Communications in Applied Analysis **13** (2009), no. 3, 341–358

MULTIPLE SOLUTIONS FOR DIRICHLET PROBLEMS WHICH ARE SUPERLINEAR AT $+\infty$ AND (SUB-)LINEAR AT $-\infty$

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ABSTRACT. We consider a semilinear Dirichlet elliptic problem with a right-hand side nonlinearity which exhibits an asymmetric growth near $+\infty$ and near $-\infty$. Namely, it is (sub-)linear near $-\infty$ and superlinear near $+\infty$. However, it need not satisfy the Ambrosetti–Rabinowitz condition on the positive semiaxis. Combining variational methods with Morse theory, we show that the problem has at least two nontrivial solutions, one of which is negative.

AMS (MOS) Subject Classification. 35J25; 35J80; 58E05

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We consider the following semilinear Dirichlet problem:

$$\begin{cases} -\Delta u(z) = f(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

The aim of this paper is to prove a multiplicity theorem for problem (1.1) when the reaction term $f(z, \cdot)$ exhibits an asymmetric behavior as $x \in \mathbb{R}$ approaches $+\infty$ and $-\infty$. More precisely, we assume that for a.a. $z \in \Omega$, $f(z, \cdot)$ grows superlinearly near $+\infty$, while near $-\infty$ it has a (sub-)linear growth. In the past, problems with asymmetric nonlinearities were investigated using the Fučík spectrum of the operator $(-\Delta, H_0^1(\Omega))$. This approach requires that $f(z, \cdot)$ exhibits linear growth near both $+\infty$ and $-\infty$ and that the limits $\lim_{x\to\pm\infty} \frac{f(z,x)}{x}$ exist and belong to $\mathbb{R}$. We mention the works of Các [4], Dancer and Zhang [7], Magalhães [12], de Paiva [16], Schechter

Received January 15, 2009

[20] and the references therein. Equations with nonlinearities which are superlinear in one direction and (sub-)linear in the other were investigated by Arcoya and Villegas [1], de Figueiredo and Ruf [9], Perera [18]. In [1], [18], the nonlinearity $f(z,\cdot)$ is linear near $-\infty$, while in [9] it is sublinear. In [9], [18], it is assumed that $N=1$, i.e., the equation is an ordinary differential equation. In [1], [18], which assume linear growth near $-\infty$, the limit $\lim_{x\to-\infty}\frac{f(z,x)}{x}$ exists. All three works express the superlinear growth near $+\infty$ using the Ambrosetti–Rabinowitz condition (AR-condition, for short). In [1], [9], the nonlinearity $f(z,x)$ is jointly continuous, while in [18], $f\in C^1([0,1]\times\mathbb{R})$. Arcoya and Villegas [1], de Figueiredo and Ruf [9] prove existence theorems, while Perera [18] has a multiplicity theorem. Here, we relax several of the above restrictions on the nonlinearity $f(z,x)$. Our nonlinearity is only measurable in $z\in\Omega$. The limit as $x\to-\infty$ of $\frac{f(z,x)}{x}$ need not exist and the growth near $-\infty$ can be linear or sublinear. Finally, for the superlinear growth near $+\infty$, we do not use the AR-condition. Recall that a function $g:\mathbb{R}\to\mathbb{R}$ is said to satisfy the AR-condition in the positive direction if there exist $\mu>2$ and $M>0$ such that

$$0<\mu G(x)\le g(x)x \quad \text{for all } x\ge M, \tag{1.2}$$

where $G(x)=\int_0^x g(s)\,ds$ (the primitive of g). Integrating (1.2), we obtain the weaker condition

$$c_0x^\mu\le G(x) \quad \text{for all } x\ge M \text{ and some } c_0>0. \tag{1.3}$$

In particular, (1.3) implies that $G(\cdot)$ is superquadratic near $+\infty$ and so it satisfies the much weaker condition

$$\lim_{x\to+\infty}\frac{G(x)}{x^2}=+\infty. \tag{1.4}$$

Here, we use (1.4) with an additional asymptotic condition (see (3.4)), which is weaker than AR-condition.

Our approach combines variational methods based on the critical point theory, together with Morse theory. In the next section, for the convenience of the reader, we present the main mathematical tools that we will use in the sequel. In Section 3, we establish the compactness property of the Euler functional for the problem (1.1) and we study its critical groups at $+\infty$ and at 0. Section 4 presents our multiplicity result.

2. MATHEMATICAL BACKGROUND

Let X be a Banach space and let X^* be its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Let $\varphi\in C^1(X)$. We say that φ satisfies the Cerami condition (the (C)-condition, for short) if every sequence $\{x_n\}_{n\ge1}\subset X$ such that $\{\varphi(x_n)\}_{n\ge1}$ is bounded and $(1+\|x_n\|)\varphi'(x_n)\to0$ in X^* as $n\to\infty$ admits a strongly convergent subsequence.

We introduce the following sets: $\varphi^c = \{x \in X : \varphi(x) \le c\}$, $\dot{\varphi}^c = \{x \in X : \varphi(x) < c\}$ $(c \in \mathbb{R})$ and $K = \{x \in X : \varphi'(x) = 0\}$. Given a topological pair (Y_1, Y_2) with $Y_2 \subset Y_1 \subset X$, for every integer $k \ge 0$, by $H_k(Y_1, Y_2)$ we denote the kth relative singular homology group for the pair (Y_1, Y_2) with integer coefficients. If $x_0 \in X$ is an isolated critical point of φ and $c = \varphi(x_0)$, then the critical groups of φ at x_0 are defined by

$$C_k(\varphi, x_0) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{x_0\}) \ \text{ for all integers } k \ge 0,$$

where U is a neighborhood of x_0 such that $K \cap \varphi^c \cap U = \{x_0\}$ (see [5], [13]). The excision property of singular homology implies that the definition of critical groups is independent of the particular choice of the neighborhood U.

Suppose that $\varphi \in C^1(X)$ satisfies the (C)-condition and $\inf \varphi(K) > -\infty$. We choose $c < \inf \varphi(K)$. The critical groups of φ at infinity are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \ \text{ for all integers } k \ge 0$$

(see, e.g., Bartsch and Li [2]). From the deformation theorem, we see that the above definition is independent of the particular choice of $c < \inf \varphi(K)$. If $c < \inf \varphi(K)$ then

$$C_k(\varphi, \infty) = H_k(X, \dot{\varphi}^c) \ \text{ for all integers } k \ge 0. \tag{2.1}$$

To see this, let $b < c < \inf \varphi(K)$. Then φ^b is a strong deformation retract of $\dot{\varphi}^c$ (see, e.g., [11]), hence $H_k(X, \varphi^b) = H_k(X, \dot{\varphi}^c)$ for all integers $k \ge 0$, which leads to (2.1).

