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DIFFERENTIAL INVARIANTS FOR NONLINEAR PDES

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ABSTRACT. We study differential invariants for scalar evolution equations such as the Kadomtsev-Petviashvili and Novikov-Veselov equations.

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1. Introduction

Many physical problems are modelled by nonlinear partial differential equations. There is no unified method by which classes of nonlinear partial differential equations can be solved. A well known nonlinear evolution equation is the Korteweg-de Vries (KdV) equation, first derived by Korteweg and de Vries as a model for shallow-water waves [1] in 1895. In 1967 a method for solving the Korteweg-de Vries Equation (KdV) was developed by Gardner, Green, Kruskal and Miura [2, 3] for initial values which decay sufficiently rapidly, through a series of linear equations. This method is called the *Inverse Scattering Method* (ISM). In 1968 Lax [4] generalized the ISM by introducing a *Lax Pair* formulation of the KdV equation. Following Lax's formulation, Zakharov and Shabat [5] solved the nonlinear Schrödinger equation. Soon thereafter the sine-Gordon equation and the modified KdV equation were solved by Ablowitz, Kaup, Newell, and Segur [6, 7] and Wadati [8]. The KdV equation is an example of an *integrable* equation.

In addition to the above methods, several methods have been developed for generating large classes of solutions to such equations. Among such methods, the most well known are the use of Bäcklund transformations, the dressing method of Zakharov-Shabat [9, 10] and the bilinear approach of Hirota [11].

Lax Equation.

Lax [4] put the inverse scattering method for solving the KdV equation into a more general framework which subsequently paved the way to generalizations of the

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technique as a method for solving other differential equations.

$$L_t + [L, M] = 0, \tag{1.1}$$

where $[L, M] = LM - ML$ is called *Lax's equation* and it is equivalent to nonlinear evolution equations for suitably chosen L and M. For example, if we take

$$L = -\frac{\partial^2}{\partial x^2} + u, \tag{1.2}$$

$$M = -4\frac{\partial^3}{\partial x^3} + 6u\frac{\partial}{\partial x} + 3u_x, \tag{1.3}$$

then L and M satisfy Lax's equation (1.1) provided u satisfies the KdV equation

$$u_t - 6uu_x + u_{xxx} = 0. \tag{1.4}$$

2. **Differential Invariants**

In this section we start by giving the definitions of the differential invariants. We define the gauge transformation $g^{-1}Lg = L^g$ where g is an arbitrary non-vanishing function acting on a given differential operator L and preserving the operator part of L. These gauge transformations produce invariants. In this paper we will call them the gauge invariants. These invariants are simply functions of the coefficients of the given operator L. We will show that the invariants of a given L form a complete set. In other words we will show that there are a complete set i.e. a set, the knowledge of which, is enough to determine the operator L completely up to gauge transformations. Additionally in this section, the definition of the Laplace Transformation (LT) for linear hyperbolic equations will be given in order to obtain the LTs of gauge invariants [12].

Gauge Transformations.

Consider linear operators L acting on $z : Lz = 0$. Linear maps $z^g = g^{-1}z$ induce gauge transformations $L \to L^g = g^{-1}Lg$. The coefficients of L^g are then related to those of L via g. A function $\mathcal{I}$ of the coefficients of L is an *invariant* (gauge invariant) if its expression is the same for both L and L^g i.e.

$$\mathcal{I}(l_1^g, l_2^g, ..) = \mathcal{I}(l_1, l_2, ..)$$

where l_i^g, respectively l_i, are the coefficients in L^g, respectively L.

Gauge transformation for linear hyperbolic system.

