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COMPETITIVE ADVERTISING IN A DUOPOLY: A STOCHASTIC DIFFERENTIAL GAME APPROACH

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ABSTRACT. We can not neglect the randomness in the dynamics of market competition. This paper develops a stochastic differential game model which incorporates advertising effects for two companies introducing a new brand of product, competing for the market share and optimizing the budget. Our model is formulated as a stochastic, two-player, noncooperative differential game. As to the choice of basic dynamics, because of the special sale growing style in the introductory period, our model disagrees with the simple decay factor in Vidale-Wolfe model, but is based on the combination of Lanchester combat model and Logistic growth model. The solution concept is *Nash Equilibrium*. We derive optimality necessary conditions for Nash Equilibrium from dynamic programming, which is a Stochastic Partial Differential Equation(SPDE). The typical work in this field is from Prasad A. (2004), where he analytically solved his model because of the special form of the model. We choose to solve the optimality conditions, which is SPDE system numerically. Management strategy and discussions based on practical considerations will be given based on numerical results.

Keywords: differential games, marketing, competitive strategy

MSc Subject classification: 91A15, 91A23, 91A80, 91B50, 91B60

1. INTRODUCTION

First we discuss some basic deterministic dynamics in Marketing. Vidale-Wolfe (1957) advertising model is one of the earliest math models in marketing, which is derived from actual market phenomena represented by cases they have observed, and consistent with experimental observations. It is simple but can describe the relationship between advertising and sales in a reasonable manner. Thus, many researchers adopt Vidale-Wolfe type dynamics in many differential game models in marketing competition. The basic Vidale-Wolfe model is the following

$$\frac{dx}{dt} = \rho\mu(1 - \frac{x}{M}) - kx, \quad x(0) = x_0 \,,$$

where x is the sales rate, M is the maximum sales potential, and the parameters ρ and k are the response constant of advertising and sales decay constant, respectively. The parameter μ is control variable, representing the rate of advertising expenditure.

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From this model, we can see that the product sales change rate depends on two factors: one is positive response to advertising that acts on the unsold portion of the market, and the other is decay caused by forgetting, which is linearly proportional to the sold portion of the market. And the general Vidale-Wolfe in duopoly is as follows:

$$\frac{dx_i}{dt} = \rho_i \mu_i (1 - f(x_i, x_1 + x_2)) - k_i x_i,$$

where $f(\cdot, \cdot)$ is strictly bigger than zero. However, Vidale-Wolfe model just reflects the fact that for some mature product, sale will decrease as time goes on, and adverting may stop this kind of natural decay to some extent.

Lanchester combat model is a basic model to describe competition among individuals, which is also used to describe competition in marketing. Kimball (1957) recognized the application of this model in advertising. The basic dynamics is as follows:

$$\frac{dx_1}{dt} = bu_1 x_2 - au_2 x_1$$
$$\frac{dx_2}{dt} = au_2 x_1 - bu_1 x_2$$

where $x_1 + x_2 = M$. Lanchester model reveals the basic fact that in competition, the driving force is the extra 'force' of one competitor over the other. In fact, by substituting $x_2 = M - x_1$ we can see that the dynamics for $i - th$ player is a Vidale-Wolfe model with time-varying decay parameter. The general Lanchester model is as follows:

$$\frac{dx}{dt} = g(u_i) x_j - h(u_j) x_i, \quad i \neq j, \quad \text{where} \quad g(\cdot), h(\cdot) > 0$$

Suresh P. Sethi (1973,1977) used optimal control model to conduct research about optimal advertising strategy, which is based on the dynamics from Vidale-Wolfe and Lanchester models. And many similar works were done by Friedman (1983), Erickson (1985), Eliashberg and Jeuland (1986). These works approach the problem as open loop, deterministic optimal control problems.

However, in all the above models, there is no competition. Kenneth R. Deal (1979) first set up deterministic differential game model to optimize advertising expenditures in a dynamic duopoly. His 'Vidale-Wolfe' type differential game is as follows:

$$\max_{u_1} J_1 = \int_{t_0}^{t_f} (c_1 x_1(t) - u_1^2(t)) dt + \omega_1 \frac{x_1(t_f)}{x_1(t_f) + x_2(t_f)}$$
$$\max_{u_2} J_2 = \int_{t_0}^{t_f} (c_2 x_2(t) - u_2^2(t)) dt + \omega_2 \frac{x_2(t_f)}{x_1(t_f) + x_2(t_f)}$$

and system dynamics:

$$\frac{dx_1}{dt} = -a_1 x_1(t) + b_1 u_1(t) \frac{M - x_1(t) - x_2(t)}{M}$$
$$\frac{dx_2}{dt} = -a_2 x_2(t) + b_2 u_2(t) \frac{M - x_1(t) - x_2(t)}{M}$$

We have analyzed some drawback of the above dynamics in [15]. Here we just repeat that competitors have no direct influence on other competitors' sales. The typical 'Lanchester' type differential game is from Case (1979):

$$\max_{u_1} J_1 = \int_{t_0}^{+\infty} e^{-rt}(q_1 x(t) - \frac{c_1}{2} u_1^2(t))dt$$

$$\max_{u_2} J_2 = \int_{t_0}^{+\infty} e^{-rt}(q_2(1 - x(t)) - \frac{c_2}{2} u_2^2(t))dt$$

and system dynamics:

$$\frac{dx}{dt} = u_1(t)(1 - x(t)) - u_2(t)x(t), \quad x(0) = x_0$$

Case used these assumptions in the above typical differential game model in advertising competition: total market potential is constant over time; the only marketing instrument used by the firm is advertising; advertising has diminishing returns since there are increasing marginal costs of advertising; there are saturation effects since u_i is employed on the market of the opponent player. Some other typical deterministic differential game models can be found in Gerhard Sorger (1989), Pradeep K Chintagunta (1992).

All of the models abovr are deterministic models. However, in reality, competition in marketing is full of uncertainity . For example, in the driving force of sale change, besides adverting and competition, there should be many other factors which are not included. So we want to extend our research on competition into stochastic environment. And we try to answer how randomness will affect the outcome of competition, and to what extend randomness will affect the results. Prasad and Sethi (2004) gave the first stochastic differential game model to describe competitive advertising in uncertain environment. Their 'Lanchester' type stochastic differential game model is as follows:

$$\max_{u_1} J_1 = \int_{t_0}^{\infty} e^{-r_1 t}(m_1 x(t) - c_1 u_1^2(t))dt$$

$$\max_{u_2} J_2 = \int_{t_0}^{\infty} e^{-r_2 t}(m_2 x(t) - c_2 u_2^2(t))dt$$

subject to:

$$dx = (\rho_1 u_1(x)\sqrt{1 - x(t)} - \rho_2 u_2(x)\sqrt{x(t)} - \delta(2x - 1))dt + \sigma(x)dw, \quad x(0) = x_0$$

In the above dynamics, we see that the driving force just comes from competition, and they did not consider the effects of sales growth or decrease in the product life cycle. Further, they solved their model analytically because of the special form of the model.

Currently, given rapid advances in technology, companies from time to time introduce newer versions of old products which incorporate not only old functions but also the newer functions, such as in electronic products and daily necessities. When such types of products come into the market, their sale usually will experience a grow-saturation-decay phase(*Figure 6.1*). In previous papers [15], [16], we conducted research in competition in different stages of a product life cycle. We have set up deterministic differential game models for different stages, and drawn practical guidelines based on numerical results. And in this paper, we will conduct research on competition in stochastic environment in marketing. In the following subsection, we will set up stochastic differential game model based on the first stage of a product life cycle. In our work, we will combine two kinds of dynamics: one is logistic growth, the other is 'Lanchester' competition. And each company has two objectives: one is market share, the other is profit. We will set up algorithm to solve our model numerically, and this numerical approach has an advantage over the approach of Prasad A. (2004) because our algorithm have more adaptability, and can be easily modified to solve more general models. Another different feature is that we choose to concentrate on research of competition at some stage of the product life cycle, which may be more useful from practical aspect.

2. MODEL

Our stochastic differential game model deals with the competition in the first stage of product life cycle, which includes 'Introduction, Growth, and Maturity' in the following graph (Figure 6.1). Suppose that there are **2** companies to sell one kind of new product in the same market. The market managers use one kind of control – advertising – to maximize profit and maximize final market share. The main notations are as follows:

$\mathbf{x}(t)$	Market share of company 1 at time t.
$\mathbf{y}(t)$	Market share of company 2 at time t.
$\mathbf{u}_i(t)$	Control/Advertising of company i at time t.
$\mathbf{c}_i$	Price of company i' product.
ω_i	Weight factor, which shows the relative importance between two objectives of sale managers.
α_i	Effectiveness of natural growth of company i' s product.
$\mathbf{k}_i$	Market limitation for company i's product.
β_i	Effectiveness of control/advertising of company i.
σ	Effectiveness of randomness on the sales.

