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ON A MIN-MAX PRINCIPLE FOR NON-SMOOTH FUNCTIONS AND APPLICATIONS

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ABSTRACT. Extensions of the seminal Ghoussoub's min-max principle [15] to non-smooth functionals given by a locally Lipschitz continuous term plus a convex, proper, lower semi-continuous function are presented and discussed in this survey paper. The problem of weakening the Palais-Smale compactness condition is also treated. Some abstract consequences as well as applications to elliptic hemivariational or variational-hemivariational inequalities are then pointed out.

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1. INTRODUCTION

The critical point theory for C^1 functions in a Banach space X is by now well established and excellent monographs devoted to various aspects of it are already available; see for instance [33, 36, 8]. A trend in today's literature is the attempt to weaken, in a fruitful way, the key assumptions of the famous Mountain Pass Theorem (briefly, MPT) by Ambrosetti-Rabinowitz [33, Theorem 2.2], namely

(a) the Mountain Pass geometry,
(b) the Palais-Smale compactness condition, and
(c) the regularity of the involved functional.

These questions have been widely investigated in latest years. As an example, [16, 35] contain meaningful generalizations of (a)–(b), while [28, 29, 14] mainly deal with (c). The book [18] represents a general reference on the subject.

Starting from the seminal papers by Chang [9] and Szulkin [37], a version of the MPT that applies to functionals $f : X \to \mathbb{R} \cup \{+\infty\}$ fulfilling the structural hypothesis

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(H_f) *$f(x) := \Phi(x) + \psi(x)$ for all $x \in X$, where $\Phi : X \to \mathbb{R}$ is locally Lipschitz continuous while $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, and lower semi-continuous*

has been established by Motreanu-Panagiotopoulos [28, Theorem 3.2]. Critical points of f are defined as solutions to the problem

$$\text{Find } x \in X \text{ such that } \Phi^0(x; z - x) + \psi(z) - \psi(x) \geq 0 \quad \forall z \in X, \tag{1.1}$$

with $\Phi^0(x; z - x)$ being the generalized directional derivative [10, p. 25] of Φ in x along the direction $z - x$. The standard Palais-Smale condition at a given level $a \in \mathbb{R}$ takes here the form:

$(\mathrm{PS})_a$ *Every sequence $\{x_n\} \subseteq X$ such that $\lim_{n\to+\infty} f(x_n) = a$ and*

$$\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geq -\epsilon_n \|z - x_n\| \quad \forall n \in \mathbb{N}, \quad z \in X,$$

where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

When $\Phi \in C^1(X, \mathbb{R})$, problem (1.1) reduces to a variational inequality, and the relevant critical point theory as well as significant applications are developed in [37]. If $\psi \equiv 0$, then (1.1) coincides with the problem treated by Chang [9], who also exploits various abstract results to study elliptic equations having discontinuous nonlinear terms. Finally, when both $\Phi \in C^1(X, \mathbb{R})$ and $\psi \equiv 0$, then problem (1.1) becomes the Euler equation $\Phi'(u) = 0$, and the theory is classical [33, 36].

To the best of our knowledge, Ghoussoub's min-max principle [15, Theorem 1] (cf. also [35]) represents a very fruitful attempt in weakening assumption (a) of the MPT. Besides the existence of critical points, it provides valuable information about their location; see [16, Chapter 5] and [13]. The interest for such a matter stems from both the natural question of whether the critical set contains saddle points (which is suggested by its construction) and the emergence of applications depending on the type of critical point rather than its mere existence. Ghoussoub's result has recently been extended in [20] to Motreanu-Panagiotopoulos' framework [28, Chapter 3] by chiefly adapting the technical approach developed in [15] for C^1 functions and using the structural hypothesis

(H'_f) *f satisfies (H_f). Moreover, ψ is continuous on any nonempty compact set $A \subseteq X$ such that $\sup_{x\in A} \psi(x) < +\infty$.*

Although less general than (H_f), this condition still works in all the most important concrete situations. The structure of min-max generated critical set is treated in Section 4 of [20], and the obtained results extend previous ones on the same subject.

The problem of studying whether the MPT holds true under conditions weaker than (c) is by now widely investigated, and [35] provides an excellent overview on this topic. Very recently, in [21], Theorem 1 of [15] has been extended to locally Lipschitz continuous functions satisfying a weak Palais-Smale hypothesis, which includes both the usual one [9, Definition 2] and the non-smooth Cerami condition [19, p. 248]; cf. [21, Theorem 3.1]. From a technical point of view, Ghoussoub's approach is adapted to the new framework and Ekeland's Variational Principle is exploited with a suitable metric of geodesic type. When the functional turns out to be C^1 while the compactness condition is that of Cerami, this idea basically goes back to Ekeland [12, p. 138].

Finally, Theorem 3.1 in [21] can actually be generalized to functions f fulfilling the structural hypothesis (H'_f); see [25, Theorem 2.1]. For this purpose, suitable versions of two auxiliary lemmas of [24] are first provided. Through them and Ekeland's Variational Principle employed as in [21], Ghoussoub's technique is then adapted to the new setting. We state here only the corresponding critical point result, where an appropriate weak Palais-Smale assumption is taken on.

The present survey collects main results from [20, 21, 25], some abstract consequences [7, 22, 6, 4], and applications to elliptic hemivariational or variational-hemivariational inequality problems [7, 22, 23, 27].

2. PRELIMINARIES

Let $(X, \|\cdot\|)$ be a real Banach space. If U is a subset of X, we write $\bar{U}$ for the closure of U and ∂U for the boundary of U. Moreover, when $x \in X$ and $r > 0$, we define $B(x,r) := \{z \in X : \|z - x\| < r\}$, $B_r := B(0,r)$, as well as

$$d(x,U) := \inf_{z \in U} \|x - z\|, \quad N_r(U) := \{z \in X : d(z,U) < r\}.$$

The symbol X^* indicates the dual space of X, while $\langle \cdot, \cdot \rangle$ stands for the duality pairing between X^* and X. A function $\Phi : X \to \mathbb{R}$ is called coercive provided

$$\lim_{\|x\| \to +\infty} \Phi(x) = +\infty.$$

If to every $x \in X$ there correspond a neighborhood V_x of x and a constant $L_x \geq 0$ such that

$$|\Phi(z) - \Phi(w)| \leq L_x \|z - w\| \quad \forall\, z, w \in X,$$

then we say that f is locally Lipschitz continuous (briefly, l.L.c.). In this case $\Phi^0(x; z)$, $x, z \in X$, indicates the generalized directional derivative of Φ at the point x along the direction z, namely

$$\Phi^0(x; z) := \limsup_{w \to x, t \to 0^+} \frac{\Phi(w + tz) - \Phi(w)}{t}.$$

It is known [10, Proposition 2.1.1] that Φ^0 is upper semi-continuous on $X \times X$. The generalized gradient of the function Φ in x, denoted by $\partial\Phi(x)$, is the set

$$\partial\Phi(x) := \{x^* \in X^* : \langle x^*, z \rangle \leq \Phi^0(x; z) \quad \forall\, z \in X\}.$$

