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POSITIVE SOLUTIONS FOR FIRST-ORDER BOUNDARY VALUE PROBLEMS AT RESONANCE

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ABSTRACT. In the paper we obtain sufficient conditions for the existence of positive solutions for first-order boundary value problems. Our result is based on a Leggett-Williams norm-type theorem for coincidences due to O'Regan and Zima.

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1. INTRODUCTION

In the paper we study the existence of positive solution of the following first-order boundary value problem (BVP)

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t)), & t \in [0, T], \\ x(0) = \alpha x(T), \end{cases} \tag{1.1}$$

where $\alpha > 0$ and $T > 0$. We are interested in the case when the problem (1.1) is at resonance, that is, the corresponding homogeneous problem

$$\begin{cases} x'(t) + a(t)x(t) = 0, & t \in [0, T], \\ x(0) = \alpha x(T), \end{cases}$$

has nontrivial solutions. Boundary value problems for first-order differential equations have been discussed for example in the papers [2], [3], [6], [7], [8], [9], [10], [13], [14], [17] and [18]. In particular, in [2], [6] and [7], the authors dealt with the nonlinear boundary condition $g(x(0), x(T)) = 0$. They obtained existence and uniqueness results by making use of the method of upper and lower solutions and of monotone iterative techniques. Note that (1.1) with $\alpha = 1$ becomes a periodic BVP. For some recent results on such problems we refer the reader to [11], [15], [19] and the references therein. The existence and multiplicity of *positive* solutions for first-order periodic BVPs have been studied for example in [3], [11], [14] and [15]. In particular, in order

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to prove the existence of a positive solution for the problem

$$\begin{cases} x'(t) + f(t, x(t)) = 0, & t \in [0, T], \\ x(0) = x(T), \end{cases}$$

Peng [15] applied the fixed point theorem on cone [4] to the equivalent non-resonant periodic BVP. A similar approach was used in [11]. In this paper we study a more general problem. Our method is based on the existence theorem for coincidence equations due to O'Regan and Zima [14]. Some results on coincidences and their applications to first and second order boundary value problems can be found for example in [1], [3], [5], [10], [12], [17] and [20]. In particular, Santanilla [17] applied his coincidence theorem of compression type for solutions in a cone to prove the existence of positive solutions for first-order periodic BVP. The key tool used in [10] to prove the existence result for first-order multi-point BVP is the well-known coincidence degree theory due to Mawhin (see for example [12]). The purpose of this paper is to extend some results from [14] and [17].

2. COINCIDENCE EQUATIONS

In this Section we recall some basic facts on Fredholm operators, coincidence equations and cones in Banach spaces. Let X and Y denote real Banach spaces. Consider a linear mapping $L : \operatorname{dom} L \subset X \to Y$ and a nonlinear operator $N : X \to Y$. We will assume that:

1° L is a Fredholm operator of index zero, that is, $\operatorname{Im} L$ is closed and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$.

This implies that there exist continuous projections

$$P : X \to X \quad \text{and} \quad Q : Y \to Y$$

such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L$ (see for example [3], [12]). Since $\dim \operatorname{Im} Q = \operatorname{codim} \operatorname{Im} L$, there exists an isomorphism

$$J : \operatorname{Im} Q \to \operatorname{Ker} L.$$

Denote by L_P the restriction of L to $\operatorname{Ker} P \cap \operatorname{dom} L$. Clearly, L_P is an isomorphism from $\operatorname{Ker} P \cap \operatorname{dom} L$ to $\operatorname{Im} L$. Thus its inverse

$$K_P : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$$

is defined. It is known (see [3], [12]) that the coincidence equation

$$Lx = Nx$$

is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Let C be a cone in X. It is well-known that C induces a partial order in X by

$$x \preceq y \text{ if and only if } y - x \in C.$$

We will also make use of the following property.

Lemma 2.1. [16] *For every $u \in C \setminus \{0\}$ there exists a positive number $\sigma(u)$ such that*

$$\|x + u\| \geq \sigma(u)\|x\|$$

for all $x \in C$.