We recall the following result from Perera [19, Lemma 2.2].

Proposition 2.1. *If $D_1 \subset D \subset E \subset E_1 \subset X$ and for some integer $k \ge 0$ we have $H_k(E, D) \ne 0$ and $H_k(E_1, D_1) = 0$, then either $H_{k+1}(E_1, E) \ne 0$ or $H_{k-1}(D, D_1) \ne 0$.*

In the analysis of problem (1.1), we will use the spaces $H_0^1(\Omega)$ endowed with the norm $\|\cdot\|$, and

$$C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : \ u|_{\partial\Omega} = 0\}.$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive cone

$$C_+ = \{u \in C_0^1(\overline{\Omega}) : \ u(z) \ge 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ : \ u(z) > 0 \text{ for } z \in \Omega \text{ and } \frac{\partial u}{\partial n}(z) < 0 \text{ for } z \in \partial\Omega \right\}.$$

Here $n(\cdot)$ stands for the outward unit normal on $\partial\Omega$.

Recall that the negative Dirichlet Laplacian (denoted by $(-\Delta, H_0^1(\Omega))$) is the operator $-\Delta \in \mathcal{L}(H_0^1(\Omega), H^{-1}(\Omega))$, where $H^{-1}(\Omega) = H_0^1(\Omega)^*$, defined by

$$\langle -\Delta u, y \rangle = \int_\Omega (Du, Dy)_{\mathbb{R}^N} \, dz \ \text{ for all } u, y \in H_0^1(\Omega)$$

(by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(H^{-1}(\Omega), H_0^1(\Omega))$). Let us briefly recall the spectral properties of $(-\Delta, H_0^1(\Omega))$. We consider the following linear eigenvalue problem:

$$\begin{cases} -\Delta u(z) = \lambda u(z) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

We know that problem (2.2) admits a sequence $\{\lambda_k\}_{k\geq 1} \subset \mathbb{R}_+$ of distinct eigenvalues satisfying $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} \to +\infty$ as $k \to +\infty$. For every integer $k \geq 1$, by $E(\lambda_k)$ we denote the eigenspace corresponding to the eigenvalue λ_k. We know that for every integer $k \geq 1$, $E(\lambda_k)$ is finite dimensional, $E(\lambda_k) \subset C_0^1(\overline{\Omega})$, and it has the unique continuation property, which means that if $u \in E(\lambda_k)$ vanishes on a set of positive measure, then $u \equiv 0$ on $\overline{\Omega}$. The space $E(\lambda_1)$ is one-dimensional. Moreover, only the eigenfunctions corresponding to the first eigenvalue λ_1 have constant sign. The eigenfunctions corresponding to the eigenvalues λ_k, $k \geq 2$, are nodal (sign-changing).

Let $k \geq 1$ be an integer and consider the following two spaces

$$\overline{H}_k = \bigoplus_{i=1}^{k} E(\lambda_i) \quad \text{and} \quad \hat{H}_k = \overline{H}_k^{\perp} = \overline{\bigoplus_{i\geq k+1} E(\lambda_i)}\,.$$

Using these spaces, we have the following variational characterizations of the eigenvalues of $(-\Delta, H_0^1(\Omega))$:

$$\lambda_1 = \min\left\{\frac{\|Du\|_2^2}{\|u\|_2^2} \;:\; u \in H_0^1(\Omega) \setminus \{0\}\right\}, \tag{2.3}$$

and for $k \geq 2$,

$$\lambda_k = \max\left\{\frac{\|D\overline{u}\|_2^2}{\|\overline{u}\|_2^2} \;:\; \overline{u} \in \overline{H}_k \setminus \{0\}\right\} = \min\left\{\frac{\|D\hat{u}\|_2^2}{\|\hat{u}\|_2^2} \;:\; \hat{u} \in \hat{H}_{k-1} \setminus \{0\}\right\}. \tag{2.4}$$

The minimum in (2.3) is attained on the eigenspace $E(\lambda_1) = \mathbb{R}u_1$, where $u_1 \in \operatorname{int} C_+$ denotes the L^2-normalized principal eigenfunction of $(-\Delta, H_0^1(\Omega))$. The maximum and the minimum in (2.4) are realized on $E(\lambda_k)$, $k \geq 2$.

We shall need the following lemma from [15].

Lemma 2.2. *If $k \geq 0$ is an integer, $\xi \in L^\infty(\Omega)_+$, $\xi(z) \leq \lambda_{k+1}$ a.e. on Ω and the inequality is strict on a set of positive measure, then there exists $\hat{\gamma}_0 > 0$ such that*

$$\|D\hat{u}\|_2^2 - \int_\Omega \xi \hat{u}^2\,dz \geq \hat{\gamma}_0 \|D\hat{u}\|_2^2 \quad \textit{for all } \hat{u} \in \hat{H}_k\,,$$

with $\hat{H}_0 := H_0^1(\Omega)$.

In what follows, 2^* denotes the critical Sobolev exponent defined by

$$2^* = \begin{cases} \frac{2N}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2\,. \end{cases}$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, and for every $x \in \mathbb{R}$ we use the notation $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$.

3. COMPACTNESS CONDITION. CRITICAL GROUPS

First, let us state the hypotheses on the nonlinearity $f(z, x)$:

(H) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $z \mapsto f(z, x)$ is measurable;
(ii) for a.a. $z \in \Omega$, $x \mapsto f(z, x)$ is C^1 and $f(z, 0) = 0$;
(iii) for a.a. $z \in \Omega$ and all $x \in \mathbb{R}$, we have

$$|f'_x(z, x)| \leq a(z) + c|x|^{r-2},$$

with $a \in L^\infty(\Omega)_+$, $c > 0$ and $2 < r < 2^*$;
(iv) there exist $\vartheta \in L^\infty(\Omega)_+$, $\vartheta(z) \leq \lambda_1$ a.e. on Ω with strict inequality on a set of positive measure, $\sigma, \beta \in \mathbb{R}$, $\xi > 0$, and $\tau \in ((r-2)\max\{1, \frac{N}{2}\}, r]$ with $\tau \geq 1$ such that for $F(z, x) = \int_0^x f(z, s)\, ds$, we have

$$\sigma \leq \liminf_{x \to -\infty} \frac{f(z, x)}{x} \leq \limsup_{x \to -\infty} \frac{f(z, x)}{x} \leq \vartheta(z) \text{ uniformly for a.a. } z \in \Omega, \tag{3.1}$$

$$\limsup_{x \to -\infty} (2F(z, x) - f(z, x)x) \leq \beta \text{ uniformly for a.a. } z \in \Omega, \tag{3.2}$$

$$\lim_{x \to +\infty} \frac{F(z, x)}{x^2} = +\infty \text{ uniformly for a.a. } z \in \Omega, \tag{3.3}$$

$$\liminf_{x \to +\infty} \frac{f(z, x)x - 2F(z, x)}{x^\tau} \geq \xi \text{ uniformly for a.a. } z \in \Omega; \tag{3.4}$$

(v) there exist $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ and an integer $m \geq 2$ such that

$$\lambda_m \leq \eta(z) \leq \hat{\eta}(z) \leq \lambda_{m+1} \text{ for a.a. } z \in \Omega,$$

the first and the last inequality are strict on sets (not necessarily the same) of positive measure, and

$$\eta(z) \leq f'_x(z, 0) = \lim_{x \to 0} \frac{f(z, x)}{x} \leq \hat{\eta}(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark 3.1. From condition (3.1), it is clear that for a.a. $z \in \Omega$, $f(z, \cdot)$ is (sub-)linear near $-\infty$, while condition (3.3) implies that for a.a. $z \in \Omega$, $f(z, \cdot)$ is superlinear near $+\infty$. However, we do not assume the AR-condition (see (1.2)). Instead, we use (3.4), which is a weaker condition. Such conditions were also used by Costa and Magalhães [6] and Fei [8].

Example 3.2. The following function satisfies hypotheses (H) (for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} -4\eta x^{-2} - 3\eta x^{-3} & \text{if } x < -1 \\ \eta x & \text{if } x \in [-1,1] \\ \frac{2\eta}{3}\left(x(\ln x + \frac{1}{2}) + 1\right) & \text{if } x > 1, \end{cases}$$

with $\eta > \lambda_1$, $\eta \notin \sigma(-\Delta, H_0^1(\Omega))$.

Example 3.3. The following function satisfies hypotheses (H) (for the sake of simplicity, we drop the z-dependence):

$$f(x) = \begin{cases} \theta(x + \ln|x|) + \theta - \frac{\eta}{2} & \text{if } x < -1 \\ \frac{\eta}{2} x^2 + \eta x & \text{if } x \in [-1,1] \\ \frac{\eta}{2} x(\ln x + 3) & \text{if } x > 1, \end{cases}$$

with $0 < \theta < \lambda_1 < \eta$, $\eta \notin \sigma(-\Delta, H_0^1(\Omega))$.

Let $\varphi : H_0^1(\Omega) \to \mathbb{R}$ be the Euler functional for problem (1.1), defined by

$$\varphi(u) = \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z, u(z))\, dz \quad \text{for all } u \in H_0^1(\Omega).$$

Hypotheses (H) imply that $\varphi \in C^2(H_0^1(\Omega))$.

Proposition 3.4. *If hypotheses* (H) *hold, then* φ *satisfies the* (C)*-condition.*

Proof. We consider a sequence $\{u_n\}_{n\geq 1} \subset H_0^1(\Omega)$ such that

$$|\varphi(u_n)| \leq M_1 \quad \text{for all } n \geq 1, \tag{3.5}$$

for some $M_1 > 0$, and

$$(1 + \|u_n\|)\varphi'(u_n) \to 0 \ \text{ in } H^{-1}(\Omega) \text{ as } n \to \infty. \tag{3.6}$$

Note that $\varphi'(u) = -\Delta u - N(u)$, with $N(u)(\cdot) = f(\cdot, u(\cdot))$ for all $u \in H_0^1(\Omega)$. From (3.6), we have

$$\left| \langle -\Delta u_n, y\rangle - \int_\Omega f(z, u_n) y\, dz \right| \leq \frac{\varepsilon_n}{1 + \|u_n\|}\|y\| \ \text{ for all } y \in H_0^1(\Omega), \tag{3.7}$$

with $\varepsilon_n \downarrow 0$.

Suppose that $\|u_n^-\| \to \infty$ as $n \to \infty$. We set $v_n = \frac{u_n^-}{\|u_n^-\|}$, $n \geq 1$. Then $\|v_n\| = 1$ for all $n \geq 1$, and so we may assume that

$$v_n \xrightarrow{\text{w}} v \text{ in } H_0^1(\Omega) \ \text{ and } \ v_n \to v \text{ in } L^r(\Omega) \text{ as } n \to \infty. \tag{3.8}$$

Choosing $y = -u_n^- \in H_0^1(\Omega)$ in (3.7), we see that

$$\|Du_n^-\|_2^2 - \int_\Omega f(z, u_n)(-u_n^-)\, dz \leq \varepsilon_n \ \text{ for all } n \geq 1. \tag{3.9}$$

Multiplying (3.9) with $\frac{1}{\|u_n^-\|^2}$, we obtain

$$\|Dv_n\|_2^2 - \int_\Omega \frac{f(z,-u_n^-)}{\|u_n^-\|}(-v_n)\,dz \le \frac{\varepsilon_n}{\|u_n^-\|^2} \quad \text{for all } n \ge 1. \tag{3.10}$$

Note that $-u_n^-(z) \to -\infty$ for a.a. $z \in \{v > 0\}$. Then using hypothesis (3.1) in (H) (iv) as well as (H) (v) (ensuring that $|f(z,x)| \le \tilde{c}|x|$ for a.a. $z \in \Omega$, all $x \le 0$, for some $\tilde{c} > 0$), and reasoning as in the proof of Proposition 5 in [14], we can show that

$$\frac{N(-u_n^-)}{\|u_n^-\|} \xrightarrow{\text{w}} -gv \quad \text{in } L^{r'}(\Omega) \text{ as } n \to \infty, \tag{3.11}$$

for $\frac{1}{r} + \frac{1}{r'} = 1$, with $g \in L^\infty(\Omega)_+$, $g(z) \le \vartheta(z)$ a.a. on Ω. Hence if in (3.10) we pass to the limit as $n \to \infty$ and we use (3.8), (3.11), we obtain

$$\|Dv\|_2^2 \le \int_\Omega gv^2\,dz \le \lambda_1 \|v\|_2^2 \tag{3.12}$$

(see hypothesis (H) (iv)). From the variational characterization of $\lambda_1 > 0$ (see (2.3)), we infer that $v = tu_1$, for some $t > 0$, or $v = 0$.