Proposition 2.1.2 of [10] ensures that $\partial\Phi(x)$ is nonempty, convex, in addition to weak* compact, and that

$$\Phi^0(x;z) = \max\{\langle x^*, z\rangle : x^* \in \partial\Phi(x)\}.$$

Hence, it makes sense to put

$$m_\Phi(x) := \min\{\|x^*\|_{X^*} : x^* \in \partial\Phi(x)\}.$$

Now, let $\psi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, proper, and lower semi-continuous. The function ψ is continuous on $\mathrm{int}(D_\psi)$, where, as usual, $D_\psi := \{x \in X : \psi(x) < +\infty\}$. If $\partial\psi(x)$ indicates the sub-differential of ψ at the point $x \in X$, $D_{\partial\psi} := \{x \in X : \partial\psi(x) \neq \emptyset\}$, and X is reflexive then

$$\mathrm{int}(D_\psi) = \mathrm{int}(D_{\partial\psi}).$$

Finally, let f be a function on X complying with hypothesis (H_f). We say that $x \in X$ is a critical point of f when

$$\Phi^0(x; z - x) + \psi(z) - \psi(x) \geq 0 \quad \forall\, z \in X.$$

Given a real number a, write

$$K_a(f) := \{x \in X : f(x) = a,\ x \text{ is a critical point of } f\},$$

besides $f^a := \{x \in X : f(x) \geq a\}$ and $f_a := \{x \in X : f(x) \leq a\}$.

3. CRITICAL POINTS OF NON-SMOOTH FUNCTIONS

In this section we shall extend the results of [15] (cf. also [17]) to functions f satisfying the structural condition (H'_f). A key role is played by the next deformation theorem, which represents a non-differentiable version of [15, Lemma 1].

Lemma 3.1. *Suppose* (H'_f) *is fulfilled,* $\varepsilon > 0$*, while* B, C *are two nonempty closed sets in* X*. If* C *is compact,* $B \cap C = \emptyset$*,* $C \subseteq D_\psi$*, and, moreover,*

(a_1) *to each* $x \in C$ *there corresponds a point* $\xi_x \in X$ *such that*

$$\Phi^0(x; \xi_x - x) + \psi(\xi_x) - \psi(x) < -\varepsilon\|\xi_x - x\|,$$

then for every $k > 1$ *there exist* $t_0 \in (0, 1]$*,* $\alpha \in C^0([0, 1] \times X, X)$*, and* $\varphi \in C^0(X, \mathbb{R}_0^+)$ *with the following properties:*

(i_1) $\alpha(t, D_\psi) \subseteq D_\psi\ \forall\, t \in [0, t_0)$ *and* $\alpha(t, x) = x\ \forall\, (t, x) \in [0, t_0) \times B$*.*
(i_2) $\|\alpha(t, x) - x\| \leq kt\ \forall\, (t, x) \in [0, t_0) \times X$*.*
(i_3) $f(\alpha(t, x)) - f(x) \leq -\varepsilon\varphi(x)t\ \forall\, (t, x) \in [0, t_0) \times D_\psi$*.*
(i_4) $\varphi(x) = 1\ \forall\, x \in C$*.*

For the proof we refer the reader to [20, Theorem 2.2].

Let B be a nonempty closed subset of X and let $\mathcal{F}$ be a class of compact sets in X. According to [15, Definition 1], we say that $\mathcal{F}$ is a *homotopy stable family with extended boundary* B when for every $A \in \mathcal{F}$ and every $\eta \in C^0([0,1] \times X, X)$ such that $\eta(t,x) = x$ in $(\{0\} \times X) \cup ([0,1] \times B)$ one has $\eta(\{1\} \times A) \in \mathcal{F}$.

Some meaningful situations are special cases of this notion. For instance, if Q denotes a compact set in X, Q_0 is a nonempty closed subset of Q, γ_0 belongs to $C^0(Q_0, X)$, $\Gamma := \{\gamma \in C^0(Q,X) : \gamma|_{Q_0} = \gamma_0\}$, and $\mathcal{F} := \{\gamma(Q) : \gamma \in \Gamma\}$, then $\mathcal{F}$ enjoys the above-mentioned property with $B := \gamma_0(Q_0)$. In particular, it holds true when Q denotes a compact topological manifold in X having a nonempty boundary Q_0 while $\gamma_0 = \mathrm{id}|_{Q_0}$.

The following assumptions will be posited in the sequel.

(a_2) *$\mathcal{F}$ is a homotopy-stable family with extended boundary B, the function f fulfils* (H'_f), *and*

$$c := \inf_{A \in \mathcal{F}} \sup_{x \in A} f(x) < +\infty. \tag{3.1}$$

(a_3) *There exists a closed subset F of X such that*

$$(A \cap F) \setminus B \neq \emptyset \quad \forall\, A \in \mathcal{F}, \tag{3.2}$$

while moreover,

$$\sup_{x \in B} f(x) \leq \inf_{x \in F} f(x). \tag{3.3}$$

Thanks to (3.3) one has

$$\inf_{x \in F} f(x) \leq c\,.$$

Theorem 3.2. *Let* (a_2) *and* (a_3) *be satisfied. Then to every sequence $\{A_n\} \subseteq \mathcal{F}$ such that $\lim_{n \to +\infty} \sup_{x \in A_n} f(x) = c$ there corresponds a sequence $\{x_n\} \subseteq X \setminus B$ having the following properties:*

(i_5) $\lim_{n \to +\infty} f(x_n) = c$.
(i_6) $\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geq -\varepsilon_n \|z - x_n\|\ \forall n \in \mathbb{N},\ z \in X$, *where* $\varepsilon_n \to 0^+$.
(i_7) $\lim_{n \to +\infty} d(x_n, F) = 0$ *provided* $\inf_{x \in F} f(x) = c$.
(i_8) $\lim_{n \to +\infty} d(x_n, A_n) = 0$.

This result furnishes a general procedure to construct Palais-Smale sequences. A straightforward, although meaningful, consequence is the next extension of Theorem 1.bis and Corollary 2 in [15]. We shall employ the following weaker form of $(\mathrm{PS})_a$, $a \in \mathbb{R}$, where U denotes a nonempty closed set in X.

$(\mathrm{PS})_{U,a}$ *Every sequence $\{x_n\} \subseteq X$ such that* $\lim_{n \to +\infty} d(x_n, U) = 0$, $\lim_{n \to +\infty} f(x_n) = a$, *and*

$$\Phi^0(x_n; z - x_n) + \psi(z) - \psi(x_n) \geq -\epsilon_n \|z - x_n\| \quad \forall\, n \in \mathbb{N}, \quad z \in X\,,$$

where $\epsilon_n \to 0^+$, possesses a convergent subsequence.

For $U := X$ it evidently coincides with condition $(\mathrm{PS})_a$.

Theorem 3.3. *Let* (a_2) *and* (a_3) *be fulfilled. Suppose that either* $(\mathrm{PS})_c$ *or* $(\mathrm{PS})_{F,c}$ *holds according to whether* $\inf_{x\in F} f(x) < c$ *or* $\inf_{x\in F} f(x) = c$. *Then* $K_c(f) \neq \emptyset$. *If, moreover,* $\inf_{x\in F} f(x) = c$, *the* $K_c(f) \cap F \neq \emptyset$.