To describe the dynamics in the fist stage of the product life cycle, we still adopt the assumptions that the sale growth of new product has an approximate logistics growth, which is called *natural growth*, and the market capability for one product is limited, so in the basic dynamics of state variables we will adopt following equations

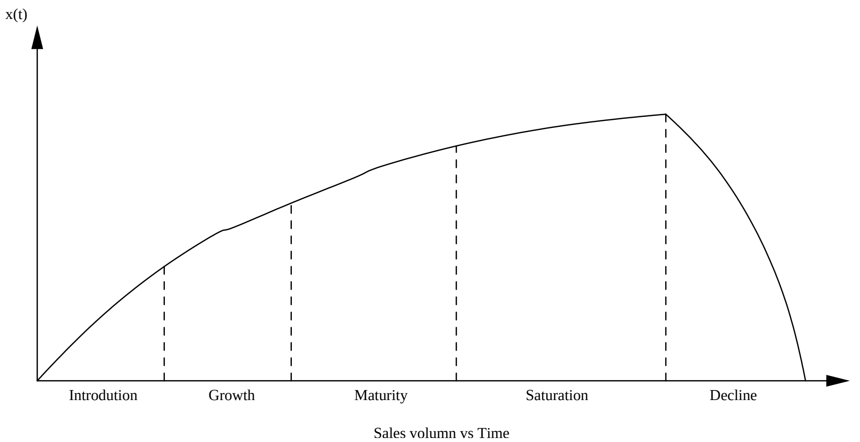

FIGURE 1

to describe the growth of sale:

$$\frac{dx}{dt} = \alpha_1(k_1 - x)x$$
$$\frac{dy}{dt} = \alpha_2(k_2 - y)y$$

Another driving force for the dynamics is from *competition*. We suppose that bigger control/advertising will lead to bigger driving force, bigger own market share will lead to less driving force, and bigger competitor's market share will lead to bigger own advertising effectiveness. Based on Lanchester competition model we integrate our assumptions about competition in following formula:

$$\frac{dx}{dt} = \beta_1 u_1 \sqrt{(k_1 - x)y} - \beta_2 u_2 \sqrt{(k_2 - y)x}$$

The third force to affect the dynamics of sale is *randomness*. Just as we mentioned in previous section, in reality, there are many factors we can not expect exactly to affect the dynamics of state variables. We model the randomness as white noise (dB_t), where dB_t is Brownian motion. In order to quanitify the randomness, we adopt the assumption that when sale is in the middle of its range, it has larger randomness, and when sale approach the broader portion of its range, it has smaller randomness, and the two companies' sales will interact to increase the randomness. Thus, we adopt following formula to describe randomness in the market competition:

$$\sigma xy(k_1 - x)(k_2 - y)dB_t$$

Based on the above analysis, we synthesize these three factors to set up stochastic differential equation for sales:

$$dx = \left(\alpha_1(k_1 - x)x + \beta_1 u_1 \sqrt{(k_1 - x)y} - \beta_2 u_2 \sqrt{(k_2 - y)x}\right) dt + \sigma xy(k_1 - x)(k_2 - y)dB_t$$

$$dy = \Big(\alpha_2(k_2 - y)y + \beta_2 u_2\sqrt{(k_2 - y)x} - \beta_1 u_1\sqrt{(k_1 - x)y}\Big)\, dt$$
$$- \sigma xy(k_1 - x)(k_2 - y)dB_t$$

Each company has its own objectives. Based on practical considerations, usually there are two goals in this competition for each company in the first stage of product life cycle, one is maximizing profit, the other is maximizing the final market share. So we integrate these two factors into each competitor's objective function as follows:

$$J_1(x(t), y(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_1} E\left\{\int_t^{t_f} (c_1 x(t) - u_1^2(t))dt + \omega_1 \frac{x(t_f)}{x(t_f) + y(t_f)}\right\}$$
$$J_2(x(t), y(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_2} E\left\{\int_t^{t_f} (c_2 x(t) - u_2^2(t))dt + \omega_2 \frac{y(t_f)}{x(t_f) + y(t_f)}\right\}$$

where $J_i(x(t), y(t), u_1^*(t), u_2^*(t), t, t_f)$ is optimal objective value when each company adopt optimal control $u_i^*(t)$. We take expectation because of the objective value is a random variable.

So we have our stochastic differential game model:

$$J_1(x(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_1} E\left\{\int_t^{t_f} (c_1 x(t) - u_1^2(t))ds + \omega_1 \frac{x(t_f)}{x(t_f) + y(t_f)}\right\}$$
$$J_2(x(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_2} E\left\{\int_t^{t_f} (c_2 x(t) - u_2^2(t))ds + \omega_2 \frac{x(t_f)}{x(t_f) + y(t_f)}\right\}$$

$s.t.$

$$dx = \Big(\alpha_1(k_1 - x)x + \beta_1 u_1\sqrt{(k_1 - x)y} - \beta_2 u_2\sqrt{(k_2 - y)x}\Big)\, dt$$
$$+ \sigma xy(k_1 - x)(k_2 - y)dB_t$$
$$dy = \Big(\alpha_2(k_2 - y)y + \beta_2 u_2\sqrt{(k_2 - y)x} - \beta_1 u_1\sqrt{(k_1 - x)y}\Big)\, dt$$
$$- \sigma xy(k_1 - x)(k_2 - y)dB_t$$

$x(t), y(t)$ given

In the following sections, we will use dynamic programming to derive optimality condition for the above stochastic differential game, then we will set up algorithm to solve optimality conditions, and then solve the above model.

3. **OPTIMALITY CONDITION**

Now we will derive optimality condition for a generalized model of our problem. The general model is as follows:

$$J_1(x(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_1} E\left\{\int_t^{t_f} f_1(x(t), u_1(t), u_2(t))ds + h_1(x(t_f))\right\}$$
$$J_2(x(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_2} E\left\{\int_t^{t_f} f_2(x(t), u_1(t), u_2(t))ds + h_2(x(t_f))\right\}$$

$$\begin{aligned} &s.t.\\ &dx = a(x(t), u_1(t), u_2(t))dt + \sigma(x(t), u_1(t), u_2(t))dB_t\\ x(t)\ \ &given \end{aligned}$$

where

$$\begin{aligned} x(t) =& [x_1(t), x_2(t), \ldots, x_n(t)]^T\\ x_i(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R}\\ f_1(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R}\\ f_2(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R}\\ a(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R}^n\\ \sigma(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R}^n\\ B(t,\omega) :& [0, t_f] \times \Omega \to \mathcal{R} \end{aligned}$$

Then we define the optimal solution for above stochastic differential game.

Definition 3.1. u_1^*, u_2^* are optimal solution for two-player stochastic differential game if following conditions are met:

$$J_1(u_1^*, u_2^*) \geq J_1(u_1, u_2^*), \quad J_2(u_1^*, u_2^*) \geq J_2(u_1^*, u_2)$$

The following lemma will be used to derive the optimality condition for above model.

Lemma 3.1. *Suppose the process $x_1(t), x_2(t), \ldots, x_n(t)$ obeys the stochastic differential equations:*

$$dx_i = a_i(t, x_1(t), \ldots, x_n(t))dt + \sigma_i(t, x_1(t), \ldots, x_n(t))dB_t$$

where $i = 1, 2, \ldots, n$. Define

$$y(t) = J(t, x_1(t), \ldots, x_n(t))$$

Then the stochastic differential equation for $y(t)$ is as follows:

$$dy = \left[\frac{\partial J}{\partial t} + \sum_i^n \frac{\partial J}{\partial x_i}\alpha_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\right]dt + \sum_{i=1}^n \frac{\partial J}{\partial x_i}\sigma_i dB_t$$

Proof. By general *Itô* formula, we have:

$$\begin{aligned} dy &= \frac{\partial J}{\partial t}dt + \sum_i \frac{\partial J}{\partial x_i}dx_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}dx_i dx_j\\ &= \frac{\partial J}{\partial t}dt + \sum_i \frac{\partial J}{\partial x_i}(a_i dt + \sigma_i dB_t) + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}(a_i dt + \sigma_i dB_t)(a_j dt + \sigma_j dB_t) \end{aligned}$$

$$= \frac{\partial J}{\partial t}dt + \sum_i \frac{\partial J}{\partial x_i}a_i dt + \sum_i \frac{\partial J}{\partial x_i}\sigma_i dB_t + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}(\sigma_i\sigma_j)dt$$

$$= [\frac{\partial J}{\partial t} + \sum_i^n \frac{\partial J}{\partial x_i}\alpha_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j]dt + \sum_{i=1}^n \frac{\partial J}{\partial x_i}\sigma_i dB_t$$

□

Now we derive optimality condition for above general model through dynamic programming.