If $v = tu_1$, for some $t > 0$, then from the first inequality in (3.12), since $g \le \vartheta$ and using the hypothesis on ϑ (see (H) (iv)) and the fact that $v(z) > 0$ for all $z \in \Omega$, we obtain $\|Du_1\|_2^2 < \lambda_1 \|u_1\|_2^2$, which is a contradiction.

Suppose $v = 0$. Choosing $y = v_n \in H_0^1(\Omega)$ in (3.7) and multiplying the equation with $\frac{1}{\|u_n^-\|}$, we have

$$\left| \|Dv_n\|_2^2 - \int_\Omega \frac{f(z,-u_n^-)}{\|u_n^-\|} v_n\,dz \right| \le \varepsilon_n', \tag{3.13}$$

with $\varepsilon_n' \downarrow 0$. By virtue of hypothesis (H) (iii) and (3.8), we have

$$\int_\Omega \frac{f(z,-u_n^-)}{\|u_n^-\|} v_n\,dz \to 0 \quad \text{as } n \to \infty,$$

and thus, due to (3.8) and (3.13), we get $\|Dv_n\|_2^2 \to 0$, which contradicts that $\|v_n\| = 1$. This proves that

$$\{u_n^-\}_{n \ge 1} \text{ is bounded in } H_0^1(\Omega). \tag{3.14}$$

Next we choose $y = u_n^+ \in H_0^1(\Omega)$ in (3.7), and then we have

$$-\|Du_n^+\|_2^2 + \int_\Omega f(z,u_n^+)\,u_n^+\,dz \le \varepsilon_n \quad \text{for all } n \ge 1. \tag{3.15}$$

In addition, from (3.5) and (3.14), we deduce that

$$\|Du_n^+\|_2^2 - \int_\Omega 2F(z,u_n^+)\,dz \le M_2 \quad \text{for all } n \ge 1, \tag{3.16}$$

for some $M_2 > 0$. Adding (3.15) and (3.16), we obtain

$$\int_\Omega (f(z,u_n^+)u_n^+ - 2F(z,u_n^+))\,dz \le M_3 \quad \text{for all } n \ge 1, \tag{3.17}$$

for some $M_3 > 0$. By virtue of (3.4) in hypothesis (H) (iv), we can find $\xi_0 > 0$ and $M_4 > 0$ such that

$$0 < \xi_0 x^\tau \leq f(z,x)x - 2F(z,x) \ \text{ for a.a. } z \in \Omega \text{ and all } x \geq M_4. \tag{3.18}$$

On the other hand, hypothesis (H) (iii) implies that

$$|f(z,x^+)x^+ - 2F(z,x^+)| \leq M_5 \ \text{ for a.a. } z \in \Omega \text{ and all } x < M_4, \tag{3.19}$$

for some $M_5 > 0$. From (3.18) and (3.19), it follows that

$$\xi_0 (x^+)^\tau - M_6 \leq f(z,x^+)x^+ - 2F(z,x^+) \ \text{ for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}, \tag{3.20}$$

with $M_6 > 0$. Using (3.20) in (3.17), we obtain

$$\{u_n^+\}_{n\geq 1} \text{ is bounded in } L^\tau(\Omega). \tag{3.21}$$

From hypothesis (H) (iii), we have

$$|f(z,x^+)x^+| \leq a(z)x^+ + \frac{c}{r-1}\,(x^+)^r \ \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.22}$$

Choosing $y = u_n^+ \in H_0^1(\Omega)$ in (3.7), we see that

$$\|Du_n^+\|_2^2 - \int_\Omega f(z,u_n^+)u_n^+\,dz \leq \varepsilon_n\,,$$

which combined with (3.22) yields

$$\|Du_n^+\|_2^2 \leq \varepsilon_n + c_1(\|u_n^+\|_1 + \|u_n^+\|_r^r) \ \text{ for all } n \geq 1, \tag{3.23}$$

for some $c_1 > 0$.

Assume for the moment that $N > 2$. Since $\tau \leq r < 2^*$ (see (H) (iv)), there is a unique $t \in [0,1)$ such that $\frac{1}{r} = \frac{1-t}{\tau} + \frac{t}{2^*}$. Invoking the interpolation inequality (see, e.g., [10, p. 905]), we have $\|u_n^+\|_r \leq \|u_n^+\|_\tau^{1-t}\,\|u_n^+\|_{2^*}^t$ for all $n \geq 1$, and using (3.21) we get

$$\|u_n^+\|_r^r \leq M_7 \|Du_n^+\|_2^{tr} \ \text{ for all } n \geq 1, \tag{3.24}$$

for some $M_7 > 0$. This also holds for $N \leq 2$ by applying the interpolation inequality with $\tau \leq r < q$, for $q > \frac{2\tau}{\tau+2-r}$. Using (3.24) in (3.23), we obtain

$$\|Du_n^+\|_2^2 \leq \varepsilon_n + c_2(\|Du_n^+\|_2 + \|Du_n^+\|_2^{tr}) \ \text{ for all } n \geq 1, \tag{3.25}$$

for some $c_2 > 0$. Note that the condition $\tau > (r-2)\max\{1, \frac{N}{2}\}$ in (H) (iv) guarantees that $tr < 2$. This and (3.25) yield that

$$\{u_n^+\}_{n\geq 1} \text{ is bounded in } H_0^1(\Omega). \tag{3.26}$$

From (3.14) and (3.26), it follows that $\{u_n\}_{n\geq 1}$ is bounded in $H_0^1(\Omega)$. Hence we may assume that

$$u_n \xrightarrow{\text{w}} u \text{ in } H_0^1(\Omega) \ \text{ and } \ u_n \to u \text{ in } L^r(\Omega) \ \text{ as } n \to \infty. \tag{3.27}$$

From (3.7) with $y = u_n - u \in H_0^1(\Omega)$, we have

$$|\langle -\Delta u_n, u_n - u\rangle - \int_\Omega f(z, u_n)(u_n - u)\, dz| \le \varepsilon_n \,. \tag{3.28}$$