Making suitable choices of $\mathcal{F}$, B, and F, more refined versions of several results can be drawn from Theorem 3.3. For instance, the result below includes Theorem 3.1 in [24] and Theorem 3.2 of [28] with (H_f) replaced by (H'_f), Theorems 1 and 2 in [34] (vide also [8, Theorem 7.3.1]), as well as Theorem (1.bis) of [17].

Theorem 3.4. *Let f satisfy the following assumptions, in addition to* (H'_f).

(a_4) $\sup_{x\in Q} f(\gamma(x)) < +\infty$ *for some* $\gamma \in \Gamma$.

(a_5) *There exists a closed subset F of X such that* $(\gamma(Q) \cap F) \setminus \gamma_0(Q_0) \neq \emptyset \ \forall \gamma \in \Gamma$ *and, moreover,* $\sup_{x\in Q_0} f(\gamma_0(x)) \leq \inf_{x\in F} f(x)$.

(a_6) *Setting*

$$c := \inf_{\gamma\in\Gamma} \sup_{x\in Q} f(\gamma(x)), \tag{3.4}$$

either $(\mathrm{PS})_c$ *or* $(\mathrm{PS})_{F,c}$ *is fulfilled, according to whether* $\inf_{x\in F} f(x) < c$ *or* $\inf_{x\in F} f(x) = c$.

Then the conclusion of Theorem 3.3 *holds true.*

We conclude this section by pointing out that the preceding results, when combined with appropriate topological arguments, yield information on the structure of $K_c(f)$. As a sample, let us state here the maybe most evocative theorem, which determines a class of non-smooth functions whose critical sets exhibit saddle points. It contains Theorem 8 of [32], concerning the C^1 framework, as well as a variant for locally Lipschitz continuous functions established in [2].

Recall that a critical point $x \in X$ is called a saddle point for f provided to every $\delta > 0$ there correspond $x', x'' \in B(x, \delta)$ such that $f(x') < f(x) < f(x'')$.

Theorem 3.5. *Suppose X is infinite dimensional, the function f satisfies* (H'_f), *and* $x_0, x_1 \in D_\psi$. *If*

(a_7) *condition* $(\mathrm{PS})_c$ *holds, while the set f^c is closed, and*

(a_8) $\max\{f(x_0), f(x_1)\} < c$,

then $K_c(f)$ possesses a saddle point.

For other structure results and the proofs of Theorems 3.2–3.5 we refer to [20].

4. FURTHER RESULTS

One may evidently ask whether classical critical point theorems can be reformulated when the involved function f fulfils (H'_f). The following versions of the Saddle Point Theorem and of the Generalized MPT are established in [7]; see [7, Theorems 2.2 and 2.3].

Theorem 4.1. *Let $X := V \oplus E$, where $V \neq \{0\}$ is finite dimensional, and let f satisfy (H'_f). Assume there exists an $r > 0$ such that*

(a_9) $\sup_{x\in\bar{B}_r\cap V} \psi(x) < +\infty$,

(a_{10}) $\sup_{x\in\partial B_r\cap V} f(x) \leq \inf_{x\in E} f(x)$, *and*

(a_{11}) *either* $(\mathrm{PS})_c$ *or* $(\mathrm{PS})_{E,c}$ *holds true, according to whether* $\inf_{x\in E} f(x) < c$ *or* $\inf_{x\in E} f(x) = c$, *where c is given by* (3.4) *written for* $Q := \bar{B}_r\cap V$, $Q_0 := \partial B_r\cap V$.

Then $K_c(f) \neq \emptyset$. If, moreover, $\inf_{x\in E} f(x) = c$ then $K_c(f)\cap E \neq \emptyset$.

Theorem 4.2. *Suppose $X := V \oplus E$, where V is finite dimensional, (H'_f) holds, and there are $r > 0$, $e \in \partial B_r \cap V$, $\rho \in]0, r[$ such that*

(a'_4) $\sup_{x\in Q} \psi(x) < +\infty$,

(a'_5) $\sup_{x\in Q_0} f(x) \leq \inf_{x\in F} f(x)$,

and (a_6) *is fulfilled for* $Q := (\bar{B}_r \cap V) \oplus [0, \rho e]$, Q_0 *the boundary of* Q *relative to* $V \oplus \mathrm{span}\{e\}$, $F := \partial B_\rho \cap E$.
Then the conclusion of Theorem 4.1 *remains true.*

These results are fruitfully employed in [7] to get other critical point theorems where no compactness assumption of Palais-Smale type is explicitly adopted, but X is supposed to be reflexive and

$$\Phi(x) := \Phi_1(x) + \Phi_2(x)\,, \quad x \in X\,, \tag{4.1}$$

for appropriate $\Phi_1, \Phi_2 : X \to \mathbb{R}$ locally Lipschitz continuous functions. In particular, Theorem 4.2 produces the following [7, Theorem 3.1]

Theorem 4.3. *Let X, f, Q, Q_0, F be as in Theorem* 4.2 *and let* (a'_4), (a'_5), (4.1) *be satisfied. Assume that*

(a_{12}) *if $\{x_n\} \subseteq D_\psi$, $x_n \rightharpoonup x$ in X, and there exists a $\{\xi_n^*\} \subseteq X^*$ fulfilling*

$$\xi_n^* \in \partial\Phi_1(x_n)\ \forall\, n \in \mathbb{N}\,, \qquad \limsup_{n\to+\infty}\langle \xi_n^*, x_n - x\rangle \leq 0\,,$$

then $\{x_n\}$ has a strongly convergent subsequence,

(a_{13}) $\limsup_{n\to+\infty} \Phi_2^0(x_n; x_n - x) \leq 0$ *provided $\{x_n\} \subseteq D_\psi$ and $x_n \rightharpoonup x$ in X.*

If either D_ψ is bounded or $\inf_{x\in F} f(x) = c$, with c given by (3.4), *then $K_c(f) \neq \emptyset$. Moreover, when $\inf_{x\in F} f(x) = c$ one has $K_c(f) \cap F \neq \emptyset$.*

Through Theorem 4.1 we achieve the next result [7, Theorem 3.4].

Theorem 4.4. *Let X be as in Theorem 4.1 and let (a_9)–(a_{10}) be satisfied for some $r > 0$. If (H'_f), (4.1), (a_{12})–(a_{13}) hold true, while*

(a_{14}) *there exist $c_1,\ c_2 \in \mathbb{R},\ \delta > 0,\ \theta : \mathbb{R}_0^+ \to \mathbb{R}$ such that*

$$\liminf_{t\to+\infty} \frac{\theta(t)}{t} > c_1$$

and to each $x \in D_\psi,\ x = \bar{x} + \tilde{x}$ with $\|\tilde{x}\| > \delta$, there corresponds a $\bar{\zeta}^* \in \partial\psi(\bar{x})$ fulfilling

$$\langle \zeta^* + \bar{\zeta}^*, \tilde{x}\rangle \geq \theta(\|\tilde{x}\|) - c_1\|\tilde{x}\| - c_2 \quad \forall\, \zeta^* \in \partial\Phi(x)\,,$$

(a_{15}) $\lim_{n\to+\infty} |f(x_n)| = +\infty$ *whenever $\{x_n\} \subseteq D_\psi,\ x_n = \bar{x}_n + \tilde{x}_n \in V \oplus E,\ n \in \mathbb{N}$, with $\|\bar{x}_n\| \to +\infty$ and $\{\tilde{x}_n\}$ bounded,*

then the function f has a critical point.