Let $\gamma : X \to C$ be a retraction, that is, a continuous mapping such that $\gamma(x) = x$ for all $x \in C$. Put

$$\Psi = P + JQN + K_P(I - Q)N$$

and

$$\Psi_\gamma = \Psi \circ \gamma.$$

In order to prove the existence of positive solution of (1.1) we will apply the following result.

Theorem 2.2. [14] *Let Ω_1, Ω_2 be open bounded subsets of X with $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that 1° is satisfied and:*

- 2° *$QN : X \to Y$ is continuous and bounded and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X,*
- 3° *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda \in (0, 1)$,*
- 4° *γ maps subsets of $\overline{\Omega}_2$ into bounded subsets of C,*
- 5° *$d_B([I - (P + JQN)\gamma]|_{\operatorname{Ker} L}, \operatorname{Ker} L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree,*
- 6° *there exists $u_0 \in C \setminus \{0\}$ such that $\|x\| \leq \sigma(u_0)\|\Psi x\|$ for $x \in C(u_0) \cap \partial\Omega_1$, where*

  $$C(u_0) = \{x \in C : \mu u_0 \preceq x \quad \text{for some} \quad \mu > 0\}$$

  and $\sigma(u_0)$ is such that $\|x + u_0\| \geq \sigma(u_0)\|x\|$ for every $x \in C$,
- 7° *$(P + JQN)\gamma(\partial\Omega_2) \subset C$,*
- 8° *$\Psi_\gamma(\overline{\Omega}_2 \setminus \Omega_1) \subset C$.*

Then the equation $Lx = Nx$ has a solution in the set $C \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. A FIRST ORDER PROBLEM

Now we state and prove the main result of the paper. Consider the problem (1.1), that is

$$\begin{cases} x'(t) + a(t)x(t) = f(t, x(t)), \quad t \in [0, T], \\ x(0) = \alpha x(T), \end{cases}$$

where $\alpha > 0$ and $T > 0$ with

$$\alpha e^{-\int_0^T a(s)ds} = 1. \tag{3.1}$$

Then (1.1) is at resonance. We set

$$\varphi(t) := e^{\int_0^t a(s)ds}, \quad t \in [0, T].$$

¿From (3.1) we get $\varphi(T) = \alpha$. Moreover, we use the following notations:

$$\psi(t) := \int_0^t \frac{ds}{\varphi(s)}, \quad t \in [0, T],$$

$$k(t, s) := \frac{\varphi(s)}{\varphi(t)} \begin{cases} 1 + \dfrac{\psi(s)}{\psi(T)}, & 0 \leq s \leq t \leq T, \\ \dfrac{\psi(s)}{\psi(T)}, & 0 \leq t < s \leq T, \end{cases}$$

and

$$G(t, s) = \frac{M\varphi(s)}{\varphi(t) \int_0^T \varphi(\tau)d\tau} + k(t, s) - \frac{\int_0^T k(t, \tau)d\tau}{\int_0^T \varphi(\tau)d\tau} \varphi(s), \quad t, s \in [0, T],$$

where $M > 0$.

Assume that:

(H1) $a : [0, T] \to [0, \infty)$ and $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

It is clear that (3.1) and (H1) imply $\alpha \geq 1$.

Moreover, assume that there exist positive constants κ, M and R such that:

(H2) $\kappa M \leq \frac{1}{\alpha\psi(T)} \int_0^T \varphi(s)ds$,
(H3) $G(t, s) \geq 0$ and $\frac{1}{\varphi(t)\psi(T)} - \kappa G(t, s) \geq 0$ for t, $s \in [0, T]$,
(H4) $f(t, R) < 0$ and $f(t, \frac{R}{\varphi(t)}) < 0$ for $t \in [0, T]$,
(H5) $f(t, x) > -\kappa x$ for $(t, x) \in [0, T] \times [0, R]$,
(H6) there exist $t_0 \in [0, T]$, $r \in (0, R/\alpha)$, $\beta > 0$, $m \in (0, 1)$ and continuous functions $g : [0, T] \to [0, \infty)$, $h : (0, r] \to [0, \infty)$ such that $f(t, x) \geq g(t)h(x)$ for $(t, x) \in [0, T] \times (0, r]$, $h(x)/x^\beta$ is non-increasing on $(0, r]$ with

$$\frac{h(r)}{r} m^\beta \int_0^T G(t_0, s)g(s)ds \geq 1 - \frac{mT}{\varphi(t_0)\psi(T)}.$$

Theorem 3.1. *Under the assumptions (H1)-(H6), the problem* (1.1) *has at least one solution, positive on* $[0, T]$.