Note that $\int_\Omega f(z, u_n)(u_n - u)\, dz \to 0$ (see (3.27)). Then from (3.28) we obtain that $\lim_{n\to\infty} \langle -\Delta u_n, u_n - u\rangle = 0$, which implies that $u_n \to u$ in $H_0^1(\Omega)$. This proves that φ satisfies the (C)-condition. □

Proposition 3.5. *If hypotheses* (H) *hold, then*

$$C_k(\varphi, \infty) = 0 \quad \textit{for all integers } k \ge 0.$$

Proof. Let $\psi = \varphi|_{C_0^1(\overline{\Omega})}$. Regularity theory (see, e.g., [10, pp. 738–739]) implies that the critical points of φ are in $C_0^1(\overline{\Omega})$. Hence ψ and φ have the same critical set. Since $C_0^1(\overline{\Omega})$ is dense in $H_0^1(\Omega)$, invoking Proposition 16 of Palais [17], we have

$$H_k(H_0^1(\Omega), \dot{\varphi}^a) = H_k(C_0^1(\overline{\Omega}), \dot{\psi}^a) \text{ for all } a \in \mathbb{R} \text{ and all integers } k \ge 0. \tag{3.29}$$

Assuming that the critical set K of φ (and of ψ) is finite (otherwise we have an infinity of solutions), then for $a < 0$ with $|a|$ large enough, by (2.1) we have

$$H_k(H_0^1(\Omega), \dot{\varphi}^a) = H_k(H_0^1(\Omega), \varphi^a) = C_k(\varphi, \infty) \quad \text{for all integers } k \ge 0 \tag{3.30}$$

and

$$H_k(C_0^1(\overline{\Omega}), \dot{\psi}^a) = H_k(C_0^1(\overline{\Omega}), \psi^a) = C_k(\psi, \infty) \text{ for all integers } k \ge 0. \tag{3.31}$$

From (3.29), (3.30) and (3.31), we see that in order to prove the proposition, it suffices to show that

$$H_k(C_0^1(\overline{\Omega}), \psi^a) = 0 \quad \text{for all } a < 0 \text{ with } |a| \text{ large and all integers } k \ge 0. \tag{3.32}$$

In order to prove (3.32), we proceed as follows. We introduce the sets

$$\partial B_1^c = \{u \in C_0^1(\overline{\Omega}) : \ \|u\|_{C_0^1(\overline{\Omega})} = 1\}$$

and

$$\partial B_{1,+}^c = \{u \in \partial B_1^c : \ u(z) > 0 \text{ for some } z \in \Omega\}.$$

We consider the map $h_+ : [0,1] \times \partial B_{1,+}^c \to \partial B_{1,+}^c$ defined by

$$h_+(t, u) = \frac{(1-t)u + tu_1}{\|(1-t)u + tu_1\|_{C_0^1(\overline{\Omega})}} \quad \text{for all } (t, u) \in [0,1] \times \partial B_{1,+}^c.$$

Clearly, h_+ is well-defined and continuous. Moreover, we have that $h_+(0, \cdot) = \mathrm{id}|_{\partial B_{1,+}^c}$ and $h_+(1, u) = \frac{u_1}{\|u_1\|_{C_0^1(\overline{\Omega})}} \in \partial B_{1,+}^c$. Therefore the set $\partial B_{1,+}^c$ is contractible in itself.

By virtue of (3.3) in hypothesis (H) (iv), given any $\gamma > 0$, we can find $M_8 = M_8(\gamma) > 0$ such that

$$F(z, x) \ge \frac{\gamma}{2} x^2 \quad \text{for a.a. } z \in \Omega \text{ and all } x \ge M_8. \tag{3.33}$$

Similarly, from (3.1) in hypothesis (H) (iv), and by choosing $M_8 > 0$ even bigger if necessary, we see that there is a number $\sigma_0 > 0$ such that

$$F(z,x) \geq -\frac{\sigma_0}{2} x^2 \quad \text{for a.a. } z \in \Omega \text{ and all } x \leq -M_8. \tag{3.34}$$

Moreover, by hypothesis (H) (iii), we have

$$|F(z,x)| \leq c_3 \quad \text{for a.a. } z \in \Omega \text{ and all } |x| < M_8, \tag{3.35}$$

for some $c_3 > 0$.

Let $u \in \partial B^c_{1,+}$. By (3.33), (3.34), (3.35), (2.3), for all $t > 0$ we can write

$$\begin{aligned}
\varphi(tu) &= \frac{t^2}{2}\|Du\|_2^2 - \int_\Omega F(z,tu)\,dz \\
&= \frac{t^2}{2}\|Du\|_2^2 - \int_{\{tu\geq M_8\}} F(z,tu)\,dz - \int_{\{tu\leq -M_8\}} F(z,tu)\,dz \\
&\quad - \int_{\{|tu|<M_8\}} F(z,tu)\,dz \\
&\leq \frac{t^2}{2}\|Du\|_2^2 - \frac{t^2}{2}\gamma \int_{\{tu\geq M_8\}} u^2\,dz + \frac{t^2}{2}\sigma_0 \int_{\{tu\leq -M_8\}} u^2\,dz + c_3|\Omega|_N \\
&\leq \frac{t^2}{2}\left[\left(1+\frac{\sigma_0}{\lambda_1}\right)\|Du\|_2^2 - \gamma \int_{\{tu\geq M_8\}} u^2\,dz\right] + c_3|\Omega|_N.
\end{aligned} \tag{3.36}$$

Since $u \in \partial B^c_{1,+}$, we can find $t_* > 0$ and $\mu > 0$ such that

$$\int_{\{tu\geq M_8\}} u^2\,dz \geq \mu \text{ for all } t \geq t_*.$$

Recalling that $\gamma > 0$ is arbitrary, we can choose $\gamma > 0$ large such that

$$\gamma\mu - \left(1+\frac{\sigma_0}{\lambda_1}\right)\|Du\|_2^2 =: \mu_0 > 0.$$

Then from (3.36), we have $\varphi(tu) \leq \frac{t^2}{2}(-\mu_0) + c_3|\Omega|_N$ for all $t \geq t_*$, and thus

$$\varphi(tu) \to -\infty \quad \text{as } t \to \infty. \tag{3.37}$$

From (3.4) in hypothesis (H) (iv), we see that there exist $\xi_0 > 0$ and $M_9 > 0$ such that

$$f(z,x)x - 2F(z,x) \geq \xi_0 x^\tau \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq M_9. \tag{3.38}$$