Theorem 3.1. *Suppose* Nash equilibrium $(u_1^*(t), u_2^*(t))$ *exits for above model. Then,* $(u_1^*(t), u_2^*(t))$ *should satisfy following equations:*

$$-\frac{\partial J_1}{\partial t} = \max_{u_1 \in [t,t_f]} \left\{ f_1(x(t), u_1(t), u_2^*(t)) + \sum_i \frac{\partial J_1}{\partial x_i}a_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_1}{\partial x_i \partial x_j}\sigma_i\sigma_j \right\}$$

$$-\frac{\partial J_2}{\partial t} = \max_{u_2 \in [t,t_f]} \left\{ f_2(x(t), u_1(t), u_2^*(t)) + \sum_i \frac{\partial J_2}{\partial x_i}a_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_2}{\partial x_i \partial x_j}\sigma_i\sigma_j \right\}$$

Proof. Suppose player 2 has reached his equilibrium $u_2^*(t)$, then player 1 will maximize his objective under the condition that player 2 has already adopted optimal control. So player 1 has following problem:

$$J_1(x(t), u_1^*(t), u_2^*(t), t, t_f) = \max_{u_1} E\left\{ \int_t^{t_f} f_1(x(t), u_1(t), u_2^*(t))ds + h_1(x(t_f)) \right\}$$

$$s.t.$$

$$dx = a(x(t), u_1(t), u_2^*(t))dt + \sigma(x(t), u_1(t), u_2^*(t))dB_t$$

$$x(t) \quad given$$

Then

$$J_1(u_1^*, u_2^*, x(t), t, t_f) = \max_{u_1} E\left\{ \int_t^{t_f} f_1(x(t), u_1(t), u_2^*(t))ds + h_1(x(t_f)) \right\}$$

$$= \max_{u_1} E\left\{ \int_t^{t+\Delta t} f_1(x(t), u_1(t), u_2^*(t))ds + \int_{t+\Delta t}^{t_f} f_1(x(t), u_1(t), u_2^*(t))ds + h_1(x(t_f)) \right\}$$

$$= \max_{u_1 \in [t,t+\Delta t]} E\Bigg\{ \int_t^{t+\Delta t} f_1(x(t), u_1(t), u_2^*(t))ds$$

$$+ \max_{u_1 \in [t+\Delta t, t_f]} E\left[\int_{t+\Delta t}^{t_f} f_1(x(t), u_1(t), u_2^*(t))ds + h_1(x(t_f)) \right] \Bigg\}$$

$$= \max_{u_1 \in [t,t+\Delta t]} E\left\{ \int_t^{t+\Delta t} f_1(x(t), u_1(t), u_2^*(t))ds + J_1(u_1^*, u_2^*, x(t+\Delta t), t+\Delta t, t_f) \right\}$$

$$= \max_{u_1 \in [t,t_f]} E\left\{ \int_t^{t+\Delta t} f_1(x(t), u_1(t), u_2^*(t))ds + J_1(u_1^*, u_2^*, x(t+\Delta t), t+\Delta t, t_f) \right\}$$

$$= \max_{u_1 \in [t,t_f]} E\Bigg\{ \int_t^{t+\Delta t} f_1(x(t), u_1(t), u_2^*(t))ds + J_1(u_1^*, u_2^*, x(t), t, t_f)$$

$$+\left(\frac{\partial J_1}{\partial t}+\sum_i \frac{\partial J_1}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\right)\Delta t+\sum_i \frac{\partial J}{\partial x_i}\sigma_i \Delta B_t+O(\Delta t)\Bigg\}$$

by Lemma 3.1

$$=\max_{u_1\in[t,t_f]} E\Bigg\{J_1(u_1^*,u_2^*,x(t),t,t_f)+\Bigg(f_1(x(t),u_1(t),u_2^*(t))$$

$$+\frac{\partial J_1}{\partial t}+\sum_i \frac{\partial J_1}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\Bigg)\Delta t+\sum_i \frac{\partial J}{\partial x_i}\sigma_i \Delta B_t+O(\Delta t)\Bigg\}$$

$\Longrightarrow$

$$0=\max_{u_1\in[t,t_f]} E\Bigg\{\Bigg(f_1(x(t),u_1(t),u_2^*(t))+\frac{\partial J_1}{\partial t}+\sum_i \frac{\partial J_1}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\Bigg)\Delta t$$

$$+\sum_i \frac{\partial J}{\partial x_i}\sigma_i \Delta B_t+O(\Delta t)\Bigg\}$$

$\Longrightarrow$

$$0=\max_{u_1\in[t,t_f]}\Bigg\{\Bigg(f_1(x(t),u_1(t),u_2^*(t))+\frac{\partial J_1}{\partial t}+\sum_i \frac{\partial J_1}{\partial x_i}a_i$$

$$+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\Bigg)\Delta t+O(\Delta t)\Bigg\} \text{ since } E[\Delta B_t]=0$$

$\Longrightarrow$

$$-\frac{\partial J_1}{\partial t}=\max_{u_1\in[t,t_f]}\left\{f_1(x(t),u_1(t),u_2^*(t))+\sum_i \frac{\partial J_1}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J}{\partial x_i \partial x_j}\sigma_i\sigma_j\right\}$$

by dividing Δt

We can use the same argument for player 2. Then the necessary conditions for optimal solution (u_1^*, u_2^*) is:

$$-\frac{\partial J_1}{\partial t}=\max_{u_1\in[t,t_f]}\left\{f_1(x(t),u_1(t),u_2^*(t))+\sum_i \frac{\partial J_1}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_1}{\partial x_i \partial x_j}\sigma_i\sigma_j\right\}$$

$$-\frac{\partial J_2}{\partial t}=\max_{u_2\in[t,t_f]}\left\{f_2(x(t),u_1^*(t),u_2(t))+\sum_i \frac{\partial J_2}{\partial x_i}a_i+\frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_2}{\partial x_i \partial x_j}\sigma_i\sigma_j\right\}$$

□

In order to use above optimality conditions to solve our stochastic differential game model, we will convert this necessary conditions into Stochastic Partial Differential Equation (SPDE) in the following way, and then solve the SPDE numerically in the following section.

From above necessary conditions for optimality, we write it out explicitly

$$
\begin{aligned}
-\frac{\partial J_1}{\partial t} &= \max_{u_1}\{f_1 + \sum_i \frac{\partial J_1}{\partial x_i}a_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_1}{\partial x_i \partial x_j}\sigma_i\sigma_j\} \\
&= \max_{u_1}\{c_1 x - u_1^2 \\
&\quad + \frac{\partial J_1}{\partial x}(\alpha_1(k_1-x)x + \beta_1 u_1\sqrt{(k_1-x)y} - \beta_2 u_2\sqrt{(k_2-y)x}) \\
&\quad + \frac{\partial J_1}{\partial y}(\alpha_2(k_2-y)y + \beta_2 u_2\sqrt{(k_2-y)x} - \beta_1 u_1\sqrt{(k_1-x)y}) \\
&\quad - \frac{1}{2}(\frac{\partial^2 J_1}{\partial x^2} + 2\frac{\partial^2 J_1}{\partial x\partial y} + \frac{\partial^2 J_1}{\partial y^2})\sigma^2 x^2 y^2 (k_1-x)(k_2-y)^2\} \\
-\frac{\partial J_2}{\partial t} &= \max_{u_2}\{f_2 + \sum_i \frac{\partial J_2}{\partial x_i}a_i + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 J_2}{\partial x_i \partial x_j}\sigma_i\sigma_j\} \\
&= \max_{u_2}\{c_2 x - u_2^2 \\
&\quad + \frac{\partial J_2}{\partial x}(\alpha_1(k_1-x)x + \beta_1 u_1\sqrt{(k_1-x)y} - \beta_2 u_2\sqrt{(k_2-y)x}) \\
&\quad + \frac{\partial J_2}{\partial y}(\alpha_2(k_2-y)y + \beta_2 u_2\sqrt{(k_2-y)x} - \beta_1 u_1\sqrt{(k_1-x)y}) \\
&\quad - \frac{1}{2}(\frac{\partial^2 J_2}{\partial x^2} + 2\frac{\partial^2 J_2}{\partial x\partial y} + \frac{\partial^2 J_2}{\partial y^2})\sigma^2 x^2 y^2 (k_1-x)(k_2-y)^2\}
\end{aligned}
$$

