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THE ∞-EIGENVALUE PROBLEM AND A PROBLEM OF OPTIMAL TRANSPORTATION

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Dedicated to Prof. Espedito De Pascale
in occasion of his retirement.

ABSTRACT. The so-called eigenvalues and eigenfunctions of the infinite Laplacian Δ_∞ are defined through an asymptotic study of that of the usual p-Laplacian Δ_p, this brings to a characterization via a non-linear eigenvalue problem for a PDE satisfied in the viscosity sense. In this paper, we obtain an other characterization of the first eigenvalue via a problem of optimal transportation, and recover properties of the first eigenvalue and corresponding positive eigenfunctions.

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1. INTRODUCTION

An eigenvalue of the p-Laplacian is a real number $\lambda \in \mathbb{R}$ such that the problem

$$\begin{cases} -div(|Du|^{p-2}Du) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

has at least one non trivial solution in $W_0^{1,p}(\Omega)$. Here solution is intended in the distributional sense and Ω is assumed to be a regular, bounded, open subset of $\mathbb{R}^N$.

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Much is unknown about the eigenvalues of the p-Laplacian and we will give a short presentation of some related open questions in section §2.

In this paper, we shall focus on the asymptotic of the above eigenvalue problem as the parameter p goes to $+\infty$. This is a standard strategy in analysis (for example in the homogenization and relaxation theories) to look at the asymptotic problem and then to try to deduce qualitative and quantitative informations on the approximating problems and the limit problem as well as reasonable conjectures.

The asymptotic as $p \to \infty$ of the p-Laplacian eigenvalue problem was introduced in [25] and then perfectioned in [26, 24, 14]. In these papers the authors proved that if $(\lambda_p)_{N<p<\infty}$ is a generalized sequence of eigenvalues of the p-Laplacian such that $\lim_{p\to\infty} \lambda_p^{1/p} = \Lambda$ and u_p are corresponding eigenfunctions such that $\|u_p\|_p \leq C$ and $u_p \to u$ uniformly, then u is a viscosity solution of

$$\begin{cases} \min\{|\nabla u| - \Lambda u\,,\, -\Delta_\infty u\} = 0 & \text{in } \{u > 0\}, \\ -\Delta_\infty u = 0 & \text{in } \{u = 0\}, \\ \max\{-|\nabla u| - \Lambda u\,,\, -\Delta_\infty u\} = 0 & \text{in } \{u < 0\}, \end{cases} \tag{1.1}$$

where the infinite Laplacian of u is given by $\Delta_\infty u = \displaystyle\sum_{i,j} u_{x_i x_j} u_{x_i} u_{x_j}$. According to the definition given in [24] this means that u is an eigenfunction of the ∞-Laplacian for the $\infty-$eigenvalue Λ.

The aim of this paper is to introduce a different asymptotic problem as $p \to \infty$ of the first eigenvalue problem which relates the problem to an optimal transportation problem, to start an analysis of the limiting problem as well as propose some related questions and a few answers. The idea that a transport equation appears in the limit as $p \to \infty$ goes back to [8]. The explicit connection of this limit with the optimal transportation problem was first exploited in [19] and in the setting of the eigenvalues problems appeared also in [22].

The main reason to focus our study on the first eigenvalue is that the restriction $u_{\lambda,V}$ of an eigenfunction u_λ (for some eigenvalue λ of the p-Laplacian operator) to one of its nodal domains V is indeed an eigenfunction for the first eigenvalue of the corresponding p-Laplacian operator for this domain V. A close study on the first eigenvalue (and related eigenfunctions) of the p-Laplacian operator is then of great help to understand the properties of the eigenfunctions of higher eigenvalues. This was in particular illustrated in [24].

The paper is organized as follows. Section §2 is devoted to review basic notions and results concerning the eigenvalues of the p-Laplacian. In section §3 we propose a new asymptotic analysis as p goes to ∞, and make the link with an optimal transport problem in section §4. In the final section §5 we show how the proposed asymptotic analysis may be applied to obtain some informations on the limits obtained.

2. DEFINITIONS AND PRELIMINARY RESULTS

Nonlinear eigenvalues of the p-Laplacian.