The result below, concerning the merely locally Lipschitz continuous case, is a consequence of Theorem 4.4. It contains [1, Theorem 2.3], where the function Φ_2 is required to be *globally* Lipschitz continuous.

Theorem 4.5. *Let $\psi \equiv 0$ and let Φ be as in* (4.1). *Suppose* (a_{12})–(a_{13}) *are satisfied, in addition to*

(a_{16}) $\Phi_1(x) \leq \beta(\tilde{x})\ \forall\, x \in X,\ x = \bar{x} + \tilde{x}$ *with* $\bar{x} \in V,\ \tilde{x} \in E$,

(a_{17}) *there exists a function $\theta : \mathbb{R}_0^+ \to \mathbb{R}$ fulfilling $\lim_{t\to+\infty} \frac{\theta(t)}{t} = +\infty$ as well as*

$$\langle \xi^*, \tilde{x}\rangle \geq \theta(\|\tilde{x}\|)\ \forall\, \xi^* \in \partial\Phi_1(x),\ x \in X\,,$$

(a_{18}) $\limsup_{\|\tilde{x}\|\to+\infty} \frac{\Phi_2^0(\bar{x}+\tilde{x};-\tilde{x})}{\|\tilde{x}\|} \leq c_3$, *where $c_3 \in \mathbb{R}$, uniformly in $\bar{x} \in V$,*

(a_{19}) *there is a constant $L \geq 0$ such that*

$$|\Phi_2(\bar{x} + \tilde{x}) - \Phi_2(\bar{x}))| \leq L\|\tilde{x}\| \quad \forall\, \bar{x} \in V\,,\ \tilde{x} \in E\,,$$

(a_{20}) $\lim_{\|\bar{x}\|\to+\infty} \Phi_2|_V(\bar{x}) = -\infty$,

(a_{21}) $\lim_{\|\tilde{x}\|\to+\infty} \frac{\Phi_1|_E(\tilde{x})}{\|\tilde{x}\|} = +\infty$.

Then the function f admits a critical point.

Still concerning splitting, Theorem 3.4 represents the main tool to achieve a non-smooth version of the famous Brézis-Nirenberg critical point theorem [5, Theorem 5]. In fact, if X is reflexive and $X = V \oplus E$, with $0 < \dim(V) < +\infty$, $\dim(E) > 0$, then the result below holds; cf. [22, Theorem 3.1]. As usual, the symbol (PS) indicates $(\mathrm{PS})_a$ for all $a \in \mathbb{R}$.

Theorem 4.6. *Assume that*

(f_1) *f turns out to be bounded below and fulfils* (PS) *besides* (H'_f),

(f_2) *$x_0 \in X$ is a global minimum point of the function f,*
(f_3) *the set $X \setminus f^a$ is open for some $a > 0$,*
(f_4) *there is an $r \in]0, \|x_0\|/2[$ such that $f_{|\bar{B}_r \cap E} \geq 0$, $f_{|\bar{B}_r \cap V} < 0$, and $f_{|\partial B_r \cap V} < 0$.*

If, moreover, $\inf_{x \in X} f(x) < f(0)$ and $f(0) = 0$ then the function f possesses at least two nontrivial critical points.

Remark 4.7. Hypothesis (f_4) is obviously satisfied in the meaningful special case when

(f_4') *for some $r > 0$ one has $f_{|\bar{B}_r \cap E} \geq 0$ as well as $f_{|\bar{B}_r \cap V \setminus \{0\}} < 0$,*

namely 0 turns out to be a local minimum of $f|_E$ and a proper local maximum for $f|_V$. If $\dim(V) \geq 2$ then (f_4) may be replaced by the more general condition:

(f_4'') *There is an $r \in]0, \|x_0\|/2[$ such that $f|_{\bar{B}_r \cap E} \geq 0$, $f|_{\bar{B}_r \cap V} \leq 0$, $f|_{\bar{B}_r \cap V \setminus \{0\}} \not\equiv 0$.*

We conclude the section by noting that Ghoussoub's min-max principle can be formulated also for symmetric non-smooth functions. As before, the first step is to construct a suitable deformation; see [6, Theorem 2.2]. This result represents a symmetric variant of Lemma 3.1.

Let B be a nonempty, closed, symmetric subset of X and let $\mathcal{F}$ be a class of compact and symmetric sets in X. We say that $\mathcal{F}$ is a *symmetric homotopy-stable family with extended boundary B* when for every $A \in \mathcal{F}$ and every $\eta \in C^0([0,1] \times X, X)$ such that $\eta(t, \cdot)$ is odd for all $t \in [0,1]$ and $\eta(t,x) = x$ in $(\{0\} \times X) \cup ([0,1] \times B)$ one has $\eta(\{1\} \times A) \in \mathcal{F}$.

The following assumptions will be posited in the sequel.

(a_{22}) *Let $\{\mathcal{F}^\alpha\}$ be a family of symmetric homotopy-stable classes with extended boundaries $\{B^\alpha\}$ and let $\widetilde{\mathcal{F}} := \bigcup_\alpha \mathcal{F}^\alpha$. The function f satisfies (H_f'), Φ and ψ are both even, while*

$$c := \inf_{A \in \widetilde{\mathcal{F}}} \sup_{x \in A} f(x) < +\infty \,.$$

(a_{23}) *There exists a closed symmetric subset F of X such that*

$$(F \cap A) \setminus B^\alpha \neq \emptyset \,, \quad A \in \widetilde{\mathcal{F}} \,,$$

for all α, and

$$\sup_{x \in B^\alpha} f(x) \leq \inf_{x \in F} f(x) \,.$$

From (a_{22})–(a_{23}) it results in

$$\inf_{x \in F} f(x) \leq c \,.$$

Theorem 4.8. *If (a_{22}) and (a_{23}) hold true then to every sequence $\{A_n\} \subseteq \widetilde{\mathcal{F}}$ such that $\lim_{n \to +\infty} \sup_{x \in A_n} f(x) = c$ there corresponds a sequence $\{x_n\} \subseteq X$ having the following properties:*

(i'_5) $\lim_{n\to+\infty} f(x_n) = c$.
(i'_6) $\Phi^0(x_n; z - x_n) + \Psi(z) - \Psi(x_n) \geq -\epsilon_n \|z - x_n\| \; \forall\, n \in \mathbb{N},\ z \in X$, *where* $\epsilon_n \to 0^+$.
(i'_7) $\lim_{n\to+\infty} d(x_n, F) = 0$ *provided* $\inf_{x\in F} f(x) = c$.
(i'_8) $\lim_{n\to+\infty} d(x_n, A_n) = 0$.

For the proof we refer the reader to [6, Theorem 3.1].

This result leads to the next non-differentiable $\mathbb{Z}_2$-symmetric version of the MPT, which extends Corollary 7.22 in [16].