Proof. Consider the Banach spaces

$$X = Y = C[0, T]$$

with

$$\|x\| = \max_{t \in [0,T]} |x(t)|.$$

Let $L : \operatorname{dom} L \to Y$ and $N : X \to Y$ with

$$\operatorname{dom} L = \{x \in X : x' \in C[0,T],\ x(0) = \alpha x(T)\}$$

be given by

$$(Lx)(t) = x'(t) + a(t)x(t)$$

and

$$(Nx)(t) = f(t, x(t)), \quad t \in [0,T].$$

Then

$$\operatorname{Ker} L = \{x \in \operatorname{dom} L : x(t) = \frac{c}{\varphi(t)},\ c \in \mathbb{R},\ t \in [0,T]\}$$

and

$$\operatorname{Im} L = \{y \in Y : \int_0^T \varphi(s)y(s)ds = 0\}.$$

Define the projections $P : X \to X$ by

$$Px(t) = \frac{1}{\varphi(t)\psi(T)} \int_0^T x(s)ds, \quad t \in [0,T],$$

and $Q : Y \to Y$ by

$$Qy = \frac{\int_0^T \varphi(s)y(s)ds}{\int_0^T \varphi(s)ds}.$$

Then $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L$ and

$$\operatorname{Ker} P = \{x \in X : \int_0^T x(s)ds = 0\}.$$

Clearly, $\operatorname{Im} L$ is closed. Note that $Y = Y_1 \oplus \operatorname{Im} L$, where

$$Y_1 = \Big\{y_1 \in Y : y_1 = \frac{\int_0^T \varphi(s)z(s)ds}{\int_0^T \varphi(s)ds},\ z \in Y\Big\}.$$

As a result, L is Fredholm of index zero, so 1° is fulfilled. For $y \in \operatorname{Im} L$ the inverse K_P of L_P is given by

$$K_P y(t) = \int_0^T k(t,s)y(s)ds.$$

Indeed, for $y \in \operatorname{Im} L$ we have

$$L_P K_P y(t) = (K_P y)'(t) + a(t)K_P y(t) = y(t) - a(t)\int_0^t \frac{\varphi(s)}{\varphi(t)}\left(1 + \frac{\psi(s)}{\psi(T)}\right) y(s)ds$$
$$- a(t)\int_t^T \frac{\varphi(s)}{\varphi(t)}\frac{\psi(s)}{\psi(T)} y(s)ds + a(t)K_P y(t) = y(t).$$

On the other hand, for $x \in \operatorname{Ker} P$ we obtain

$$\int_0^T \varphi(s)\psi(s)x'(s)ds = \varphi(T)\psi(T)x(T) - \int_0^T [a(s)\varphi(s)\psi(s) + 1]\, x(s)ds$$
$$= \alpha\psi(T)x(T) - \int_0^T a(s)\varphi(s)\psi(s)x(s)ds.$$

Hence

$$
\begin{aligned}
K_P L_P x(t) &= \int_0^T k(t,s)(x'(s)+a(s)x(s))ds \\
&= \int_0^T \frac{\varphi(s)\psi(s)}{\varphi(t)\psi(T)} x'(s)ds + \int_0^t \frac{\varphi(s)}{\varphi(t)} x'(s)ds + \int_0^T k(t,s)a(s)x(s)ds \\
&= \frac{1}{\varphi(t)\psi(T)} \left(\alpha\psi(T)x(T) - \int_0^T a(s)\varphi(s)\psi(s)x(s)ds \right) \\
&\quad + x(t) - \frac{1}{\varphi(t)} x(0) - \int_0^t \frac{\varphi(s)}{\varphi(t)} a(s)x(s)ds + \int_0^T k(t,s)a(s)x(s)ds = x(t).
\end{aligned}
$$