We shall denote by $\|\cdot\|_p$ the usual norm of $L^p(\Omega)$ (or $L^p(\Omega;\mathbb{R}^N)$ when dealing with the gradient of some element of $W_0^{1,p}(\Omega)$).

An eigenvalue of the $p-$Laplacian operator $-\Delta_p$ is a real number λ for which the problem

$$(\mathrm{P}_p^\lambda) \qquad \begin{cases} -\Delta_p u := -div(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

has a non-zero solution in $W_0^{1,p}(\Omega)$. This problem (and its generalizations to monotone elliptic operators) has been widely studied in the literature and for more detailed treatment we refer to [3, 9, 15, 20, 21, 24, 27]. Much is still unknown about the eigenvalues of the $p-$Laplacian operator. A good understanding of the set of the eigenvalues would permit some progress on more general nonlinear equations involving the p-Laplacian (e.g. a good definition of jumping nonlinearity) as well as some progress on parabolic equations involving the p-Laplacian. Let us report some classical results. It is known that λ is an eigenvalue if and only if it is a critical value for the Rayleigh quotient

$$v \mapsto \frac{\int_\Omega |\nabla v|^p dx}{\int_\Omega |v|^p dx} \quad \left(= \frac{\|\nabla v\|_p^p}{\|v\|_p^p} \right)$$

which is a Gateaux differentiable functional on $W_0^{1,p}(\Omega)$ outside the origin. Moreover, a sequence $(\lambda_p^k)_{k\geq 1}$ of eigenvalues can be obtained as follows (we refer to [20] and [27] for details). Denote by $\Sigma_p^k(\Omega)$ the set of those subsets G of $W_0^{1,p}(\Omega)$ which are symmetric (i.e. $G = -G$), contained in the set $\{v : \|v\|_p = 1\}$, strongly compact in $W_0^{1,p}(\Omega)$ and with Krasnoselskii genus $\gamma(G) \geq k$ (we refer to [29] for more details on the Krasnoselskii genus), and set

$$\lambda_p^k \ = \inf_{G\in\Sigma_k^p(\Omega)} \ \sup_{u\in G} \|\nabla u\|_p^p.$$

Then each λ_p^k defined as above is an eigenvalue of the p-Laplacian operator and $\lambda_p^k \to +\infty$ as $k \to \infty$. Moreover λ_p^1 is the smallest eigenvalue of $-\Delta_p$, it is simple (see [2] for the original proof or [7] for a short proof) and the operator $-\Delta_p$ doesn't have any eigenvalue between λ_p^1 and λ_p^2.

A second sequence $(\mu_p^k)_k$ of eigenvalues was introduced in Theorem 5 of [18]. This sequence is also obtained by a $\inf - \sup$ operation but in this case the inf operation is performed on a smaller class of sets than Σ_p^k (we refer the reader to [18] for more details). It is only known that $\lambda_p^1 = \mu_p^1$ and $\lambda_p^2 = \mu_p^2$. Some interesting questions related to our analysis are the following: does it hold $\lambda_p^k = \mu_p^k$ for all p and k? Is it true that $\{\lambda_p^k\}_{k\geq 1}$ is the entire set of eigenvalues?

The relevance of these questions may be also understood in the light of a theorem of Fredholm alternative for the p-Laplacian which appear in [3] (namely theorem 12.12 therein).

Finally let us report a basic estimate for the first eigenvalue which is a consequence of the following characterization:

$$\lambda_p^1 = \min\left\{\int_\Omega |\nabla u|^p dx \mid u \in W_0^{1,p}(\Omega),\ \|u\|_p = 1\right\}. \tag{2.2}$$

Denote by

$$R_1 = \sup\{r|\ \exists x_0 \text{ s.t. } B(x_0, r) \subset \Omega\},$$

the radius of the biggest ball inscribed in Ω then

Lemma 2.1. *For each $p \in [1, \infty)$, we have $(\lambda_p^1)^{1/p} \leq \frac{1}{R_1}$ and then*

$$\limsup_{p\to\infty} (\lambda_p^1)^{1/p} \leq \frac{1}{R_1}.$$

Proof. Let $B(\overline{x}, R_1)$ be a ball inscribed in Ω, then $v(x) := \max\{R_1 - |x - \overline{x}|, 0\}$ belongs to $W_0^{1,p}(\Omega)$ and it is enough to test the minimality in (2.2) against $v/\|v\|_p$ to obtain the desired estimate. $\square$