Let us consider the second order, linear hyperbolic system in canonical form

$$Lz = z_{,xy} + az_{,x} + bz_{,y} + cz = 0 \tag{2.1}$$

where $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$ and a, b and c are real functions of x and y. The above equation can be written in the form

$$Lz = (\partial_x\partial_y + a\partial_x + b\partial_y + c)z = 0. \tag{2.2}$$

The gauge transformation is $z \to z^g = g^{-1}z$, equivalently $L^g = g^{-1}Lg$, where g is a function of x and y. Therefore if we substitute $z = gz^g$ in the equation (2.1), we obtain

$$z^g_{,xy} + a^g z^g_{,x} + b^g z^g_{,y} + c^g z^g = 0 \tag{2.3}$$

where

$$\begin{aligned} a^g &= a + g^{-1}g_{,y}, \\ b^g &= b + g^{-1}g_{,x}, \\ c^g &= c + ag^{-1}g_{,x} + bg^{-1}g_{,y} + g^{-1}g_{,xy}. \end{aligned}$$

By eliminating g between the above expressions we have

$$a^g_{,x} + a^g b^g - c^g = a_{,x} + ab - c,$$
$$b^g_{,y} + a^g b^g - c^g = b_{,y} + ab - c.$$

So we may choose

$$h = a_{,x} + ab - c \tag{2.4}$$
$$k = b_{,y} + ab - c \tag{2.5}$$

where h and k are gauge invariants [13].

Laplace Transformation for linear hyperbolic equation.

Let us consider the equations

$$z_{,xy} + az_{,x} + bz_{,y} + cz = 0 \tag{2.6}$$
$$z^{\sigma_1}_{,xy} + a^{\sigma_1} z^{\sigma_1}_{,x} + b^{\sigma_1} z^{\sigma_1}_{,y} + c^{\sigma_1} z^{\sigma_1} = 0 \tag{2.7}$$

We define Laplace Transformation (LT) as a mapping between two copies of linear hyperbolic equations:

$$z \to z^{\sigma_1} = z_y + az \tag{2.8}$$

with

$$\begin{aligned} a^{\sigma_1} &= a - h^{-1}h_{,y}, \\ b^{\sigma_1} &= b, \\ c^{\sigma_1} &= c + b_{,y} - a_{,x} - bh^{-1}h_{,y}. \end{aligned}$$

where $h = a_x + ab - c$ is the gauge invariant defined in (2.4) . Hence we obtain

$$\begin{aligned} a^{\sigma_1}_{,x} + a^{\sigma_1}b^{\sigma_1} - c^{\sigma_1} &= 2h - k - \left(h^{-1}h_{,y}\right)_{,x}, \\ b^{\sigma_1}_{,y} + a^{\sigma_1}b^{\sigma_1} - c^{\sigma_1} &= h. \end{aligned}$$

So the LT transforms the gauge invariants h and k as follows

$$h^{\sigma_1} = 2h - k - (\ln h)_{,xy}, \tag{2.9}$$

$$k^{\sigma_1} = h. \tag{2.10}$$

If a solution $z(x, y)$ of the equation (2.6) is given with the potentials $a(x, y)$, $b(x, y)$ and $c(x, y)$ then we can construct a new solution to the equation (2.7) with new potentials a^{σ_1}, b^{σ_1} and c^{σ_1} by using the LT: $z^{\sigma_1} = z_y + az$.

Similarly, another Laplace map, σ_2, is defined by setting

$$z^{\sigma_2} = z_x + bz.$$

We obtain the following Laplace Transformations of gauge invariants h and k:

$$\begin{aligned} h^{\sigma_2} &= k, \\ k^{\sigma_2} &= 2k - h - (\ln k)_{,xy}. \end{aligned}$$

One may check and see that $\sigma_1\sigma_2 = \sigma_2\sigma_1 = \text{id}$ [13].

3. **Invariant Form For Scalar Evolution Equations**

The aim of this section is to obtain scalar evolution equations in invariant forms. We will consider two scalar differential operators L and M. We call them scalar because their coefficients are functions rather than matrices. We will show that the Lax equation $[L, M] = 0$ and 'L-M-f triad' representation $[L, M] + fL = 0$ give the KP and the NV equations respectively where L and M are both scalar differential operators and f is an operator function to be found. We will also discuss the completeness for the Lax pair L and M. We will show that the invariants of L and M are an almost complete set.

The Kadomtsev-Petviashvili (KP) Equation.

Let L and M be two operator functions such that

$$L = \partial_t + \partial_x^2 + u\partial_x + v \tag{3.1}$$

$$M = \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c \tag{3.2}$$

where u, v, a, b and c are functions of x, y and t.