On the right sides of above equations, we have concave functions of u_1, u_2 respectively, so first order derivative with respect to u_1, u_2 will vanish at the optimal trajectories. Then, we differentiate right hand sides and get optimal u_1, u_2 as follows:

$$
\begin{aligned}
u_1^* &= \frac{1}{2}\beta_1\sqrt{(k_1-x)y}(\frac{\partial J_1}{\partial x} - \frac{\partial J_1}{\partial y}) \\
u_2^* &= \frac{1}{2}\beta_2\sqrt{(k_2-y)x}(\frac{\partial J_2}{\partial y} - \frac{\partial J_2}{\partial x})
\end{aligned}
$$

Now let

$$\bar{p} = (p_1 \quad p_2) \triangleq \begin{pmatrix} \frac{\partial J_1}{\partial x} & \frac{\partial J_1}{\partial y} \end{pmatrix}$$

$$\bar{r} = (r_1 \quad r_2) \triangleq \begin{pmatrix} \frac{\partial J_2}{\partial x} & \frac{\partial J_2}{\partial y} \end{pmatrix}$$

and put u_1^*, u_2^* back to above partial differential equations system to get:

$$
\begin{aligned}
-\frac{\partial J_1}{\partial t} &= H_1^*(x, y, \bar{p}, \bar{r}, \frac{\partial \bar{p}}{\partial x}, \frac{\partial \bar{p}}{\partial y}, \frac{\partial \bar{r}}{\partial x}, \frac{\partial \bar{r}}{\partial y}) \\
-\frac{\partial J_2}{\partial t} &= H_2^*(x, y, \bar{p}, \bar{r}, \frac{\partial \bar{p}}{\partial x}, \frac{\partial \bar{p}}{\partial y}, \frac{\partial \bar{r}}{\partial x}, \frac{\partial \bar{r}}{\partial y})
\end{aligned}
$$

and from

$$p_1(t) = \frac{\partial J_1}{\partial x}(x(t), t, t_f), \quad p_2(t) = \frac{\partial J_1}{\partial y}(x(t), t, t_f)$$

$$r_1(t) = \frac{\partial J_2}{\partial x}(x(t), t, t_f), \quad r_2(t) = \frac{\partial J_2}{\partial y}(x(t), t, t_f)$$

we apply *Itô* formula to $p_1(t), p_2(t), r_1(t), r_2(t)$ to get following related stochastic differential equations:

$$\begin{aligned}
dp_1 &= -\frac{\partial H_1^*}{\partial x}dt + (\frac{\partial^2 J_1}{\partial x \partial x} + \frac{\partial^2 J_1}{\partial x \partial y})\sigma dB \\
dp_2 &= -\frac{\partial H_1^*}{\partial y}dt + (\frac{\partial^2 J_1}{\partial y \partial x} + \frac{\partial^2 J_1}{\partial y \partial y})\sigma dB \\
dr_1 &= -\frac{\partial H_2^*}{\partial x}dt + (\frac{\partial^2 J_2}{\partial x \partial x} + \frac{\partial^2 J_2}{\partial x \partial y})\sigma dB \\
dr_2 &= -\frac{\partial H_2^*}{\partial y}dt + (\frac{\partial^2 J_2}{\partial y \partial x} + \frac{\partial^2 J_2}{\partial y \partial y})\sigma dB
\end{aligned}$$

We put the above optimal u_1^*, u_2^* back into the state equation, and get following state and co-state system as follows, which is Stochastic Partial Differential Equations (SPDE):

$$\begin{aligned}
dx &= (\alpha_1(k_1 - x)x + \frac{1}{2}\beta_1^2(k_1 - x)y(p_1 - p_2) - \frac{1}{2}\beta_2^2(k_2 - y)x(r_2 - r_1))dt \\
&\quad + \sigma xy(k_1 - x)(k_2 - y)dB \\
dy &= (\alpha_2(k_2 - y)y + \frac{1}{2}\beta_2^2(k_2 - y)x(r_2 - r_1) - \frac{1}{2}\beta_1^2(k_1 - x)y(p_1 - p_2))dt \\
&\quad - \sigma xy(k_1 - x)(k_2 - y)dB \\
dp_1 &= \{-c_1 - \frac{1}{4}\beta_1^2(p_1 - p_2)^2 \\
&\quad - p_1[\alpha_1 k_1 - 2\alpha_1 x - \frac{1}{2}y(p_1 - p_2)^2 - \frac{1}{2}\beta_2^2(k_2 - y)(r_2 - r_1)] \\
&\quad - p_2[\frac{1}{2}\beta_2^2(k_2 - y)(r_2 - r_1) + \frac{1}{2}\beta_1^2(p_1 - p_2)] \\
&\quad + \sigma^2 x(k_1 - x)(k_2 - 2x)y^2(k_2 - y)^2[\frac{\partial p_1}{\partial x} + 2\frac{\partial p_1}{\partial y} + \frac{\partial p_2}{\partial y}]\}dt \\
&\quad + [\frac{\partial p_1}{\partial x} - \frac{\partial p_1}{\partial y}]\sigma xy(k_1 - x)(k_2 - y)dB \\
dp_2 &= \{\frac{1}{4}\beta_1^2(k - x)(p_1 - p_2)^2 \\
&\quad - p_1[\frac{1}{2}\beta_1^2(k - x)(p_1 - p_2) + \frac{1}{2}\beta_2^2 x(r_2 - r_1)] \\
&\quad - p_2[\frac{1}{2}\alpha_2(k_2 - 2y) - \frac{1}{2}\beta_2^2 x(r_1 - r_2) - \frac{1}{2}\beta_1^2(k_1 - x)(p_1 - p_2)] \\
&\quad + \sigma^2 y(k_2 - y)(k_2 - 2y)x^2(k_1 - x)^2[\frac{\partial p_1}{\partial x} + 2\frac{\partial p_2}{\partial y} + \frac{\partial p_2}{\partial y}]\}dt \\
&\quad + [\frac{\partial p_2}{\partial x} - \frac{\partial p_2}{\partial y}]\sigma xy(k_1 - x)(k_2 - y)dB
\end{aligned}$$

$$
\begin{aligned}
dr_1 = \{&\frac{1}{4}\beta_2^2(k-y)(r_1-r_2)^2 \\
&- r_1[\alpha_1(k_1-2x) - \frac{1}{2}\beta_1^2 y(p_1-p_2) - \frac{1}{2}\beta_2^2(k_2-y)(r_2-r_1)] \\
&- r_2[\frac{1}{2}\beta_2^2(k_2-y)(r_2-r_1) + \frac{1}{2}\beta_1^2 y(p_1-p_2)] \\
&+ \sigma^2 x(k_1-x)(k_1-2x)y^2(k_2-y)^2[\frac{\partial r_1}{\partial x} + 2\frac{\partial r_1}{\partial y} + \frac{\partial p_2}{\partial y}]\}dt \\
&+ [\frac{\partial r_1}{\partial x} - \frac{\partial x_1}{\partial y}]\sigma xy(k_1-x)(k_2-y)dB \\
dr_2 = \{&-c_2 - \frac{1}{4}\beta_2^2(r_2-r_1)^2 \\
&- r_1[\frac{1}{2}\beta_1^2(k_1-x)(p_1-p_2) + \frac{1}{2}\beta_2^2 x(r_2-r_1)] \\
&- r_2[\alpha_2(k_2-2y) - \frac{1}{2}\beta_2^2 x(r_2-r_1) - \frac{1}{2}\beta_1^2(k_1-x)(p_1-p_2)] \\
&+ \sigma^2 y(k_2-y)(k_2-2y)x^2(k_2-x)^2[\frac{\partial r_1}{\partial x} + 2\frac{\partial r_2}{\partial x} + \frac{\partial r_2}{\partial y}]\}dt \\
&+ [\frac{\partial r_1}{\partial x} - \frac{\partial r_2}{\partial y}]\sigma xy(k_1-x)(k_2-y)dB
\end{aligned}
$$

Boundary condition:

$$
\begin{aligned}
p_1(x,y,t_f) &= \omega_1\frac{y}{(x+y)^2}|_{t_f} \\
p_2(x,y,t_f) &= -\omega_2\frac{x}{(x+y)^2}|_{t_f} \\
r_1(x,y,t_f) &= -\omega_2\frac{y}{(x+y)^2}|_{t_f} \\
r_2(x,y,t_f) &= \omega_2\frac{x}{(x+y)^2}|_{t_f}
\end{aligned}
$$

What we will do in next section is to design algorithm to solve the above specific SPDE system.