As the main focus of the paper will be on the generalized sequence of the first eigenvalue we will simplify the notations and write λ_p for λ_p^1. Up to subsequences we may then assume that $(\lambda_p)^{1/p} \to \Lambda_\infty$ and we will in fact prove that $\Lambda_\infty = \frac{1}{R_1}$. This has already been proved in [25] and then in [24, 14]. Here we deduce this equality from a minimality property of u_p and from the Monge-Kantorovich (or optimal transportation) problem obtained in the limit as $p \to \infty$.

Γ-convergence. A crucial tool in the analysis of this paper will be the following concept of Γ-convergence.

Let X be a metric space, a sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ is said to Γ-converge to F_∞ at x if

$$F_\infty(x) \ = \Gamma - \liminf F_n(x) \ = \Gamma - \limsup F_n(x), \tag{2.3}$$

where

$$\begin{cases} \Gamma - \liminf F_n(x) &= \inf \left\{ \liminf F_n(x_n) : \quad x_n \to x \textit{ in } X \right\}, \\ \Gamma - \limsup F_n(x) &= \inf \left\{ \limsup F_n(x_n) : \quad x_n \to x \textit{ in } X \right\}. \end{cases} \tag{2.4}$$

The Γ-convergence was introduced in [16], for an introduction to this theory we refer to [17] and [5]. We report a classical theorem which includes some properties of the Γ-convergence that we shall use in the following.

Theorem 2.2. *Assume that the sequence $(F_n)_{n\in\mathbb{N}}$ of functionals Γ-converges to F_∞ on X. Assume in addition that the sequence $(F_n)_n$ is equi-coercive on X. Then*

$$\lim_{n\to+\infty}\left(\inf_{x\in X} F_n(x)\right) = \inf_{x\in X} F_\infty(x)$$

and one has $F_\infty(x_\infty) = \inf_{x\in X} F_\infty(x)$ for any cluster point x_∞ of a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$$\forall n\in\mathbb{N} \qquad F_n(x_n) \le \inf_{x\in X} F_n(x) + \varepsilon_n$$

with $\varepsilon_n \to 0$ as $n\to\infty$.

3. THE ASYMPTOTIC BEHAVIOR AS $p\to\infty$.

Recall that, for any $p > N$, λ_p stands for the first eigenvalue of the p-Laplace operator. We shall denote by u_p the unique corresponding eigenfunction which is positive in Ω and such that

$$\|u_p\|_p = \left(\int_\Omega u_p^p(x)dx\right)^{1/p} = 1. \tag{3.1}$$

We also introduce the following measures:

$$\sigma_p := \frac{|\nabla u_p|^{p-2}\nabla u_p}{\lambda_p}dx, \quad f_p := u_p^{p-1}dx, \quad \mu_p := \frac{|\nabla u_p|^{p-2}}{\lambda_p}dx. \tag{3.2}$$

Lemma 3.1. *The above measures satisfy the following inequalities for $p > 2$:*

$$\int_\Omega \left|\frac{\nabla u_p}{\lambda_p^{1/p}}\right|^p dx = 1, \qquad \int_\Omega d|f_p| \le |\Omega|^{1/p},$$
$$\int_\Omega d|\mu_p| \le |\Omega|^{2/p}, \qquad \int_\Omega d|\sigma_p| \le |\Omega|^{1/p}.$$

Then there exists $u_\infty \in \mathrm{Lip}(\Omega)\cap\mathcal{C}_0(\Omega)$ with $\|u_\infty\|_\infty = 1$, $f_\infty \in \mathcal{M}_b^+(\overline{\Omega})$ a probability measure, $\mu_\infty \in \mathcal{M}_b^+(\overline{\Omega})$ and $\xi_\infty \in L^1_{\mu_\infty}(\Omega)^d$ such that, up to subsequences:

$$u_p \to u_\infty \text{ uniformly on } \overline{\Omega}, \quad f_p \overset{*}{\rightharpoonup} f_\infty \text{ in } \mathcal{M}_b(\overline{\Omega}),$$
$$\mu_p \overset{*}{\rightharpoonup} \mu_\infty \text{ in } \mathcal{M}_b^+(\overline{\Omega}), \quad \sigma_p \overset{*}{\rightharpoonup} \sigma_\infty := \xi_\infty\mu_\infty \text{ in } \mathcal{M}_b(\overline{\Omega},\mathbb{R}^N).$$