Theorem 4.9. *Let* $X = Y \oplus Z$, *where* $\dim(Y) = k < +\infty$, *and let* f *satisfy* (H'_f), *with* Φ *and* ψ *both even, as well as* (PS). *Assume that* $f(0) = 0$ *in addition to:*

(a_{24}) *For suitable* $\rho > 0$, $\beta \geq 0$ *one has* $\inf_{x\in\partial B_\rho\cap Z} f(x) \geq \beta$.
(a_{25}) *There are* $R > \rho$ *and a subspace* E *of* X *such that* $Y \subseteq E$, $\dim(E) = n > k$, $\sup_{x\in\partial B_R\cap E} f(x) \leq 0$.

Then, there exist critical values c_j $(1 \leq j \leq n-k)$ *for* f *such that*

(i'_9) $0 \leq c_1 \leq \cdots \leq c_{n-k}$, *and*
(i'_{10}) f *possesses at least* $n-k$ *distinct pairs of non-trivial symmetric critical points.*

Moreover, when $c_j = \beta$ *for some* $j \in \{1, 2, \ldots, n-k\}$ *we also have*

$$\gamma(K_\beta(f) \cap \partial B_\rho \cap Z) \geq j\,,$$

with γ *being the* $\mathbb{Z}_2$*-index of Krasnoselski.*

Finally, the result stated below (see [6, Theorem 4.2]) guarantees the existence of an unbounded sequence of critical values.

Theorem 4.10. *Let* $X = Y \oplus Z$, *where* $\dim(Y) = k < +\infty$, *and let* f *fulfil* (H'_f), *with* Φ *and* ψ *both even, as well as* (PS). *Suppose that* $f(0) = 0$, (a_{24}) *of Theorem* 4.9 *holds, and, moreover,*

(a'_{25}) *there exists an increasing sequence* $\{E_n\}$ *of finite-dimensional subspaces of* X *such that* $\lim_{n\to+\infty} \dim(E_n) = +\infty$, $Y \subseteq E_n$, $\sup_{x\in\partial B_{R_n}\cap E_n} f(x) \leq 0$ *for some* $R_n > \rho$ *and all* $n \in \mathbb{N}$.

Then f *has an unbounded sequence of critical values.*

5. WEAKENING THE PALAIS-SMALE CONDITION: THE l.L.c. CASE

Throughout this section, $(X, \|\cdot\|)$ denotes a real Banach space while $f : X \to \mathbb{R}$ is locally Lipschitz continuous, i.e., $\psi \equiv 0$ in (H_f). Let $h : [0, +\infty[\to [0, +\infty[$ be a continuous function enjoying the following property:

$$\int_0^{+\infty} \frac{1}{1 + h(\xi)}\, d\xi = +\infty\,. \tag{5.1}$$

We say that f satisfies a *weak Palais-Smale condition at the level* $c \in \mathbb{R}$ when for some h as above one has:

$(\mathrm{PS})_c^h$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\lim_{n\to+\infty} f(x_n) = c$ *and*

$$\lim_{n\to+\infty}(1 + h(\|x_n\|))m_f(x_n) = 0$$

possesses a convergent subsequence.

Remark 5.1. If $h(\xi) \equiv 0$ then $(\mathrm{PS})_c^h$ reduces to [9, Definition 2]. Setting $h(\xi) := \xi$, $\xi \in [0, +\infty[$, we obtain a non-smooth version of Cerami's compactness assumption [18, Section 13.1].

Given $x, z \in X$, write $\mathcal{P}(x, z)$ for the family of all piecewise C^1 paths $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = z$. Moreover, put

$$l_h(p) := \int_0^1 \frac{\|p'(t)\|}{1 + h(\|p(t)\|)}\, dt\,, \quad p \in \mathcal{P}(x, z)\,,$$

as well as

$$\delta_h(x, z) := \inf\{l_h(p) : p \in \mathcal{P}(x, z)\}\,. \tag{5.2}$$

For $h(\xi) := \xi$, $\xi \in [0, +\infty[$, the function $\delta_h : X \times X \to \mathbb{R}$ defined by (5.2) coincides with the geodesic distance introduced in [12, p. 138]. Exploiting (5.1) and the arguments of [12, p. 138] (cf. besides [11, Section 4], where a more general situation is treated) yields the following basic properties of δ_h.

(p_1) $\delta_h(x, z) \le \|x - z\|$ *for all* $x, z \in X$.

(p_2) *If* U *is a nonempty bounded subset of* X *then there exists a constant* $c_U > 0$ *such that*

$$\delta_h(x, z) \ge c_U\|x - z\| \quad \forall\, x, z \in U\,.$$

(p_3) δ_h *turns out to be a distance on* X *and the metric topology derived from* δ_h *coincides with the norm topology.*

(p_4) δ_h*-bounded and norm-bounded sets in* X *are the same.*

Through (p_1), (p_2), and (p_4) one easily verifies that the metric space (X, δ_h) is complete.

The following assumptions will be posited in the sequel.

(b_1) $\mathcal{F}$ *denotes a homotopy-stable family with extended boundary* B.

(b_2) *There exists a nonempty closed subset* F *of* X *such that* (3.2) *holds and, moreover,*

$$\sup_{x\in B} f(x) \le \inf_{x\in F} f(x)\,. \tag{5.3}$$

(b_3) $h : [0, +\infty[\to [0, +\infty[$ *is a continuous function fulfilling* (5.1)*, while* δ_h *indicates the metric defined by* (5.2).

Set, as usual,

$$c := \inf_{A\in\mathcal{F}} \max_{x\in A} f(x)\,. \tag{5.4}$$

Thanks to (5.3) one has

$$\inf_{x\in F} f(x) \le c\,.$$

Theorem 5.2. *Let* (b_1)–(b_3) *be satisfied. Then to every sequence* $\{A_n\} \subseteq \mathcal{F}$ *such that* $\lim_{n\to+\infty} \max_{x\in A_n} f(x) = c$ *there corresponds a sequence* $\{x_n\} \subseteq X \setminus B$ *having the following properties:*

- (i_1) $\lim_{n\to+\infty} f(x_n) = c$.
- (i_2) $(1 + h(\|x_n\|))f^0(x_n; z) \ge -\epsilon_n\|z\|$ *for all* $n \in \mathbb{N}$, $z \in X$, *where* $\epsilon_n \to 0^+$.
- (i_3) $\lim_{n\to+\infty} \delta_h(x_n, F) = 0$ *provided* $\inf_{x\in F} f(x) = c$.
- (i_4) $\lim_{n\to+\infty} \delta_h(x_n, A_n) = 0$.

For the proof we refer the reader to [21, Theorem 3.1].

The next critical point result is an almost direct but meaningful consequence of Theorem 5.2.

Theorem 5.3. *Suppose* (b_1)–(b_3) *and* $(PS)_c^h$, *with* c *given by* (5.4), *hold true. Then* $K_c(f) \ne \emptyset$. *If, moreover,* $\inf_{x\in F} f(x) = c$ *then* $K_c(f) \cap F \ne \emptyset$.