Consider the commutativity representation $[L, M] = 0$, i.e. L and M are a Lax pair. Our aim is to obtain some gauge invariant equations. In order to achieve this goal we firstly find the gauge invariants of L and M by applying the gauge transformation on L and M. And then by using the Lax equation, $[L, M] = 0$, we have some differential equations in the invariant forms. After that we derive the Kadomtsev-Petviashvili (KP) equation as an example.

Invariants for $L = \partial_t + \partial_x^2 + u\partial_x + v$.

We apply the gauge transformation on $L = \partial_t + \partial_x^2 + u\partial_x + v$ which is $g^{-1}Lg = L^g$, where g is a function of x, y and t. Thus $g^{-1}Lg = L^g$,

$$g^{-1}(\partial_t + \partial_x^2 + u\partial_x + v)g = \partial_t + \partial_x^2 + u^g\partial_x + v^g,$$

gives us the coefficients of L^g:

$$u^g = u + 2g^{-1}g_x$$

$$v^g = v + g^{-1}g_t + ug^{-1}g_x + g^{-1}g_{xx}$$

If we eliminate g in the above expressions, we obtain

$$u^g{}_t + u^g{}_{xx} + u^g u^g{}_x - 2v^g{}_x = u_t + u_{xx} + uu_x - 2v_x$$

We choose

$$J = u_t + u_{xx} + uu_x - 2v_x. \tag{3.3}$$

This is an invariant for the differential operator L where L is defined in (3.1).

Completeness for L.

We now turn our attention to *completeness*: We will show that the invariant J is almost sufficient i.e. it is almost a complete set. In other words we will prove

$$L' = g^{-1}Lg \Longleftrightarrow J' = J,$$

where $L' = \partial_t + \partial_x^2 + u'\partial_x + v'$ up to an undetermined function of t and y. We have already shown '$\Longrightarrow$' part. We need to prove '$\Longleftarrow$' part now. Assume $J' = J$ i.e.

$$u'_t + u'_{xx} + u'u'_x - 2v'_x = u_t + u_{xx} + uu_x - 2v_x.$$

If we rearrange the above equation, we have

$$(v' - v)_x = \frac{1}{2}(u' - u)_t + \frac{1}{4}\left[(u' - u)(u' + u)\right]_x + \frac{1}{2}(u' - u)_{xx}.$$

Let us choose g such that $u' - u = 2(\ln g)_x$. Then we obtain

$$\begin{aligned} u' &= u + 2g^{-1}g_x \\ v' &= v + g^{-1}g_t + ug^{-1}g_x + g^{-1}g_{xx} + c(y,t) \end{aligned}$$

where c is a function of time and y. So

$$L' = \partial_t + \partial_x^2 + u'\partial_x + v' = g^{-1}(\partial_t + \partial_x^2 + u\partial_x + v)g + c(y,t)$$

i.e.

$$L' = g^{-1}Lg + c(y,t).$$

The proof is complete. J is a complete set up to additive undetermined functions of y and t.

Invariants for $M = \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c$.

This time we use the gauge transformation on $M = \partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c$ which is $g^{-1}Mg = M^g$, where g is a function of x, y and t.

$$g^{-1}(\partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c)g = \partial_y + \partial_x^3 + a^g\partial_x^2 + b^g\partial_x + c^g$$

This gives us the coefficients of the operator M^g:

$$\begin{aligned} a^g &= a + 3g^{-1}g_x \\ b^g &= b + 2ag^{-1}g_x + 3g^{-1}g_{xx} \\ c^g &= c + g^{-1}g_y + bg^{-1}g_x + ag^{-1}g_{xx} + g^{-1}g_{xxx} \end{aligned}$$

From the above expressions we can eliminate the function $g(x,y,t)$ and then have

$$\begin{aligned} {a^g}_x + \frac{1}{3}{a^g}^2 - b^g &= a_x + \frac{1}{3}a^2 - b \\ {a^g}_y + ({a^g}_{xx} - \frac{2}{9}{a^g}^3 + a^g b^g)_x - 3{c^g}_x &= a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x \end{aligned}$$

Let us choose

$$I = a_x + \frac{1}{3}a^2 - b \tag{3.4}$$

$$K = a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x \tag{3.5}$$

where I and K are invariants for the differential operator M.

