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WAVE EQUATIONS THAT ARE RADIALLY SYMMETRIC

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ABSTRACT. We study the periodic semilinear problem for the rotationally invariant wave equation. Our hypotheses are given in terms of the primitive of the nonlinearity.

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1. INTRODUCTION

In this paper we study periodic solutions of the rotationally invariant Dirichlet problem for the semilinear wave equation

$$\Box u - \mu u = p(t, x, u), \quad t \in \mathbb{R}, \quad x \in B_R \tag{1.1}$$

$$u(t, x) = 0, \quad t \in \mathbb{R}, \quad x \in \partial B_R \tag{1.2}$$

$$u(t + T, x) = u(t, x), \quad t \in \mathbb{R}, \quad x \in B_R, \tag{1.3}$$

where

$$\Box u := u_{tt} - \Delta u, \tag{1.4}$$

$$B_R = \{x \in \mathbb{R}^n : |x| < R\}, \tag{1.5}$$

and

$$p(t, x, u) = p(t, |x|, u), \quad x \in B_R.$$

Our basic assumption is that the ratio R/T is rational. Thus, we can write

$$8R/T = a/b, \tag{1.6}$$

where a, b are relatively prime positive integers. We show that

$$n \not\equiv 3 \qquad (\text{mod } (4, a)) \tag{1.7}$$

implies that the linear problem corresponding to (1.1)–(1.3) has no essential spectrum. If

$$n \equiv 3 \qquad (\text{mod } (4, a)), \tag{1.8}$$

then the essential spectrum of the linear operator consists of precisely one point

$$\lambda_0 = -(n - 3)(n - 1)/4R^2. \tag{1.9}$$

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We consider the nonlinear case for $p(t,r,s)$ satisfying

$$|p(t,r,s)| \le C(|s|+1), \quad s \in \mathbb{R},\ r = |x|. \tag{1.10}$$

We assume that the point μ is in the resolvent set of $\Box$ and

$$(m_- - \mu)s^2 - W_1(t,r) \le 2P(t,r,s) \le (m_+ - \mu)s^2 + W_2(t,r), \tag{1.11}$$

where

$$P(t,r,s) = \int_0^s p(t,r,\sigma)\,d\sigma, \tag{1.12}$$

$\mu \in (m_-, m_+)$ is contained in the resolvent set and the functions W_1, W_2 are in $L^1(Q,\rho)$ with $Q = [0,T] \times [0,R]$ and $\rho = r^{n-1}$. We also assume that

$$H(t,r,s) := 2P(t,r,s) - sp(t,r,s) \tag{1.13}$$

satisfies

$$\limsup_{|s|\to\infty} H(t,r,s)/|s| \le h(t,r) < 0. \tag{1.14}$$

Our main theorem is

Theorem 1.1. *If* (1.7) *holds, then* (1.1)–(1.3) *has a weak rotionally invariant solution. If* (1.8) *holds and* $m_- \ge \lambda_0$, *assume that* $p(t,r,s)$ *is nondecreasing in* s. *If* $m_+ \le \lambda_0$, *assume that* $p(t,r,s)$ *is nonincreasing in* s. *Then* (1.1)–(1.3) *has a weak rotationally invariant solution.*

For the definition of essential spectrum, cf., e.g., [12].

Beginning with Smiley [13], several authors have examined the radially symmetric problem (1.1)–(1.3) (cf. [13, 14, 2, 3, 4, 1, 10, 5, 7, 8] and the references cited in them). The complications for this problem depend on the values of R and T. Only in [10] were all possible rational values of R/T considered.

In [13, 2, 1, 8] the hypotheses included inequalities of the form

$$p \le \liminf \frac{f(u)}{u} \le \limsup \frac{f(u)}{u} \le q.$$

In [2] the authors examine radially symmetric solutions to the problem

$$u_{tt} - \Delta u + g(u) = f(t,x),$$

$$u(t+T,\cdot) = u(t,\cdot),$$

where x belongs to a bounded ball B in $\mathbb{R}^n$ with radius R, u satisfies the homogeneous Dirichlet boundary conditions on ∂B, and R/T is rational. The existence of at least one weak solution is proved provided that g is asymptotically linear and the behaviour of $g(u)/u$ for u tending to $\pm\infty$ is suitably related to the eigenvalues of the operator $Lv = v_{tt} - \Delta v, v(t+T,\cdot) = v(t,\cdot)$.

In [4] irrational values of R/T are considered.

In [10] we proved Theorem 1.1 under the assumption

$$|p(t,r,s)| \leq C(|s|^{\theta} + 1), \quad s \in \mathbb{R}, \tag{1.15}$$

holding for some $\theta < 1$. This assumption is a far greater restriction than (1.10) and (1.11).

What distinguishes the present paper from the results of others is that we cover all rational values of R/T, and our hypotheses are given in terms of the primitive

$$P(t,r,s) = \int_0^s p(t,r,\sigma)\, d\sigma, \tag{1.16}$$

of $p(t,r,s)$ rather than the function $p(t,r,s)$ itself.

2. THE SPECTRUM OF THE LINEAR OPERATOR

In dealing with problem (1.1)–(1.3), one needs to calculate the spectrum of the linear operator $\square$ applied to periodic rotationally symmetric functions. Specifically, we shall need the following theorem proved in [10].