Using hypothesis (H) (iii) and (3.2) in hypothesis (H) (iv), we obtain

$$2F(z,x) - f(z,x)x \leq c_4 \quad \text{for a.a. } z \in \Omega \text{ and all } x < M_9, \tag{3.39}$$

for some $c_4 > 0$. By (3.38), (3.39), for any $u \in H_0^1(\Omega)$ we have

$$\begin{aligned}
&\int_\Omega (2F(z,u) - f(z,u)u)\,dz \\
&= \int_{\{u\geq M_9\}} (2F(z,u) - f(z,u)u)\,dz + \int_{\{u<M_9\}} (2F(z,u) - f(z,u)u)\,dz \\
&\leq -\xi_0 \int_{\{u\geq M_9\}} u^\tau\,dz + c_5, \qquad (3.40)
\end{aligned}$$

where $c_5 = c_4|\Omega|_N$. Let $i : C_0^1(\overline{\Omega}) \to H_0^1(\Omega)$ be the continuous embedding map. Let $\langle\cdot,\cdot\rangle_0$ denote the duality brackets for the pair $(C_0^1(\overline{\Omega})^*, C_0^1(\overline{\Omega}))$. We have $\psi = \varphi \circ i$, and so

$$\psi'(u) = i^*\varphi'(i(u)) \text{ for all } u \in C_0^1(\overline{\Omega}). \tag{3.41}$$

Let $u \in \partial B_{1,+}^c$. Using (3.41), (3.40), for all $t > 0$ we have

$$\begin{aligned}
\frac{d}{dt}\psi(tu) &= \langle\psi'(tu), u\rangle_0 = \langle\varphi'(tu), u\rangle = t\|Du\|_2^2 - \int_\Omega f(z,tu)u\,dz \\
&\leq \frac{1}{t}\left(2\varphi(tu) + c_5\right).
\end{aligned}$$

Then, owing to (3.37), we obtain

$$\frac{d}{dt}\psi(tu) < 0 \text{ for all } t > 0 \text{ large such that } \varphi(tu) < -\frac{c_5}{2}. \tag{3.42}$$

From hypothesis (H) (iii) and (3.1) in hypothesis (H) (iv), we see that given $\varepsilon > 0$, we can find $\xi_\varepsilon > 0$ such that

$$F(z,x) \leq \frac{1}{2}(\vartheta(z) + \varepsilon)x^2 + \xi_\varepsilon \text{ for a.a. } z \in \Omega \text{ and all } x \leq 0. \tag{3.43}$$

Using (3.43) and Lemma 2.2, we have

$$\begin{aligned}
\varphi(v) &\geq \frac{1}{2}\|Dv\|_2^2 - \frac{1}{2}\int_\Omega \vartheta v^2\,dz - \frac{\varepsilon}{2}\|v\|_2^2 - \xi_\varepsilon|\Omega|_N \\
&\geq \frac{1}{2}\left(\hat{\gamma}_0 - \frac{\varepsilon}{\lambda_1}\right)\|Dv\|_2^2 - \xi_\varepsilon|\Omega|_N \text{ for all } v \in -C_+.
\end{aligned}$$

Taking $\varepsilon \in (0, \hat{\gamma}_0\lambda_1)$, it follows that $\varphi|_{-C_+}$ is coercive, thus we can find $c_6 > 0$ such that $\varphi|_{-C_+} \geq -c_6$. We pick

$$a < \min\left\{-\frac{c_5}{2},\ -c_6,\ \inf_{\partial B_1^c}\psi\right\}.$$

Then (3.42) implies that we can find unique $k(u) > 1$ such that

$$\begin{cases}
\psi(tu) > a & \text{if } t \in [0, k(u)), \\
\psi(tu) = a & \text{if } t = k(u), \\
\psi(tu) < a & \text{if } t > k(u).
\end{cases}$$

Moreover, the implicit function theorem implies that $k \in C(\partial B_{1,+}^c, [1,\infty))$.

By the choice of a, we have

$$\psi^a = \{tu : \ u \in \partial B^c_{1,+}\,, \ t \geq k(u)\}. \tag{3.44}$$

We introduce the set $E_+ = \{tu : \ u \in \partial B^c_{1,+}\,, \ t \geq 1\}$. The map $\hat{h}_+ : [0,1] \times E_+ \to E_+$ defined by

$$\hat{h}_+(s, tu) = \begin{cases} (1-s)tu + sk(u)u & \text{if } 1 \leq t < k(u) \\ tu & \text{if } t \geq k(u) \end{cases}, \quad s \in [0,1], \tag{3.45}$$

is a continuous deformation of E_+, $\hat{h}_+(1, E_+) \subset \psi^a$ and $\hat{h}_+(s, \cdot)|_{\psi^a} = \mathrm{id}|_{\psi^a}$ for all $s \in [0,1]$ (see (3.44) and (3.45)). Therefore, ψ^a is a strong deformation retract of E_+. Moreover, using the radial retraction, we see that E_+ and $\partial B^c_{1,+}$ are homotopically equivalent. Hence we have

$$H_k(C^1_0(\overline{\Omega}), \psi^a) = H_k(C^1_0(\overline{\Omega}), E_+) = H_k(C^1_0(\overline{\Omega}), \partial B^c_{1,+}) \ \text{ for all } k \geq 0. \tag{3.46}$$

Recall that in the first part of the proof, we established that $\partial B^c_{1,+}$ is contractible. This yields

$$H_k(C^1_0(\overline{\Omega}), \partial B^c_{1,+}) = 0 \ \text{ for all integers } k \geq 0$$

(see [11, p. 389]). Combining with (3.46) leads to (3.32), which completes the proof. □

Proposition 3.6. *If hypotheses* (H) *hold and* $d = \dim \overline{H}_m$ *, then*

$$C_d(\varphi, 0) \neq 0.$$

Proof. By virtue of hypotheses (H) (v), given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{2}\,(\eta(z) - \varepsilon)x^2 \leq F(z,x) \ \text{ for a.a. } z \in \Omega \text{ and all } |x| \leq \delta. \tag{3.47}$$

Since $\overline{H}_m = \overset{m}{\underset{i=1}{\oplus}} E(\lambda_i)$ is finite dimensional, all norms are equivalent. Thus we can find $\rho_1 > 0$ small such that

$$\|u\| \leq \rho_1 \iff \|u\|_\infty \leq \delta \quad \text{for all } u \in \overline{H}_m\,. \tag{3.48}$$