Completeness for M.

We now look at *completeness* for M: We will prove that the invariants I and K are almost sufficient i.e they are almost a complete set. In other words we will show

$$M' = g^{-1}Mg \Longleftrightarrow I' = I,\ K' = K,$$

where $M' = \partial_y + \partial_x^3 + a'\partial_x^2 + b'\partial_x + c'$ up to additive functions of y and t. We have already shown' $\Longrightarrow$' part. We need to show ' $\Longleftarrow$' part now. Assume $I' = I$, $K' = K$ i.e.

$$a'_x + \frac{1}{3}a'^2 - b' = a_x + \frac{1}{3}a^2 - b, \tag{3.6}$$

$$a'_y + (a'_{xx} - \frac{2}{9}a'^3 + a'b')_x - 3c'_x = a_y + (a_{xx} - \frac{2}{9}a^3 + ab)_x - 3c_x. \tag{3.7}$$

If we rearrange the equation (3.6), we have

$$b' = b + (a' - a)_x + \frac{1}{3}(a' - a)(a' + a).$$

Let us choose g such that $a' - a = 3g^{-1}g_x$ i.e.

$$a' = a + 3g^{-1}g_x.$$

By substituting this in the above equation we obtain

$$b' = b + 2ag^{-1}g_x + 3g^{-1}g_{xx},$$

and by rearranging the equation (3.7) we have

$$c'_x = c_x + \frac{1}{3}(a' - a)_y + \frac{1}{3}\left[(a' - a)_{xx} - \frac{2}{9}(a'^3 - a^3) + a'b' - ab\right]_x.$$

If we substitute a', b' in the above equation and then integrate with respect to x we obtain

$$c' = c + g^{-1}g_y + bg^{-1}g_x + ag^{-1}g_{xx} + g^{-1}g_{xxx} + f(y,t)$$

where f is an arbitrary function of y and t. So

$$M' = \partial_y + \partial_x^3 + a'\partial_x^2 + b'\partial_x + c' = g^{-1}(\partial_y + \partial_x^3 + a\partial_x^2 + b\partial_x + c)g + f$$

i.e.

$$M' = g^{-1}Mg + f(y,t).$$

The proof is complete. I and K are an almost complete set.

The Commutativity Equations.

$$[L, M] = 0$$

gives us the equations

$$2a_x - 3u_x = 0 \tag{3.8}$$

$$a_t + a_{xx} + 2b_x + ua_x - 3u_{xx} - 3v_x - 2au_x = 0 \tag{3.9}$$

$$b_t + b_{xx} + 2c_x + ub_x - u_y - u_{xxx} - 3v_{xx} - au_{xx} - 2av_x - bu_x = 0 \tag{3.10}$$

$$c_t + c_{xx} + uc_x - v_y - v_{xxx} - av_{xx} - bv_x = 0 \tag{3.11}$$

From the above equations, we obtain the following invariant relations:

$$J = \frac{4}{3} I_x, \tag{3.12}$$

$$I_t = I_{xx} - \frac{2}{3} K, \tag{3.13}$$

$$K_t = 2 \left(I_y + I_{xxx} - I I_x - \frac{1}{2} K_x \right)_x . \tag{3.14}$$

The importance of these equations is that they are *universal* for systems of the form $[L, M] = 0$ where L and M (3.1-3.2) are of orders 2 and 3 in ∂_x. Their form is pleasingly simple. By introducing potentials $K = 2\varphi_x$, $I = \psi_x$ the two time evolutions become

$$\psi_t = \psi_{xx} - \frac{4}{3} \varphi, \tag{3.15}$$

$$\varphi_t = \psi_{xxxx} + \psi_{xy} - \psi_x \psi_{xx} - \varphi_{xx}. \tag{3.16}$$

Let us rewrite the above invariant equations $(3.12 - 3.14)$ in the following forms

$$I_x = \frac{3}{4} J \tag{3.17}$$

$$K = \frac{3}{2} \left(I_{xx} - I_t \right) \tag{3.18}$$

$$K_t + K_{xx} = 2 \left(I_y + I_{xxx} - I.I_x \right)_x \tag{3.19}$$

By substituting $K = \frac{3}{2} \left(I_{xx} - I_t \right)$ in the equation (3.19), we obtain

$$\left(4I_y - 4I.I_x + I_{xxx} \right)_x + 3I_{tt} = 0 \tag{3.20}$$

which is the KP equation up to scaling of I by a constant and relabeling of y and t. It is interesting that the KP arises in this way where the potential itself is a gauge invariant of the linear problem, i.e. *the KP is a gauge invariant form of the evolution equation.*

The Novikov-Veselov(NV) Equation.

