frac{X(t)}{\sqrt{2V(t)\log\log(V(t))}} = +1\right\}\right) = 1,$$

$$\mathbb{P}\left(\left\{\liminf_{t\to+\infty}\frac{X(t)}{\sqrt{2V(t)\log\log(V(t))}} = -1\right\}\right) = 1.$$

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