Taking (3.47) and (3.48) into account, for all $u \in \overline{H}_m$ with $\|u\| \leq \rho_1$ we have

$$\varphi(u) \leq \frac{1}{2}\int_\Omega (\lambda_m - \eta(z))u(z)^2\,dz + \frac{\varepsilon}{2}\,\|u\|_2^2\,. \tag{3.49}$$

Consider the functional

$$\zeta(u) = \int_\Omega (\eta(z) - \lambda_m)u(z)^2\,dz \ \text{ for all } u \in \overline{H}_m \cap \partial B_1^{L^2},$$

where $\partial B_1^{L^2} = \{u \in L^2(\Omega) : \ \|u\|_2 = 1\}$. Evidently, $\zeta(\cdot)$ is continuous, while the set $\overline{H}_m \cap \partial B_1^{L^2}$ is compact (recall that $\overline{H}_m$ is finite dimensional). Therefore we can find $u_0 \in \overline{H}_m \cap \partial B_1^{L^2}$ such that

$$m_0 := \inf\left\{\zeta(u) : \ u \in \overline{H}_m \cap \partial B_1^{L^2}\right\} = \zeta(u_0).$$

From the unique continuation property and the hypothesis on η (see (H) (v)), we have $m_0 = \zeta(u_0) > 0$.

Choose $0 < \varepsilon < m_0$. Then from (3.49) and for $u \in \overline{H}_m$ with $\|u\| \le \rho_1$, $u \neq 0$, we have

$$\frac{1}{\|u\|_2^2}\,\varphi(u) \le \frac{1}{2}\int_\Omega (\lambda_m - \eta(z))\,\frac{u(z)^2}{\|u\|_2^2}\,dz + \frac{\varepsilon}{2} \le -\frac{1}{2}\,m_0 + \frac{\varepsilon}{2} < 0\,,$$

and thus

$$\varphi(u) \le 0 \ \text{ for all } u \in \overline{H}_m \text{ with } \|u\| \le \rho_1. \tag{3.50}$$

On the other hand, combining hypotheses (H) (iii) and (v), given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F(z,x) \le \frac{\hat{\eta}(z)+\varepsilon}{2}\,x^2 + c_\varepsilon |x|^r \ \text{ for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}. \tag{3.51}$$

By (3.51), (2.4) and Lemma 2.2, we have

$$\begin{aligned} \varphi(u) &\ge \frac{1}{2}\,\|Du\|_2^2 - \frac{1}{2}\int_\Omega \hat{\eta}u^2\,dz - \frac{\varepsilon}{2\lambda_{m+1}}\,\|Du\|_2^2 - \hat{c}_\varepsilon\|Du\|_2^r \\ &\ge \frac{1}{2}\left(\hat{\gamma}_0 - \frac{\varepsilon}{\lambda_{m+1}}\right)\|Du\|_2^2 - \hat{c}_\varepsilon\|Du\|_2^r \ \text{ for all } u \in \hat{H}_m\,, \end{aligned} \tag{3.52}$$

for some $\hat{c}_\varepsilon > 0$. Choosing $\varepsilon < \hat{\gamma}_0\lambda_{m+1}$ and since $r > 2$, from (3.52), we infer that for $\rho_2 \in (0,1)$ small we have

$$\varphi(u) > 0 \ \text{ for all } u \in \hat{H}_m \text{ with } 0 < \|u\| \le \rho_2\,. \tag{3.53}$$

Let $\rho = \min\{\rho_1, \rho_2\}$. From (3.50) and (3.53), it follows that

$$\varphi(u) \begin{cases} \le 0 & \text{if } u \in \overline{H}_m\,,\ \|u\| \le \rho \\ > 0 & \text{if } u \in \hat{H}_m\,,\ 0 < \|u\| \le \rho\,, \end{cases}$$

i.e., φ has a local linking at 0. Then invoking Proposition 2.3 of Bartsch and Li [2], we conclude that $C_d(\varphi, 0) \neq 0$. □

4. MULTIPLICITY THEOREM

First we produce a nontrivial smooth solution of constant sign (negative).

Proposition 4.1. *If hypotheses* (H) *hold, then problem* (1.1) *has a solution* $v_0 \in -\operatorname{int} C_+$ *which is a local minimizer of* φ.

Proof. We consider the negative truncation of the nonlinearity $f(z,\cdot)$, namely

$$f_-(z,x) = \begin{cases} f(z,x) & \text{if } x < 0 \\ 0 & \text{if } x \ge 0\,. \end{cases}$$

Evidently, this is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $z \mapsto f_-(z,x)$ is measurable, and for a.a. $z \in \Omega$, $x \mapsto f_-(z,x)$ is continuous). We set $F_-(z,x) = \int_0^x f_-(z,s)\,ds$ and consider the C^1-functional $\varphi_- : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{2}\,\|Du\|_2^2 - \int_\Omega F_-(z,u(z))\,dz \quad \text{for all } u \in H_0^1(\Omega).$$

Hypotheses (H) (iii) together with (3.1) in hypothesis (H) (iv) imply that given $\varepsilon > 0$, we can find $c_\varepsilon > 0$ such that

$$F_-(z,x) \le \frac{1}{2}\,(\vartheta(z)+\varepsilon)x^2 + \xi_\varepsilon \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}$$

(note that $F_-(z,x) = 0$ for a.a. $z \in \Omega$ and all $x \ge 0$). Hence, by Lemma 2.2 and (2.3), we have

$$\begin{aligned} \varphi_-(u) &\ge \frac{1}{2}\,\|Du\|_2^2 - \frac{1}{2}\int_\Omega \vartheta u^2\,dz - \frac{\varepsilon}{2}\,\|u\|_2^2 - \xi_\varepsilon|\Omega|_N \\ &\ge \frac{1}{2}\left(\hat{\gamma}_0 - \frac{\varepsilon}{\lambda_1}\right)\|Du\|_2^2 - \xi_\varepsilon|\Omega|_N \quad \text{for all } u \in H_0^1(\Omega). \end{aligned} \tag{4.1}$$

Choosing $\varepsilon \in (0, \hat{\gamma}_0\lambda_1)$, from (4.1) we infer that φ_- is coercive. In addition, exploiting the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, we see that φ_- is sequentially weakly lower semicontinuous. Thus we can find $v_0 \in H_0^1(\Omega)$ such that

$$\varphi_-(v_0) = \inf \varphi_- =: m_-. \tag{4.2}$$

By hypothesis (H) (v), given $\varepsilon \in (0, \lambda_m - \lambda_1)$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{2}\,(\eta(z)-\varepsilon)x^2 \le F_-(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in [-\delta, 0]. \tag{4.3}$$