4. **NUMERICAL CALCULATION**

We first rewrite the above SPDE system using simpler notation:

$$
\begin{aligned}
dx &= f_1(x,y,p_1,p_2,r_1,r_2)dt + g_1(x,y)dB_t \\
dy &= f_2(x,y,p_1,p_2,r_1,r_2)dt + g_2(x,y)dB_t \\
dp_1 &= f_3(x,y,p_1,p_2,r_1,r_2,\frac{\partial p_i}{\partial x},\frac{\partial p_i}{\partial y})dt + g_3(x,y,\frac{\partial p_i}{\partial x},\frac{\partial p_i}{\partial y})dB_t
\end{aligned}
$$

$$dp_2 = f_4(x, y, p_1, p_2, r_1, r_2, \frac{\partial p_i}{\partial x}, \frac{\partial p_i}{\partial y})dt + g_4(x, y, \frac{\partial p_i}{\partial x}, \frac{\partial p_i}{\partial y})dB_t$$
$$dr_1 = f_5(x, y, p_1, p_2, r_1, r_2, \frac{\partial r_i}{\partial x}, \frac{\partial r_i}{\partial y})dt + g_5(x, y, \frac{\partial r_i}{\partial x}, \frac{\partial r_i}{\partial y})dB_t$$
$$dr_2 = f_6(x, y, p_1, p_2, r_1, r_2, \frac{\partial r_i}{\partial x}, \frac{\partial r_i}{\partial y})dt + g_6(x, y, \frac{\partial r_i}{\partial x}, \frac{\partial r_i}{\partial y})dB_t$$
$$x(0), y(0) \text{ given}$$
$$p_i(x, y, t_f), r_i(x, y, t_f) \text{ known}$$
$$\text{where } i = 1, 2$$

so we can see that in above SPDE system to be solved, the initial value for state $x(0), y(0)$, terminal value for $p_i(x, y, t_f), r_i(x, y, t_f)$ are known. We can call this specific SPDE *Two-Point Boundary Value Stochastic Partial Differential Equation* (TP-BVSPDE).

The idea to solve the above problem comes from the observation of following graph (*Figure 6.2*), which is xy-plane v.s. *time*. We discretize time interval $[0, t_f]$ into N subintervals, and at time $0, \Delta t, 2\Delta t, \ldots, (N+1)\Delta t$, we have $N+1$ parallel planes. And on each xy-plane, we can do discretization for each xy-plane, dividing each plane into $N_x \times N_y$ sub-rectangles. On the $xy - t_f$ plane, we know every value of $p_i(x, y, t_f)$, $r_i(x, y, t_f)$, which means we know every discretized $p_i(x(i),\ y(j), t_f)$, $r_i(x(i), y(j), t_f)$ at time t_f. So if we can figure out $p_i(x(i), y(j),\ k\Delta t)$, $r_i(x(i), y(j), k\Delta t)$ at each xy-plane, then we may solve our problem. We try to use backward Euler method to integrate the SPDE backward.

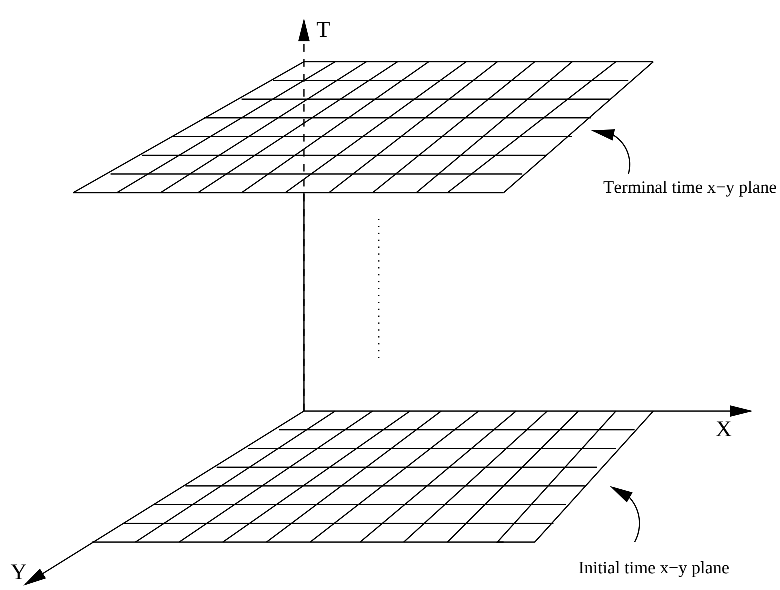

FIGURE 2. xy-time plane

The first problem we'd like to solve is how to evaluate $\frac{\partial p_i}{\partial x}$, $\frac{\partial p_i}{\partial y}$, $\frac{\partial r_i}{\partial x}$, $\frac{\partial r_i}{\partial y}$. At one discretized xy-plane, if we know the value of $p_i(x,y,t)$, $r_i(x,y,t)$ at each corner of the sub-rectangular plane, then we can approximate them by $\frac{p_i(i+1,j,k)-p_i(i,j,k)}{\Delta x}$, etc. The second problem is how to do stochastic integrals. As to an autonomous SDE:

$$dX(t) = f(x(t))dt + g(X(t))dB(t), \quad 0 \le t \le t_f$$

where $X(0)$ is given, the Euler-Maruyama(EM) Method takes the following form to solve it numerically:

$$X(i+1) = X(i) + f(X(i))\Delta t + g(X(i))(B(i+1) - B(i)), \quad i = 1, \dots, N$$

where $B(1), \dots, B(N)$ is discretized Brownian path. These random variables are simulated with a random number generator. Based on EM method to solve SDE, once we get value for $\frac{\partial p_i}{\partial x}(x(i), y(j), k\Delta t)$, we can use backward Euler-Maruyama method to integrate the costate equation backward equation. So, after we get all the values of $p_i(x(i),\ y(j), k)$, $r_i(x(i),\ y(j), k)$, we still use Euler-Maruyama (EM) Method to integrate the state equations forward. And then using state and costate values we evaluate controls. Based on above idea, we design *Algorithm 6.1* to solve the above TPBVSPDE.

Algorithm 4.1. Step 1. Discretize time interval $[0, t_f]$ into N subintervals. Discretize xy-plane into $N_x \times N_y$ sub-rectangles.
Step 2. Generate discretized Brownian path at each sub-time intervals: $dB(1), \dots,$ $dB(N)$.
Step 3. Evaluate $p_i(i,j,N+1), r_i(i,j,N+1)$.
Step 4. Use Backward Euler method to integrate costate equations backward:
for $k = N : 1$
approximate $\frac{\partial p_i}{\partial x}$, $\frac{\partial p_i}{\partial y}$, $\frac{\partial r_i}{\partial x}$, $\frac{\partial r_i}{\partial y}$ by $\frac{p_i(i+1,j,k)-p_i(i,j,k)}{\Delta x}$, $\frac{p_i(i,j+1,k)-p_i(i,j,k)}{\Delta y}$, $\frac{r_i(i+1,j,k)-r_i(i,j,k)}{\Delta x}$, $\frac{r_i(i,j+1,k)-r_i(i,j,k)}{\Delta y}$.

for $i = 1 : N_x, j = 1 : N_y$

$p_i(i,j,k) = p_i(i,j,k+1)$
$-f_i(x(i),\ y(j),\ p_1(i,j,k+1),\ p_2(i,j,k+1),\ r_1(i,j,k+1),\ r_2(i,j,k+1),$
$\frac{\partial p_i}{\partial x}(i,j),\ \frac{\partial p_i}{\partial y}(i,j))\Delta t$
$-g_i(x(i),\ y(j),\ \frac{\partial p_i}{\partial x}(i,j),\ \frac{\partial p_i}{\partial y}(i,j))\Delta B(k)$
$r_i(i,j,k) = r_i(i,j,k+1)$
$-f_i(x(i),\ y(j),\ p_1(i,j,k+1),\ p_2(i,j,k+1),\ r_1(i,j,k+1),\ r_2(i,j,k+1),$
$\frac{\partial r_i}{\partial x}(i,j),\ \frac{\partial r_i}{\partial y}(i,j))\Delta t$
$-g_i(x(i),\ y(j),\ \frac{\partial r_i}{\partial x}(i,j),\ \frac{\partial r_i}{\partial y}(i,j))\Delta B(k)$

end

end

Step 5. Use forward Euler-Maruyama method to integrate state equations forward:

for $k = 1 : N$

$$x(k+1) = x(k) + f_1(x(k),\, y(k),\, p_1(x(k), y(k), k),\, p_2(x(k), y(k), k),\, r_1(x(k), y(k), k),\, r_2(x(k), y(k), k))\Delta t + g_1(x(k), y(k))\Delta B(k)$$

$$y(k+1) = y(k) + f_2(x(k),\, y(k),\, p_1(x(k), y(k), k),\, p_2(x(k), y(k), k),\, r_1(x(k), y(k), k), r_2(x(k), y(k), k))\Delta t + g_2(x(k), y(k))\Delta B(k)$$

end

Step 6. Evaluate controls u_1, u_2:

for $k = 1 : N + 1$

$$u_1(k) = \frac{1}{2}\beta_1\sqrt{(k_1 - x(k))y(k)}(p_1(x(k), y(k), k) - p_2(x(k), y(k), k))$$

$$u_2(k) = \frac{1}{2}\beta_2\sqrt{(k_2 - y(k))x(k)}(r_1(x(k), y(k), k) - r_2(x(k), y(k), k))$$

end □

5. NUMERICAL RESULTS

Using *Algorithm 6.1*, we solved our model. In this section, we will discuss some important numerical results based on practical considerations. The result we will get is in terms of strong solution of SDE, which means the solution is based on the path of the underlying Brownian motion. Each time we generate a Brownian motion path, we will get one sample solution path. So, in the following experiment, we run each case three times, from which we can see some common features, which is what we expected.