Proof. Let $\{x_n\} \subseteq X \setminus B$ fulfil (i_1)–(i_4) of Theorem 5.2. Conclusion (i_2) actually means

$$\lim_{n\to+\infty} (1 + h(\|x_n\|))m_f(x_n) = 0\,. \tag{5.5}$$

In fact, due to [37, Lemma 1.3], for any $n \in \mathbb{N}$ there exists a $z_n^* \in X^*$ such that $\|z_n^*\|_{X^*} \le 1$ and

$$\epsilon_n^{-1}(1 + h(\|x_n\|))f^0(x_n; z) \ge \langle z_n^*, z\rangle \quad \forall\, z \in X\,.$$

Hence,

$$\epsilon_n(1 + h(\|x_n\|))^{-1} z_n^* \in \partial f(x_n)\,,$$

which gives

$$(1 + h(\|x_n\|))m_f(x_n) \le \epsilon_n\|z_n^*\|_{X^*} \le \epsilon_n\,, \quad n \in \mathbb{N}\,.$$

Now (5.5) immediately comes out from $\epsilon_n \to 0^+$. Thanks to $(PS)_c^h$ we may thus assume that $x_n \to x$ in X, where a subsequence is considered when necessary. At this point, (i_2) and the upper semi-continuity of f^0 yield $f^0(x; z) \ge 0$ for all $z \in X$, namely $x \in K_c(f)$, because $f(x) = c$ by (i_1). Next, suppose that $\inf_{x\in F} f(x) = c$. On account of (p_3) the set F turns out to be δ_h-closed. So, (i_3) forces $x \in F$, i.e., $K_c(f) \cap F \ne \emptyset$. □

Theorem 5.3 might be employed to get valuable information about the nature of min-max generated critical points. For instance one has [4, Theorems 2.3 and 2.4]

Theorem 5.4. *Let* (b_1), (b_3), *and* $(PS)_c^h$, *with* c *as in* (5.4), *be satisfied. Assume also that:*

(b_4) *The members of* $\mathcal{F}$ *are path-wise connected and contain* B.
(b_5) $\sup_{x\in B} f(x) < c$.

Then $K_c(f)$ *possesses a nonlocal minimum point.*

Theorem 5.5. *Suppose* (b_3) *and* $(PS)_a^h$, $a \in \mathbb{R}$, *hold true. If* $x_0, x_1 \in X$ *are two local minima of* f *then it has at least three critical points.*

Theorem 5.4 extends [16, Corollary 4.14] while Theorem 5.5 represents a non-smooth variant of the famous [31, Corollary 1]. Other structure results can be found in [2].

Combining [3, Theorem 3.1] with Theorem 5.5 yields the multiplicity result below. We shall assume that:

(b_6) X *is reflexive. Moreover, there exists a Banach space* $\tilde{X}$ *such that* X *compactly embeds in* $\tilde{X}$.
(b_7) $\Phi, \Psi : \tilde{X} \to \mathbb{R}$ *turn out to be locally Lipschitz continuous,* $\Phi(0) = \Psi(0) = 0$, *and* $\Phi|_X$ *is coercive.*
(b_8) *For some* $r > 0$, $x_1 \in X$ *one has* $r < \Phi(x_1)$ *as well as*

$$\sup_{z\in\Phi_r} \Psi(z) < r\frac{\Psi(x_1)}{\Phi(x_1)} .$$

To shorten notation, define

$$f_\lambda := \Phi - \lambda\Psi , \quad \lambda \in \Lambda_r , \tag{5.6}$$

where

$$\Lambda_r := \left] \frac{\Phi(x_1)}{\Psi(x_1)}, \frac{r}{\sup_{z\in\Phi_r} \Psi(z)} \right[.$$

(b_9) f_λ *fulfils* $(PS)_a^h$, $a \in \mathbb{R}$, *and is coercive for all* $\lambda \in \Lambda_r$.

Theorem 5.6. *Let* (b_3) *and* (b_6)–(b_9) *be satisfied. Then the function* f_λ *given by* (5.6) *possesses at least three critical points (two local minima and a nonlocal minimum point) provided* $\lambda \in \Lambda_r$.

The proof of this result is analogous to that of [4, Theorem 2.6]. So, we omit it.

6. THE MOTREANU-PANAGIOTOPOULOS CASE

Let $(X, \|\cdot\|)$ be a real reflexive Banach space, let $f : X \to \mathbb{R} \cup \{+\infty\}$ comply with (H'_f), and let $h : [0, +\infty[\to [0, +\infty[$ be continuous and enjoying property (5.1). We say that the function f fulfils a *weak Palais-Smale condition at the level* $c \in \mathbb{R}$ when for some h as above one has:

$(\mathbf{PS})_c^h$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\lim_{n\to+\infty} f(x_n) = c$ *and there exist* $y_n^* \in \partial\Phi(x_n)$, $z_n^* \in \partial\psi(x_n)$, $n \in \mathbb{N}$, *satisfying*

$$\lim_{n\to+\infty}(1 + h(\|x_n\|))\|y_n^* + z_n^*\|_{X^*} = 0$$

possesses a convergent subsequence.

Remark 6.1. For $\psi \equiv 0$, namely in the locally Lipschitz continuous framework, $(\mathbf{PS})_c^h$ reduces to hypothesis $(\mathrm{PS})_c^h$.

Theorem 6.2. *Suppose* (b_1)–(b_3) *of Section* 5 *and* (3.1) *hold true for* f *as above. If, moreover,* f *fulfils* $(\mathbf{PS})_c^h$, *and there exist* $r, \mu > 0$ *such that* $N_r(f^{c+\mu}) \subseteq D_\psi$, *then* $K_c(f) \neq \emptyset$. *When* $inf_{x\in F} f(x) = c$ *one actually has* $K_c(f) \cap F \neq \emptyset$.

For the proof we refer the reader to [25, Theorem 2.2].

Remark 6.3. If $\inf_{x\in F} f(x) = c$ condition $(\mathbf{PS})_c^h$ that appears in Theorem 6.2 can be replaced by the following weaker one, as an elementary argument shows.

$(\mathbf{PS})_{F,c}^h$ *Every sequence* $\{x_n\} \subseteq X$ *such that* $\lim_{n\to+\infty} \delta_h(x_n, F) = 0$, $\lim_{n\to+\infty} f(x_n) = c$, *and there exist* $y_n^* \in \partial\Phi(x_n)$, $z_n^* \in \partial\psi(x_n)$, $n \in \mathbb{N}$, *satisfying*

$$\lim_{n\to+\infty}(1 + h(\|x_n\|))\|y_n^* + z_n^*\|_{X^*} = 0$$

possesses a convergent subsequence.

Remark 6.4. Theorem 6.2 might be exploited to get further information on the critical set $K_c(f)$, as already made in [20, 4] for $h \equiv 0$ or $\psi \equiv 0$, respectively.

7. SOME APPLICATIONS

The results of Sections 3–6 can be used to study elliptic variational-hemivariational inequalities in the sense of Panagiotopoulos [30] besides hemivariational inequality problems.