It follows from (H1) that 2° is satisfied. Now define an isomorphism between $\operatorname{Im} Q$ and $\operatorname{Ker} L$ by

$$J(c)(t) = \frac{Mc}{\varphi(t)}, \quad t \in [0,T],$$

and consider the sets

$$C = \{x \in X : x(t) \geq 0 \text{ on } [0,1]\},$$

$$\Omega_1 = \{x \in X : r > |x(t)| > m\|x\|, \ t \in [0,1]\}$$

and

$$\Omega_2 = \{x \in X : \|x\| < R\}.$$

Clearly, C is a cone in X, Ω_1 and Ω_2 are open and bounded and (see [14])

$$\overline{\Omega}_1 = \{x \in X : r \geq |x(t)| \geq m\|x\|, \ t \in [0,1]\} \subset \Omega_2.$$

Note that $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. To show that 3° holds suppose that there exist $x_0 \in C \cap \partial\Omega_2 \cap \operatorname{dom} L$ and $\lambda_0 \in (0,1)$ such that $Lx_0 = \lambda_0 N x_0$. Then

$$x_0'(t) + a(t)x_0(t) = \lambda_0 f(t, x_0(t)), \quad t \in [0,T].$$

Let $t^* \in [0,T]$ be such that $x_0(t^*) = R$. Then in view of (H1) and (H4) we have

$$0 \leq a(t^*)R = \lambda_0 f(t^*, R) < 0,$$

a contradiction. Let $(\gamma x)(t) = |x(t)|$ for $x \in X$. Then γ is a retraction and maps subsets of $\overline{\Omega}_2$ into bounded subsets of C. Next we show that 5° is satisfied. In order to do this, for $x \in \operatorname{Ker} L \cap \Omega_2$, $\lambda \in [0,1]$ and $t \in [0,T]$ define

$$H(x,\lambda)(t) = x(t) - \frac{\lambda}{\varphi(t)} \Big[\frac{1}{\psi(T)} \int_0^T |x(s)|ds + \frac{M}{\int_0^T \varphi(s)ds} \int_0^T f(s,|x(s)|)\varphi(s)ds \Big].$$

Suppose that $H(x,\lambda)=0$ for $x\in \operatorname{Ker} L\cap\partial\Omega_2$, that is, for $x(t)=\dfrac{c}{\varphi(t)}$ with $\|x\|=R$. By (H2) and (H5) we get

$$\begin{aligned} c &= \lambda\Big[\frac{1}{\psi(T)}\int_0^T \frac{|c|}{\varphi(s)}ds + \frac{M}{\int_0^T\varphi(s)ds}\int_0^T f(s,\frac{|c|}{\varphi(s)})\varphi(s)ds\Big] \\ &\geq \lambda\Big[\frac{1}{\psi(T)}\int_0^T \frac{|c|}{\varphi(s)}ds - \frac{\kappa M}{\int_0^T\varphi(s)ds}\int_0^T \frac{|c|}{\varphi(s)}\varphi(s)ds\Big] \\ &= \lambda|c|\Big[1-\frac{\kappa MT}{\int_0^T\varphi(s)ds}\Big]\geq 0. \end{aligned}$$

Therefore $c=R$. This gives

$$\begin{aligned} R &= \lambda\Big[\frac{1}{\psi(T)}\int_0^T \frac{R}{\varphi(s)}ds + \frac{M}{\int_0^T\varphi(s)ds}\int_0^T f(s,\frac{R}{\varphi(s)})\varphi(s)ds\Big] \\ &= \lambda R + \frac{\lambda M}{\int_0^T\varphi(s)ds}\int_0^T f(s,\frac{R}{\varphi(s)})\varphi(s)ds. \end{aligned}$$

Hence

$$0\leq R(1-\lambda)=\frac{\lambda M}{\int_0^T\varphi(s)ds}\int_0^T f(s,\frac{R}{\varphi(s)})\varphi(s)ds,$$

contrary to (H4). This gives $H(x,\lambda)\neq 0$ for $x\in\partial\Omega_2$ and $\lambda\in[0,1]$. As a consequence we have,

$$d_B(H(x,0),\operatorname{Ker} L\cap\Omega_2,0)=d_B(H(x,1),\operatorname{Ker} L\cap\Omega_2,0).$$