In **case 1**, the main assumptions about company X and company Y are 1) company X's sale increases more quickly, that is $\alpha_1 > \alpha_2$; 2) Company Y's product is more competitive, that is $\beta_1 < \beta_2$; 3) Company Y emphasizes final market more than company X, that is $\omega_1 < \omega_2$. These assumptions have been reflected in the following data set:

Case 1:

$\omega_1 = 0.15$	$\omega_1 = 0.25$
$k_1 = 0.9$	$k_2 = 0.7$
$c_1 = 1$	$c_2 = 1.5$
$\alpha_1 = 0.02$	$\alpha_2 = 0.01$
$\beta_1 = 1$	$\beta_2 = 1.2$
$x_0 = 0.018$	$y_0 = 0.01$
$\sigma = 10$	

The numerical results are as follows. The expected objective value is $J_1 = 0.0812, J_2 = 0.1476$. The state and control trajectories are in *Figure 3* and *Figure 4*. Regarding the state trajectories we can clearly see from these three sample pathes that company X's sale is decreasing and company Y's sale is increasing. This result comes from a combination of two reasons, one is company Y emphasize more final market share, the other is company Y's product is more competitive. So, many customers change to buy company Y's product. In this case, competition effect dominates natural growth effect. As to control trajectories, both of them use relatively bigger controls at first, then decrease controls. This phenomena have been reflected in the deterministic differential game, which is the competitors in a system always compete furiously (use bigger controls) at first until they eventually find that they can not get more from each other and comprise to some equilibrium.

Objective values:

J_1	J_2
0.08131269347293	0.14770122394696
0.08120381779516	0.14767756426844
0.08116196526100	0.14740749943551

In **case** 2, all the main assumptions about company X and company Y are the same as case 1 except that company X's competitive capability is bigger than that of company Y, that is, $\beta_1 > \beta_2$. So the related data set is as follows:

Case 2:

$\omega_1 = 0.15$	$\omega_1 = 0.25$
$k_1 = 0.9$	$k_2 = 0.7$
$c_1 = 1$	$c_2 = 1.5$
$\alpha_1 = 0.02$	$\alpha_2 = 0.01$
$\beta_1 = 1.2$	$\beta_2 = 1$
$x_0 = 0.018$	$y_0 = 0.01$
$\sigma = 10$	

Numerical results are as follows. The expected objective values are $J_1 = 0.1030$, $J_2 = 0.11894$. The state trajectories are in *Figure 5*. In this case, company X's sale is almost keeping at some constant, and company Y's sale is just increasing a little bit. This result is clearly from the fact that company X's competitive capability has been increased, although company Y wants final market share more. And as to controls, the results are also as expected. Because of competitive capability and less emphasis on final market share, company X's control is approximately decreasing first and then keeping at some constant low level. However, company Y will struggle for his goal, more final market share under the condition that his competitive capability is not as big as company X, so company Y's control is increasing first and then keeping at some higher level.

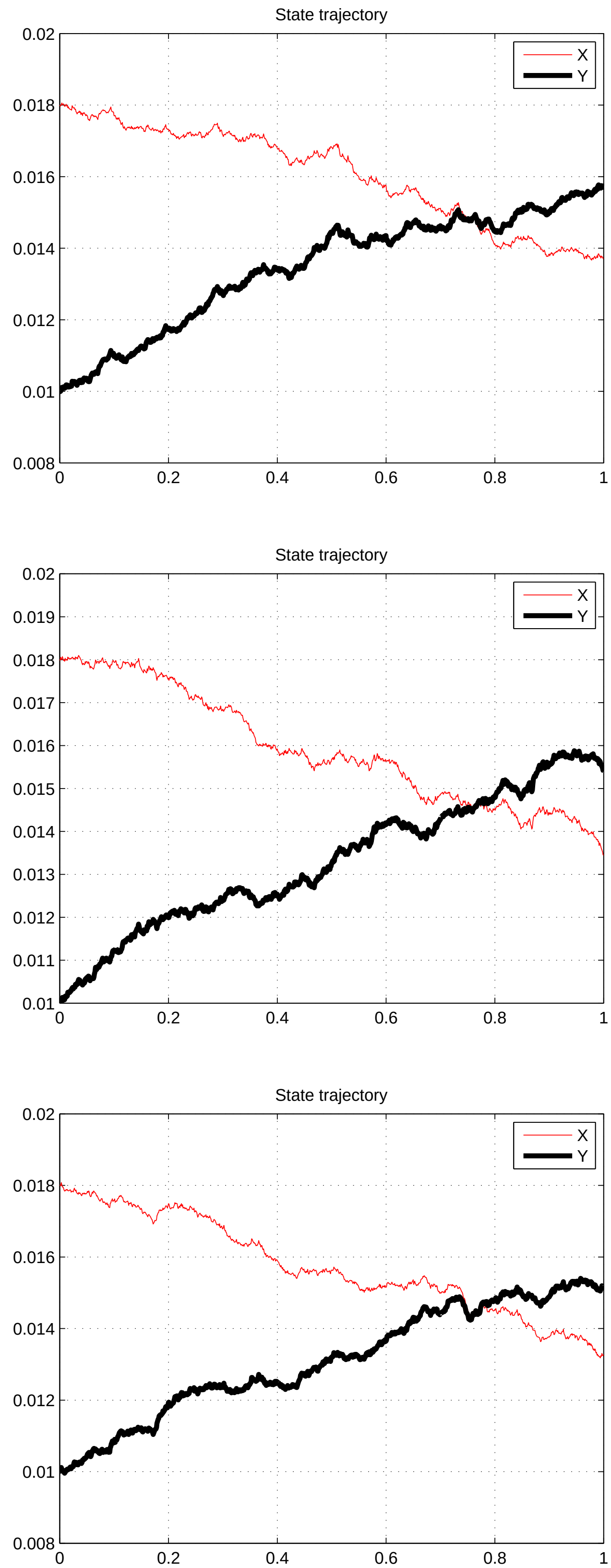

FIGURE 3. State trajectories(Case 1)