Let L and M be two operator functions such that

$$L = \partial_x\partial_y + a\partial_x + b\partial_y + c, \tag{3.21a}$$

$$M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w. \tag{3.21b}$$

We will consider the 'L-M-f triad' representation [14] :

$$[L, M] + fL = 0$$

where f is an operator function to be found.

Invariants for L and M.

We will use a gauge transformation on L and M in order to get invariants for L and M:

$$g^{-1}Lg = L^g,$$

$$g^{-1}Mg = M^g.$$

The gauge transformation on $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$ gives us

$$I = a_x - b_y \tag{3.22}$$

$$J = a_x + ab - c \tag{3.23}$$

where I and J are invariants for the differential operator L.
Similarly applying the gauge transformation on $M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w$, we have

$$P = u_x + \frac{1}{3}u^2 - v, \tag{3.24}$$

$$R = u_t + (u_{xx} - \frac{2}{9}u^3 + uv)_x - 3w_x. \tag{3.25}$$

where P and R are invariants for the differential operator M.

***L*-*M*-*f* triad representation.**

$$[L, M] + fL = 0 \tag{3.26}$$

where $L = \partial_x\partial_y + a\partial_x + b\partial_y + c$, $M = \partial_t + \partial_x^3 + u\partial_x^2 + v\partial_x + w$ and
$f = (3b_x - u_x)\partial_x - v_x + 2ub_x + 3b_{xx} - 3bb_x + bu_x$, gives the following equations:

$$P_y = 3J_x,$$

$$\int I_t dy - \tfrac{1}{3}R + \left[\int I_{xx} dy + 3\int I dy \int I_x dy - P\int I dy + \left(\int I dy\right)^3\right]_x = 0,$$

$$R_y = 9\left[J\int I dy\right]_{xx},$$

$$J_t + J_{xxx} - (JP)_x - 3\left[J_x\int I dy\right]_x + 3\left[J\left(\int I dy\right)^2\right]_x = 0.$$

We can rewrite the above equations in simpler forms if we substitute $I = I'_y$:

$$
\begin{aligned}
P_y &= 3J_x, &(3.27)\\
R_y &= 9\left(I'J\right)_{xx}, &(3.28)\\
I'_t - \frac{1}{3}R + \left(I'_{xx} + 3I'I'_x - I'P + {I'}^3\right)_x &= 0, &(3.29)\\
J_t + \left(J_{xx} - JP - 3I'J_x + 3{I'}^2J\right)_x &= 0. &(3.30)
\end{aligned}
$$

By letting $I' = 0$, we obtain

$$J_t + J_{xxx} - (JP)_x = 0 \qquad (3.31)$$

which is the special case of the Novikov-Veselov(NV) equation, where $P_y = 3J_x$.

4. Conclusion

In this paper we have discussed the completeness for the given Lax pair L and M. We have shown that the invariants of L and M are an almost complete set i.e. a set, the knowledge of which, is enough to determine the operators L and M completely up to gauge transformations. We also have shown that one might obtain scaler evolution equations in invariant forms such as KP and NV equations. To derive the NV equation from the Lax pair see [14], where Konopelchenko gives a method which transforms the equation $[T_1, T_2] = BT_1$ into the equation $[T_1^M, \tilde{T}_2^M] = 0$, where T_i and T_i^M denote scalar and matrix operators respectively $(i = 1, 2)$.

Athorne and Yilmaz [15] have extended the idea of the differential invariants to the matrix case. They have suggested an algebraic solution procedure for the $(2+1)$-dimensional AKNS system (It is beyond the scope of this paper). One should know that the Laplace Transformation plays an important role in this procedure together with gauge invariants.

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