Let Ω a nonempty bounded domain of the real Euclidean N-space $(\mathbb{R}^N, |\cdot|)$, $N \geq 3$ having a smooth boundary $\partial\Omega$. The symbol $|\Omega|$ stands for the Lebesgue measure of Ω, while $H_0^1(\Omega)$ indicates the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega |\nabla u(x)|^2 dx\right)^{1/2}.$$

Denote by 2^* the critical exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$. Recall that $2^* = \frac{2N}{N-2}$, if $p \in [1, 2^*]$ then there exists a positive constant c_p such that

$$\|u\|_{L^p(\Omega)} \leq c_p\|u\| \tag{7.1}$$

for all $u \in H_0^1(\Omega)$ and, in particular, the embedding is compact whenever $p \in [1, 2^*[$; see [33, Proposition B.7]. Now, let $\{\lambda_n\}$ be the sequence of eigenvalues of the operator

$-\Delta$ in $H_0^1(\Omega)$, with $0 < \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots$, and let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$\|\varphi_n\|^2 = 1 = \lambda_n \|\varphi_n\|^2_{L^2(\Omega)}, \quad n \in \mathbb{N},$$

$$\int_\Omega \nabla\varphi_m(x) \cdot \nabla\varphi_n(x)dx = \int_\Omega \varphi_m(x)\varphi_n(x)dx = 0 \qquad \text{provided } m \neq n.$$

Define

$$J(x,\xi) := \int_0^\xi -j(x,t)dt, \qquad (x,t) \in \Omega \times \mathbb{R}, \tag{7.2}$$

where $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies the conditions

(j_1) *j is measurable with respect to each variable separately,*
(j_2) *there exist $a_1 > 0$, $p \in [1, 2^*]$ such that*

$$|j(x,t)| \le a_1(1 + |t|^{p-1}) \quad \forall\, (x,t) \in \Omega \times \mathbb{R}.$$

Under (j_1)–(j_2), the function J turns out well defined, $J(\cdot,\xi)$ is measurable, while $J(x,\cdot)$ is locally Lipschitz continuous. So it make sense to consider its generalized directional derivative J_x^0 with respect to the variable ξ.

Finally, given $k \in \mathbb{N}$ such that $\lambda_k < \lambda_{k+1}$, consider the orthogonal decomposition $H_0^1(\Omega) = V \oplus E$, where

$$V := \operatorname{span}\{\varphi_1, \varphi_2, \ldots, \varphi_k\}, \qquad E := V^\perp.$$

Using Theorem 4.3 yields the following result [7, Theorem 4.2].

Theorem 7.1. *Let $\lambda \in [\lambda_k, \lambda_{k+1}]$ and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfil* (j_1)–(j_2) *with $p \in\,]2, 2^*[$. If, moreover,*

(j_3) *for some $a_2 > 0$, $q \in\,]2, p]$ one has $J(x,\xi) \le \min\{0, a_2(1 + |\xi|^q)\}$ in $\Omega \times \mathbb{R}$, and*
(j_4) *there exists a $\rho > 0$ such that $\inf_{\tilde{u} \in \partial B_\rho \cap E} \int_\Omega J(x, \tilde{u}(x))dx \ge -\frac{1}{2}\left(1 - \frac{\lambda}{\lambda_{k+1}}\right)$,*

where the function J is given by (7.2), then to any sufficiently large $R > 0$ there corresponds a function $u \in \bar{B}_R \setminus \{0\} \subseteq H_0^1(\Omega)$ satisfying

$$-\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)dx + \lambda \int_\Omega u(x)(v(x) - u(x))dx \le \int_\Omega J_x^0(u(x); v(x) - u(x))dx$$

for all $v \in \bar{B}_R$.

Through Theorem 4.4 we obtain the result [7, Theorem 4.3] below, where P_V indicates the projection of $H_0^1(\Omega)$ onto V.

Theorem 7.2. *Let $\lambda \in [\lambda_k, \lambda_{k+1}[$ and let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy* (j_1)–(j_2) *with $p \in\,]2, 2^*[$. If, moreover,*

(j_3') *for some $a_2 > 0$, $q \in\,]2, p]$ one has $J(x,\xi) \le a_2(1 + |\xi|^q)\}$ in $\Omega \times \mathbb{R}$,* *where J is given by (7.2), then there exists a function $u \in H_0^1(\Omega)$ such that*

$$- \int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)dx + \lambda \int_\Omega u(x)(v(x)-u(x))dx$$

$$\leq \int_\Omega J_x^0(P_V(u)(x); P_V(v-u)(x))dx \quad \forall\, v \in H_0^1(\Omega)\,.$$

The next application (see [7, Theorem 4.4]) stems from Theorem 4.1.

Theorem 7.3. *Suppose B is a nonempty, convex, closed subset of $H_0^1(\Omega)$ such that $(\bar{B}_r \cap V) \oplus E \subseteq B$ for some $r > 0$, while the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfils* (j_1) *and* (j_2) *with $a_1 \in\,]0, \lambda_{k+1} - \lambda_k\,[,\ p = 2$. If, moreover,*

(j_5) *for suitable $a_3 \in\,]0, (\lambda_{k+1} - \lambda_k)/2]$ one has $-a_3\xi^2 \leq J(x,\xi) \leq 0$ in $\Omega \times \mathbb{R}$,*

where J is given by (7.2), then there exists an $\tilde{u} \in B \cap E$ such that

$$-\int_\Omega \nabla \tilde{u}(x) \cdot \nabla(v-\tilde{u})(x)dx + \lambda \int_\Omega \tilde{u}(x)(v(x)-\tilde{u}(x))dx \leq \int_\Omega J_x^0(\tilde{u}(x); v(x)-\tilde{u}(x))dx$$

for all $v \in B$.

Now, let the function $j : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following condition.

(j_6) *j is locally bounded and measurable in $\Omega \times \mathbb{R}$. Moreover,*

$$-\infty < \liminf_{|t|\to+\infty} \frac{j(x,t)}{t} \leq \limsup_{|t|\to+\infty} \frac{j(x,t)}{t} < \lambda_1$$

uniformly in $x \in \Omega$.

Using (j_6) provides constants $\varepsilon \in\,]0, \lambda_1[,\ r > 0$ such that

$$j(x,t) < (\lambda_1 - \varepsilon)t$$

for all $|t| \geq r$ and $x \in \Omega$. Define

$$M := \sup_{(x,t)\in\Omega\times[-r,r]} |j(x,t)|\,.$$

Clearly, $M < +\infty$. As a consequence of Theorem 4.6 we obtain the following [27, Theorem 4.1]

Theorem 7.4. *Let $j : \Omega \times \mathbb{R} \to \mathbb{R}$ fulfil* (j_6) *and let K be a nonempty, convex, closed subset of $H_0^1(\Omega)$ such that $\bar{B}_{r_\kappa} \subseteq K$ for some $\kappa > 0$, where $r_\kappa := \sqrt{\frac{2}{\varepsilon}\lambda_1\left(\kappa + Mr|\Omega|\right)}$. Assume also that*

$$\limsup_{\xi\to 0} \frac{J(x,\xi)}{|\xi|^2} < -\frac{\lambda_k}{2}\,, \qquad \limsup_{|\xi|\to 0} \frac{j(x,\xi)}{\xi} < \lambda_{k+1}$$

uniformly in $x \in \Omega$, with J being as in (7.2). Then there exist $u_1, u_2 \in K \setminus \{0\}$ such that

$$\int_\Omega \nabla u_i(x) \cdot \nabla(v-u_i)(x)dx \leq \int_\Omega J^0(u(x); v(x)-u_i(x))dx\,, \quad i = 1, 2,$$

for all $v \in K$.