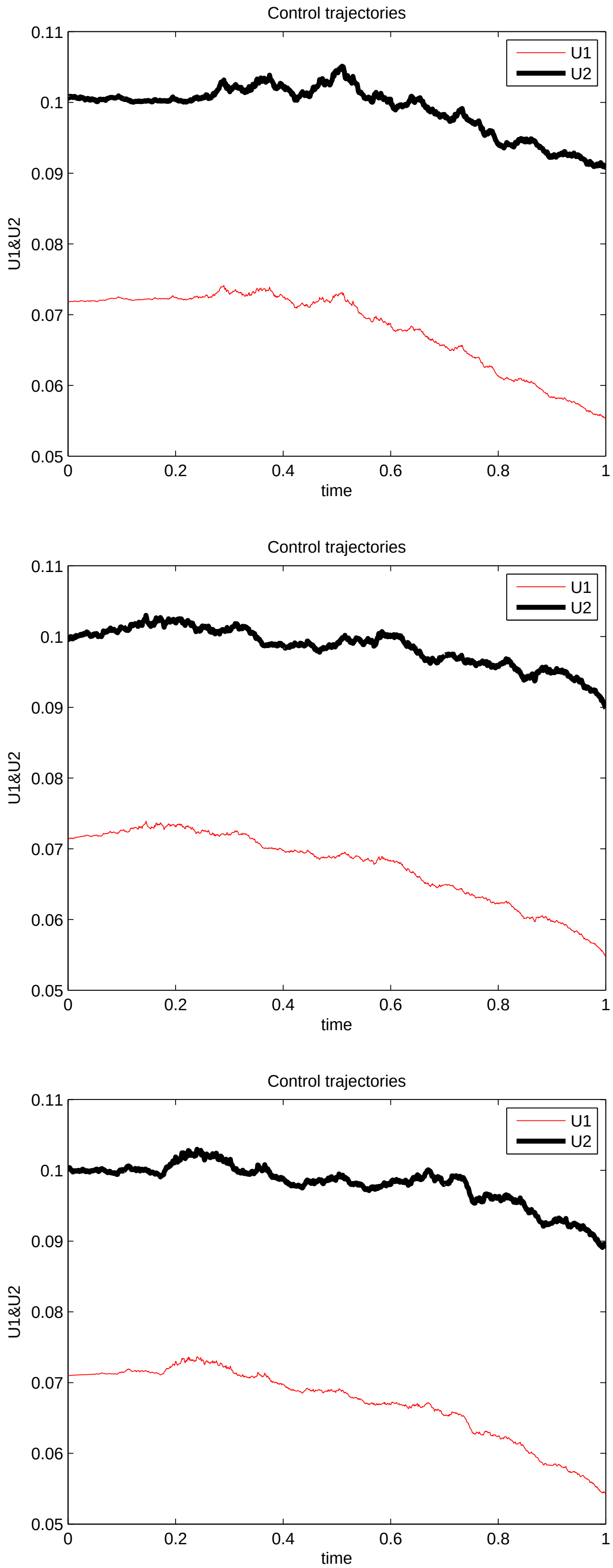

FIGURE 4. Control trajectories(Case 1)

Objective values:

J_1	J_2
0.10241283869068	0.11997014633432
0.10307904939597	0.11892865268987
0.10420939135293	0.11630669702601

In **case** 3, we are interested in the situation that weaker company X wants more final market. Here all main assumptions about company X and company Y are still the same as case 1 except that company X in this case will emphasize final market share more than company Y, that is, $\omega_1 > \omega_2$. So the related data set are as follows:

Case 3:

$\omega_1 = 0.25$ $\quad \omega_1 = 0.15$
$k_1 = 0.9$ $\quad k_2 = 0.7$
$c_1 = 1$ $\quad c_2 = 1.5$
$\alpha_1 = 0.02$ $\quad \alpha_2 = 0.01$
$\beta_1 = 1.2$ $\quad \beta_2 = 1$
$x_0 = 0.018$ $\quad y_0 = 0.01$
$\sigma = 10$

Numerical results are as follows. The expected objective values are $J_1 = 0.1605$, $J_2 = 0.079$. The trajectories for state variables are in *Figure 7* below. Now in this case, company Y's sale approximately increase a little and company X's sale approximately decrease a little as time goes on. And as to controls(*Figure 8* below), company X's control is obviously bigger than that of company Y's, both of them use big control first and decrease to some level. This can be obviously explained by company X's want for more final market share but his competitive capability is not as good as company Y. So he always uses bigger control all the time. However even with bigger control, his sale sometime will decrease a little. On the side of company Y, he has better competitive capability, and does not care much about final market share, so he can just use smaller control to keep his sale on some level.

Objective values:

J_1	J_2
0.16074372560537	0.07883976295665
0.16084549921485	0.08017856767953
0.16027936192200	0.07913855277273

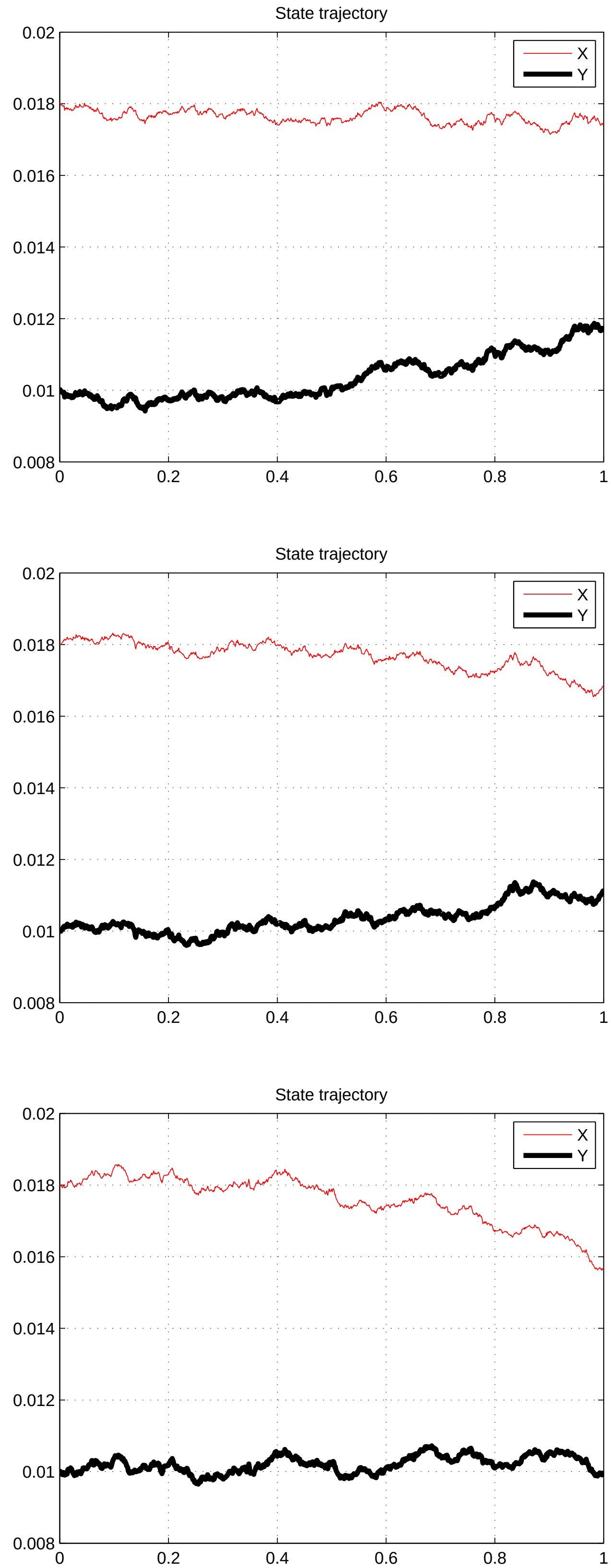

FIGURE 5. State trajectories(Case 2)

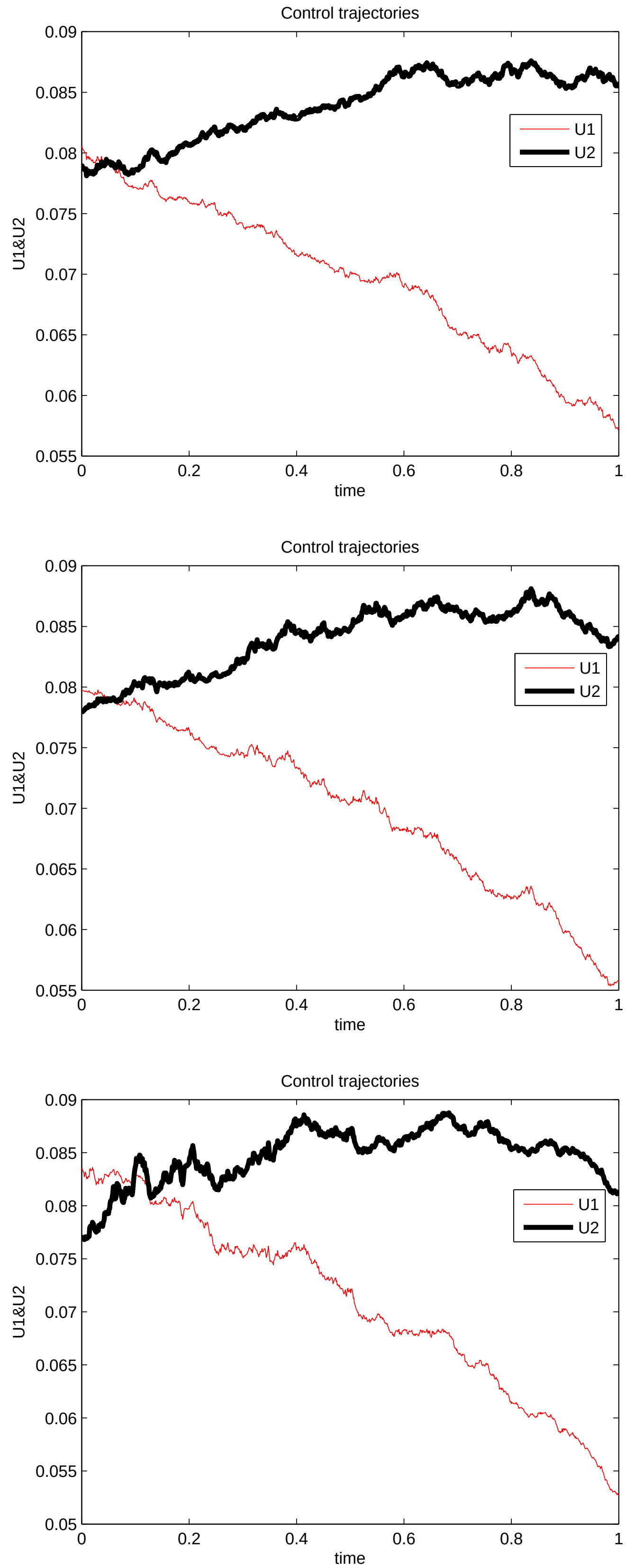

FIGURE 6. Control trajectories(Case 2)