Theorem 2.1. *Let L_0 be the operator*

$$L_0 u = u_{tt} - u_{rr} - r^{-1}(n-1)u_r \tag{2.1}$$

applied to functions $u(t,r)$ in $C^{\infty}(\bar{Q})$ satisfying

$$u(T,r) = u(0,r),\ u_t(T,r) = u_t(0,r), \quad 0 \leq r \leq R \tag{2.2}$$

$$u(t,R) = u_r(t,0) = 0, \quad t \in \mathbb{R} \tag{2.3}$$

where $Q = [0,T] \times [0,R]$. Then L_0 is symmetric on $L^2(Q,\rho)$, where $\rho = r^{n-1}$. Assume that $8R/T = a/b$, where a, b are relatively prime integers (i.e., $(a,b) = 1$). Then L_0 has a selfadjoint extension L having no essential spectrum other than the point $\lambda_0 = -(n-3)(n-1)/4R^2$. If $n \not\equiv 3\ (mod(4,a))$, then L has no essential spectrum. If $n \equiv 3\ (mod(4,a))$, then the essential spectrum of L is precisely the point λ_0.

3. THE NONLINEAR CASE

We now turn to the problem solving (1.1)–(1.3). If one is searching for rotationally invariant solutions, the problem reduces to

$$Lu = f(t,r,u), \quad u \in D(L), \tag{3.1}$$

where L is the selfadjoint extension of the operator L_0 given in Theorem 2.1. Under the hypotheses of that theorem the spectrum of L is discrete. We assume that $f(t,r,s)$ is a Carathéodory function on $Q \times \mathbb{R}$ such that

$$|f(t,r,s)| \leq C(|s| + 1), \quad s \in \mathbb{R}. \tag{3.2}$$

We have

Theorem 3.1. *Let $f(t,r,s)$ satisfy* (1.11), (1.14), (3.2), *and assume the hypotheses of Theorem* 2.1. *If*

$$n \not\equiv 3 \ (mod(4,a)) \tag{3.3}$$

make no further assumptions. If

$$n \equiv 3 \ (mod(4,a)) \tag{3.4}$$

and $m_- \geq \lambda_0$, assume in addition that there is a point $\mu \in (m_-, m_+)$ such that

$$p(t,r,s) = f(t,r,s) - \mu s \tag{3.5}$$

is nondecreasing in s. If $m_+ \leq \lambda_0$, assume that there is such a point such that $p(t,r,s)$ is nonincreasing in s. Then (3.1) *has at least one weak solution.*

The following theorem is used in the proof. We believe it is of interest in its own right. It was proved in [10].

Theorem 3.2. *Let N be a closed separable subspace of a Hilbert space E. Let G be a continuously differentiable functional on E such that*

$$v_n = Pu_n \to v \text{ weakly in } E, \quad w_n = (I-P)u_n \to w \text{ strongly in } E$$

implies

$$G'(v_n + w_n) \to G'(v+w) \text{ weakly in } E, \tag{3.6}$$

where P is the projection of E onto N. Assume

$$a_0 := \sup_N G < \infty, \quad b_0 : \inf_M G > -\infty. \tag{3.7}$$

Then there is a sequence $\{u_k\} \subset E$ such that

$$G(u_k) \to c, \ b_0 \leq c \leq a_0, \ G'(u_k) \to 0. \tag{3.8}$$

Proof of Theorem 3.1. Let

$$G(u) = ([L-\mu]u, u) - 2\int\int_Q P(t,r,u)\rho \, dt \, dr, \quad u \in E \tag{3.9}$$

where

$$P(t,r,s) = \int_0^s p(t,r,\sigma)\, d\sigma \tag{3.10}$$

and the scalar product is that of $L^2(Q,\rho)$. One checks readily that G is a C^1 functional on E with

$$(G'(u), v)/2 = ([L-\mu]u, v) - (p(u), v), \quad u, v \in E, \tag{3.11}$$

where we write $p(u)$ in place of $p(t,r,u)$. This shows that u is a weak solution of (3.1) iff $G'(u) = 0$. Let N be the subspace of E spanned by the eigenvectors corresponding to those eigenvalues $< \mu$, and let M denote the subspace of E spanned by the rest. Thus $M = N^{\perp}$ in E. Then

$$\begin{aligned} G(v) &= ([L-\mu]v, v) - 2\int\int_Q P(t,r,v)\rho \, dt\, dr \\ &\leq (m_- - \mu)\|v\|^2 + (\mu - m_-)\|v\|^2 + B_1 = B_1, \quad v \in N, \end{aligned}$$

where

$$B_j = \int\int_Q W_j(t,r)\rho \, dt\, dr.$$

Also,

$$G(w) \geq (m_+ - \mu)\|w\|^2 - (m_+ - \mu)\|w\|^2 - B_2 = -B_2, \quad w \in M. \tag{3.12}$$

If $\{u_k\} \subset E$ is a sequence converging weakly to u in E, then $\{u_k\}$ has a renamed subsequence which converges strongly in $L^2(Q,\rho)$ and a.e. in Q. This follows from the fact that the embedding of E in $L^2(Q,\rho)$ is compact. Now

$$(G'(u_k), v)/2 = (w_k, v)_E - (v_k, v)_E - \mu(u_k, v) - (p(u_k), v), \quad v \in E, \tag{3.13}$$

where $u_k = v_k + w_k,\ v_k \in N,\ w_k \in M$. It follows that $G'(u_k) \to G'(u)$ weakly in E. Hence all of the hypotheses of Theorem 3.2 are satisfied, and we can conclude that there is a sequence $\{u_k\}$ satisfying (3.8). A compactness argument shows that

$$\|u_k\|_E \leq C. \tag{3.14}$$

Consequently, there is a renamed subsequence which converges weakly to u in E, a.e. in Q and strongly in $L^2(Q,\rho)$. Taking the limit in

$$(G'(u_k), v)/2 = ([L-\mu]u_k, v) - (p(u_k), v), \tag{3.15}$$

we obtain a weak solution of (3.1). □

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