Remark 7.5. Write, provided $(x,t) \in \Omega \times \mathbb{R}$,

$$j^-(x,t) := \lim_{\delta\to 0} \inf_{|\xi-t|<\delta} j(x,\xi)\,, \qquad j^+(x,t) := \lim_{\delta\to 0} \sup_{|\xi-t|<\delta} j(x,\xi)\,,$$

and suppose $j^-, j^+ : \Omega \times \mathbb{R} \to \mathbb{R}$ superposition measurable. Choosing $K := H_0^1(\Omega)$, Theorem 7.4 gives at least two nontrivial solutions of the following multi-valued Dirichlet problem:
Find $u \in H_0^1(\Omega)$ such that

$$\begin{cases} -\Delta u \in [j^-(x,u), j^+(x,u)] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Next, let $a \in L^\infty(\Omega)$, let $\lambda_1,\, \lambda_2, \ldots$ the eigenvalues of $-\Delta + a$ in $H_0^1(\Omega)$, and let $\{\varphi_n\}$ be a corresponding sequence of eigenfunctions normalized as follows:

$$\int_\Omega (|\nabla\varphi_n(x)|^2 + a(x)\varphi_n(x)^2)dx = \lambda_n \int_\Omega \varphi_n(x)^2 dx = \lambda_n, \quad n \in \mathbb{N}\,;$$

$$\int_\Omega (\nabla\varphi_m(x) \cdot \nabla\varphi_n(x) + a(x)\varphi_m(x)\varphi_n(x))dx = \int_\Omega \varphi_m(x)\varphi_n(x)dx = 0$$

provided $m \neq n$.
Assume that

$$\lambda_1 < \lambda_2 \le \cdots \le \lambda_s < 0 < \lambda_{s+1} \le \cdots \tag{7.3}$$

and consider the orthogonal decomposition $H_0^1(\Omega) = V \oplus E$, where

$$V := \operatorname{span}\{\lambda_1, \lambda_2, \ldots, \lambda_s\}, \qquad E := V^\perp\,.$$

If $\hat{j} : \mathbb{R} \to \mathbb{R}$ satisfies the conditions

($\hat{\mathrm{j}}_1$) *$\hat{j}$ is measurable,*
($\hat{\mathrm{j}}_2$) *there exist $a_1 > 0$, $p \in]2, 2^*[$ such that* $|\hat{j}(t)| \le a_1(1 + |t|^{p-1})\ \forall\, t \in \mathbb{R}$,

then the function $\hat{J} : \mathbb{R} \to \mathbb{R}$ defined by

$$\hat{J}(\xi) := \int_0^\xi -\hat{j}(t)dt\,, \quad \xi \in \mathbb{R}\,,$$

turns out well defined and locally Lipschitz continuous. So, it make sense to consider its generalized directional derivative $\hat{J}^0$. For our application, we will further suppose that

($\hat{\mathrm{j}}_3$) $\lim_{t\to 0} \frac{\hat{j}(t)}{t} = 0$,
($\hat{\mathrm{j}}_4$) $\limsup_{|t|\to +\infty} \frac{\hat{j}(t)}{t} < 0$,
($\hat{\mathrm{j}}_5$) *there exists a $\xi_0 \in \mathbb{R}$ such that $\hat{J}(\xi_0) < 0$.*

Through $(\hat{j}_4)$ one easily finds two positive constants β, γ satisfying

$$\hat{j}(t) \le -\beta t - \gamma\ \forall\, t \le 0, \qquad \hat{j}(t) \le -\beta t\gamma\ \forall\, t \ge 0.$$

Define, for every let $\lambda, \mu > 0$, $r_{\lambda,\mu} := \lambda\gamma c_1 + \sqrt{(\lambda\gamma c_1)^2 + 2\mu}$, with c_1 given by (7.1). Finally, a set $K_\lambda \subseteq H_0^1(\Omega)$ is called of type ($\mathrm{K}_\lambda^{\hat{j}}$) provided

($K_\lambda^{\hat{j}}$) *K_λ is convex closed in $H_0^1(\Omega)$. Moreover, there is a $\mu > 0$ such that $\bar{B}_{r_{\lambda,\mu}} \subseteq K_\lambda$.*

Given $\lambda > 0$ and K_λ fulfilling ($K_\lambda^{\hat{j}}$), the following problem, say $(\hat{P}_\lambda)$, can be treated via Theorem 4.6:

Find $u \in K_\lambda$ such that

$$-\int_\Omega \nabla u(x) \cdot \nabla(v-u)(x)dx - \int_\Omega a(x)u(x)(v-u)(x)dx \leq \lambda \int_\Omega \hat{J}^0(u(x);(v-u)(x))dx$$

for all $v \in K_\lambda$.

In particular, one has [22, Theorem 4.1]

Theorem 7.6. *Suppose $(\hat{j}_1)-(\hat{j}_5)$ and ($K_\lambda^{\hat{j}}$) hold true. Then, for every λ sufficiently large, problem $(\hat{P}_\lambda)$ possesses at least two nontrivial solutions.*

Finally, Theorem 4.6 provides also the existence of multiple weak solutions to the hemivariational inequality problem, say (P), with p-Laplacian

$$\begin{cases} -\Delta_p u \in \partial J(u) & \text{in } \Omega, \\ \frac{\partial u}{\partial n_p} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2 \leq p < +\infty$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, the potential $J : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous, while $\frac{\partial u}{\partial n_p} = |\nabla u|^{p-2}\nabla u \cdot n$, with $n(x)$ being the outward unit normal vector to $\partial\Omega$ at the point $x \in \partial\Omega$.

Let $p' := p/(p-1)$. A function $u \in W^{1,p}(\Omega)$ is called a weak solution to (P) provided there exists $v \in L^{p'}(\Omega)$ such that $v(x) \in \partial J(u(x))$ almost everywhere in Ω and

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla w(x)dx = \int_\Omega v(x)w(x)dx \quad \forall\, w \in W^{1,p}(\Omega)\,,$$

namely u turns out to be a solution of the hemivariational inequality

$$\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x) \cdot \nabla w(x)dx \leq \int_\Omega J^0(u(x);w(x))dx, \quad w \in W^{1,p}(\Omega)\,.$$

The following result is established in [23]; see [23, Theorem 3.1].

Theorem 7.7. *Assume that:*

(J_1) *J turns out to be locally Lipschitz continuous and $J(0) = 0$.*

(J_2) *There exists a constant $a_1 > 0$ such that for every $\xi \in \mathbb{R}$ one has*

$$|y| \leq a_1(1 + |\xi|^{p-1}) \quad \forall\, y \in \partial J(\xi)\,.$$

(J_3) $\limsup_{|\xi|\to 0} \frac{J(\xi)}{|\xi|^p} = 0$.

(J_4) $\limsup_{|\xi|\to +\infty} \frac{J(\xi)}{|\xi|^p} < 0$.

(J_5) *There are $\mu > p$, $M > 0$ such that $\mu J(\xi) \leq -J^0(\xi; -\xi)$ for any $|\xi| \geq M$.*

(J_6) *There exists a $\rho > 0$ such that $J(\xi) > 0$ provided $0 < |\xi| < \rho$.*

Then (P) possesses at least two nontrivial weak solutions $u_1, u_2 \in W^{1,p}(\Omega)$.

Further existence results concerning problem (P) may be found in [26].

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