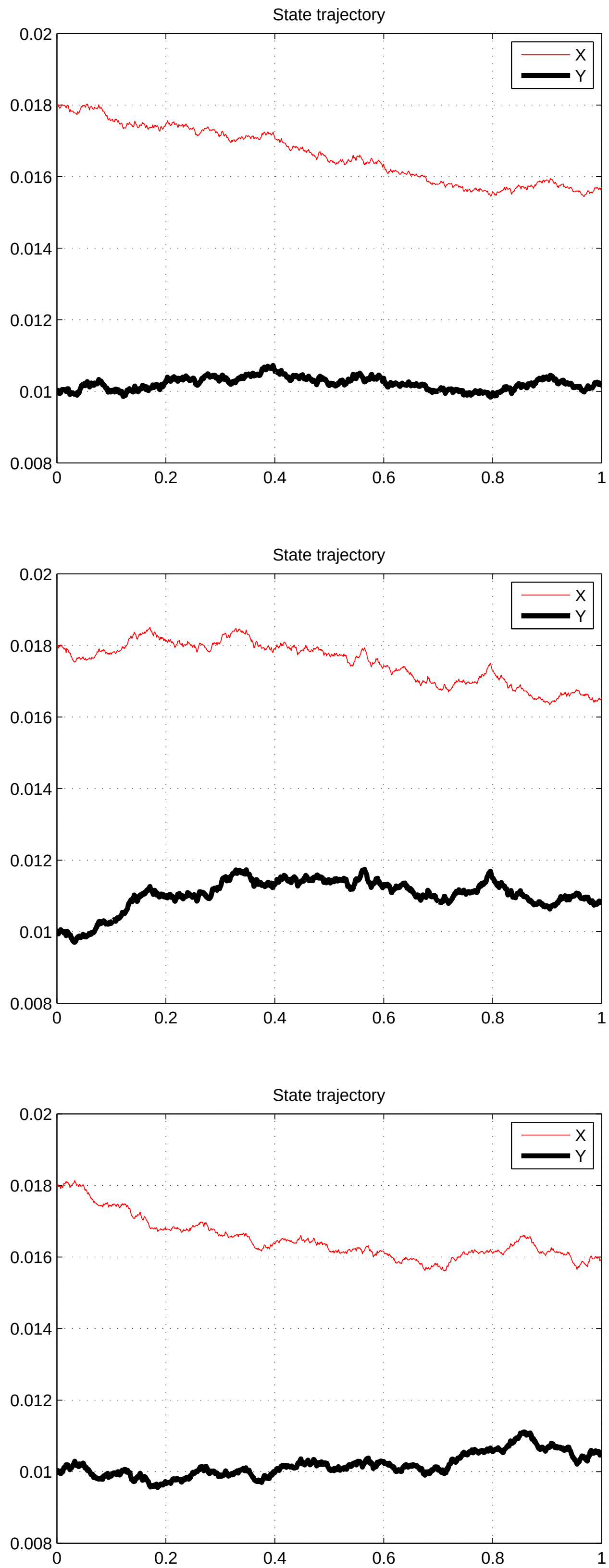

FIGURE 7. State trajectories(Case 3)

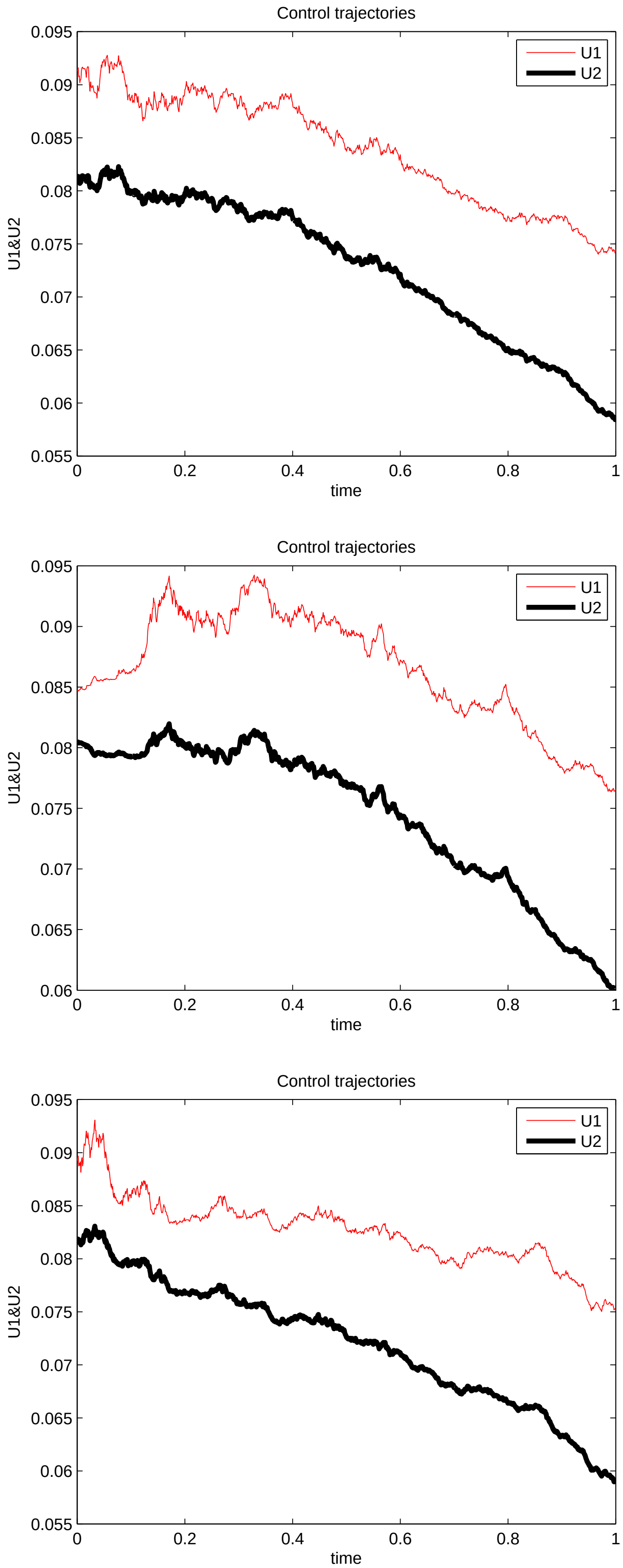

FIGURE 8. Control trajectories(Case 3)

6. CONCLUSION

In this paper, we set up stochastic differential game model to explore competition in a market. We used dynamic programming to derive the necessary optimality conditions for general two-person Stochastic differential game model. The optimality condition consists of a system of Stochastic Partial Differential Equation with separated boundary conditions. Because of the nonlinearity of the equations, analytic solution is hard to get. We appeal to numerical methods to solve it. Specific algorithm is set up to solve the system of optimality conditions. The key technique used in the algorithm is 1) simulating random variables, discretized Brownian path; 2) integrating backwards costate equations on every state planes at each discretized time points, which means we will calculate all values of costate variables at all grid point of the xy-plane. This is necessary to approximate the partial derivative costate variables. And this will induce much more computation than the usual Euler backwards method. 3) Approximating the partial derivatives at each grid point on each xy-plane. 4) Integrating stochastic differential equations of state variables based on Euler-Maruyama method. The numerical solution of our model is in terms of a strong solution of Stochastic Differential Equation, which mean every specific solution trajectories come from specific generated Brownian path.

We use our algorithm to solve our model based on some practical considerations. The numerical results can be explained well from these practical aspects. Comparing stochastic and deterministic differential game from metaphysical level, we have found that 1) competition is always fierce at first and then settle at a lower equilibrium level, and 2) competitor's objective and characteristics will determine the outcome of the competition. So based on these two rules, we can approximately anticipate the state and control trajectories before numerical calculations in both stochastic and deterministic situations.

Our algorithm is not hard to be extended to solve general n-person stochastic differential game models. However before doing that, much research work should be expected in analyzing stability issues under specific situations.

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