This implies

$$d_B\big([I-(P+JQN)\gamma]\big|_{\operatorname{Ker} L},\operatorname{Ker} L\cap\Omega_2,0\big)\neq 0.$$

To show that 6° is fulfilled, set $u_0(t)\equiv 1$ on $[0,T]$. Then $u_0\in C\setminus\{0\}$, $C(u_0)=\{x\in C: x(t)>0 \text{ on } [0,T]\}$ and we can choose $\sigma(u_0)=1$. For $x\in C(u_0)\cap\partial\Omega_1$ we have $x(t)>0$ on $[0,T]$, $0<\|x\|\leq r$ and $x(t)\geq m\|x\|$ on $[0,T]$. Hence, by (H6), we get for all $x\in C(u_0)\cap\partial\Omega_1$

$$\begin{aligned} (\Psi x)(t_0) &= \frac{1}{\varphi(t_0)\psi(T)}\int_0^T x(s)ds + \int_0^T G(t_0,s)f(s,x(s))ds \\ &\geq \frac{1}{\varphi(t_0)\psi(T)}\int_0^T m\|x\|ds + \int_0^T G(t_0,s)g(s)h(x(s))ds \\ &\geq \frac{1}{\varphi(t_0)\psi(T)}Tm\|x\| + \int_0^T G(t_0,s)g(s)\frac{h(x(s))}{x^\beta(s)}x^\beta(s)ds \\ &\geq \frac{1}{\varphi(t_0)\psi(T)}Tm\|x\| + \int_0^T G(t_0,s)g(s)\frac{h(r)}{r^\beta}m^\beta\|x\|^\beta ds \\ &= \frac{1}{\varphi(t_0)\psi(T)}Tmr + h(r)m^\beta\int_0^T G(t_0,s)g(s)ds\geq r=\|x\|. \end{aligned}$$

¿From (H2) and (H5) we have for $x \in \partial\Omega_2$

$$
\begin{aligned}
(P+JQN)\gamma x(t) &= \frac{1}{\varphi(t)\psi(T)}\int_0^T |x(s)|ds + \frac{M}{\varphi(t)\int_0^T \varphi(s)ds}\int_0^T f(s,|x(s)|)\varphi(s)ds \\
&\geq \frac{1}{\varphi(t)}\Big[\frac{1}{\psi(T)}\int_0^T |x(s)|ds - \frac{\kappa M}{\int_0^T \varphi(s)ds}\int_0^T |x(s)|\varphi(s)ds\Big] \\
&\geq \frac{1}{\varphi(t)}\int_0^T \Big[\frac{1}{\psi(T)} - \frac{\kappa M\alpha}{\int_0^T \varphi(\tau)d\tau}\Big]|x(s)|ds \geq 0.
\end{aligned}
$$

This means that 7° holds.

Finally, from (H3) and (H5) we obtain for all $x \in \overline{\Omega}_2 \setminus \Omega_1$ and $t \in [0,T]$,

$$
\begin{aligned}
\Psi_\gamma x(t) &= \frac{1}{\varphi(t)\psi(T)}\int_0^T |x(s)|ds + \int_0^T G(t,s)f(s,|x(s)|)ds \\
&\geq \frac{1}{\varphi(t)\psi(T)}\int_0^T |x(s)|ds - \kappa\int_0^T G(t,s)|x(s)|ds \geq 0,
\end{aligned}
$$

which implies 8°. This completes the proof. □

Remark 3.2. Observe that the assumption (H4) is fulfilled if the function f satisfies the following condition

(H4') $f(t,x) < 0$ for $(t,x) \in [0,T] \times [R/\alpha, R]$.

Remark 3.3. It is to be noted that for $T = 1$, $\alpha = 1$ and $a(t) \equiv 0$ on $[0,1]$, Theorem 3.1 extends the existence results for the problem

$$
\begin{cases} x'(t) = f(t,x(t)), \quad t \in [0,1], \\ x(0) = x(1), \end{cases}
$$

obtained in [14] and [17]. In this case the assumption (H4) reduces to one condition $f(t,R) < 0$ for $t \in [0,1]$. The use of the constants M and m allows us to relax the conditions imposed on κ and f.

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