Since $u_1 \in \operatorname{int} C_+$, there exists $\theta > 0$ small such that $-\theta u_1(z) \in [-\delta, 0]$ for all $z \in \overline{\Omega}$. Then, taking into account (4.3), (2.3), that $\|u_1\|_2 = 1$, (H) (v), and the choice of $\varepsilon > 0$, we obtain

$$\begin{aligned} \varphi_-(-\theta u_1) &= \frac{\theta^2}{2}\,\lambda_1 - \int_\Omega F_-(z,-\theta u_1)\,dz \\ &\le \frac{\theta^2}{2}\,\lambda_1 - \frac{\theta^2}{2}\int_\Omega \eta u_1^2\,dz + \frac{\varepsilon\theta^2}{2} \le \frac{\theta^2}{2}(\lambda_1 - \lambda_m + \varepsilon) < 0, \end{aligned}$$

and thus, by (4.2), $\varphi_-(v_0) \le \varphi_-(-\theta u_1) < 0 = \varphi_-(0)$, so $v_0 \ne 0$.

From (4.2), we have $\varphi_-'(v_0) = 0$, that is,

$$-\Delta v_0 = N_-(v_0), \tag{4.4}$$

where $N_-(u)(\cdot) = f_-(\cdot, u(\cdot))$ for all $u \in H_0^1(\Omega)$. Acting on (4.4) with $v_0^+ \in H_0^1(\Omega)$, we obtain $\|Dv_0^+\|_2^2 = 0$, i.e., $v_0^+ = 0$, and so $v_0 \le 0$, $v_0 \ne 0$. Therefore from (4.4) we have

$$\begin{cases} -\Delta v_0(z) = f_-(z, v_0(z)) = f(z, v_0(z)) & \text{in } \Omega, \\ v_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

thus v_0 solves (1.1), and from regularity theory, $v_0 \in -C_+$. From (3.1) in hypothesis (H) (iv) and from hypothesis (H) (v), we have

$$|f_-(z,x)| \le \tilde{c}|x| \text{ for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}, \tag{4.6}$$

for some $\tilde{c} > 0$. Then from (4.5) and (4.6), we have

$$\Delta(-v_0)(z) \le \tilde{c}(-v_0)(z) \text{ for a.a. } z \in \Omega.$$

From this inequality and the maximum principle of Vázquez [21], we have that $v_0 \in -\operatorname{int} C_+$. Noting that $\varphi_-|_{-C_+} = \varphi|_{-C_+}$, it follows that v_0 is a local $C_0^1(\overline{\Omega})$-minimizer of φ. Invoking the result of Brezis and Nirenberg [3], we conclude that v_0 is a local $H_0^1(\Omega)$-minimizer of φ. $\square$

Since $v_0 \in -\operatorname{int} C_+$ is a local minimizer of φ, from the characterization of the critical groups of a C^1-functional at a local minimizer (see [5, p. 33] or [13, p. 175]), we have:

Corollary 4.2. *If hypotheses* (H) *hold and* $v_0 \in -\operatorname{int} C_+$ *is the solution obtained in Proposition* 4.1*, then*

$$C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all integers } k \ge 0.$$

Now we are ready for the multiplicity result concerning problem (1.1).

Theorem 4.3. *If hypotheses* (H) *hold, then problem* (1.1) *has at least two nontrivial smooth solutions*

$$v_0 \in -\operatorname{int} C_+ \quad \text{and} \quad x_0 \in C_0^1(\overline{\Omega}).$$

Proof. From Proposition 4.1 we have one solution $v_0 \in -\operatorname{int} C_+$ of (1.1) and

$$C_k(\varphi, v_0) = \delta_{k,0}\mathbb{Z} \text{ for all integers } k \ge 0 \tag{4.7}$$

(see Corollary 4.2). Let $\theta \in \mathbb{R}$, $\varepsilon > 0$ be such that $\theta < m_- = \varphi(v_0) < -\varepsilon$ (see (4.2)). We consider the sublevel sets

$$\varphi^\theta \subset \varphi^{-\varepsilon} \subset \varphi^\varepsilon \subset H_0^1(\Omega).$$

Suppose that 0 and v_0 are the only critical points of φ. Otherwise, we have a second nontrivial smooth (by regularity theory) solution and so we are done. From the definition of critical groups at infinity and Proposition 3.5, we have

$$H_k(H_0^1(\Omega), \varphi^\theta) = C_k(\varphi, \infty) = 0 \text{ for all integers } k \ge 0. \tag{4.8}$$

We know that φ satisfies the (C)-condition (see Proposition 3.4). Hence choosing $\varepsilon > 0$ small, we have

$$H_d(\varphi^\varepsilon, \varphi^{-\varepsilon}) = C_d(\varphi, 0) \neq 0 \tag{4.9}$$

(see Proposition 3.6 and [2, Remark 2.2]). Because of (4.8) and (4.9), applying Proposition 2.1, we obtain

$$H_{d+1}(H_0^1(\Omega), \varphi^{\varepsilon}) \neq 0 \quad \text{or} \quad H_{d-1}(\varphi^{-\varepsilon}, \varphi^{\theta}) \neq 0.$$

If $H_{d+1}(H_0^1(\Omega), \varphi^{\varepsilon}) \neq 0$, then there is a critical point $x_0 \in H_0^1(\Omega)$ of φ such that

$$\varphi(v_0) = m_- < 0 = \varphi(0) < \varepsilon \leq \varphi(x_0),$$

so $x_0 \neq 0, v_0$, it solves (1.1), and, by regularity theory, $x_0 \in C_0^1(\overline{\Omega})$.

If $H_{d-1}(\varphi^{-\varepsilon}, \varphi^{\theta}) \neq 0$, then there is a critical point $x_0 \in H_0^1(\Omega)$ of φ such that

$$C_{d-1}(\varphi, x_0) \neq 0. \tag{4.10}$$

Since $d \geq 2$, from (4.7) and (4.10), we see that $x_0 \neq v_0$. In addition, we have $\theta \leq \varphi(x_0) \leq -\varepsilon < 0 = \varphi(0)$, and thus $x_0 \neq 0$. Therefore x_0 is a solution of (1.1) distinct from $0, v_0$, and, by regularity theory, $x_0 \in C_0^1(\overline{\Omega})$. □

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