Proof. The second bound is an easy consequence of Hölder's inequality and of the assumption $\int |u_p|^p\, dx = 1$. To obtain the remaining estimates, it is sufficient to show the first equality and then apply Hölder's inequality. As u_p solves (2.1), by multiplying the PDE (2.1) by u_p and integrating by parts we get

$$\int_\Omega |\nabla u_p|^p\, dx = \lambda_p \int_\Omega |u_p|^p\, dx = \lambda_p.$$

By the above estimates, for any $N \le q < +\infty$, $(u_p)_{p>q}$ is bounded in $W_0^{1,q}(\Omega)$, more precisely, using Holder's inequality, we get:

$$\int_\Omega |\nabla u_p(x)|^q\, dx \le (\int_\Omega |\nabla u_p|^p dx)^{\frac{q}{p}}|\Omega|^{1-q/p} = (\lambda_p^{1/p})^q|\Omega|^{1-q/p}.$$

As a consequence, fixing $q > N$, we obtain that $(u_p)_{p>q}$ is precompact in $\mathcal{C}(\overline{\Omega})$ and, up to subsequences, the uniform convergence to some u_∞ holds.

Using again the estimates above, we get (up to subsequences) the existence of a weak* limit f_∞ for $(f_p)_p$, σ_∞ for $(\sigma_p)_p$ and μ_∞ for $(\mu_p)_p$ in $\mathcal{M}_b(\overline{\Omega})$. Note that, as we are on a compact set, the convergence of $(f_p)_p$ is tight. ¿From this convergence it comes that $|f_\infty(\overline{\Omega})| \leq 1$. To obtain the reverse inequality we observe that for all p one has $\int u_p df_p = 1$ so that in the limit $\int u_\infty df_\infty = 1$. On the other hand it follows from the Holder inequality applied with $1 < q < p$ that

$$\|u_p\|_q \;\leq\; \|u_p\|_p \, |\Omega|^{\frac{1}{q}-\frac{1}{p}} \;=\; |\Omega|^{\frac{1}{q}-\frac{1}{p}}.$$

Taking the limit as $p \to +\infty$ and then as $q \to +\infty$ yields $\|u_p\|_\infty \leq 1$. Therefore

$$1 \;=\; \int u_\infty df_\infty \;\leq\; \|u_\infty\|_\infty \, |f_\infty(\overline{\Omega})| \;\leq\; 1$$

so that f_∞ is a probability measure on $\overline{\Omega}$. Moreover, thanks to lemma 3.1 of [11], we can write $\sigma_\infty = \xi_\infty \mu_\infty$ for some $\xi_\infty \in L^1_{\mu_\infty}(\Omega)^d$.

□

We devote the rest of the paper to the properties of the limits $u_\infty,\ f_\infty,\ \sigma_\infty,\ \mu_\infty$.

A first Γ-convergence approach. If we consider f_p as known, we may introduce the following variational problem:

$$(\mathcal{P}_p) \qquad \min_{u \in W_0^{1,p}(\Omega)} \left\{ \frac{1}{p\lambda_p} \int_\Omega |\nabla u(x)|^p \, dx - \langle f_p, u \rangle \right\}.$$

By the definitions of u_p and f_p, it follows that u_p is the unique minimizer of $(\mathcal{P}_p)$. Moreover, since the solution set of the problem $(\mathrm{P}_p^{\lambda_p})$ is spanned by u_p, we may consider $(\mathcal{P}_p)$ as a variational formulation of (2.1) for $\lambda = \lambda_p$. Then we have:

Proposition 3.2. *The sequence* $(\min(\mathcal{P}_p))_p$ *converges to the minimum of the following optimization problem:*

$$(\mathcal{P}_\infty) \quad \min\{- < f_\infty, u >: \ u \in \mathrm{Lip}(\Omega),\ |\nabla u| \leq \Lambda_\infty \ a.e.,\ u = 0 \ on \ \partial\Omega\},$$

and u_∞ *minimizes* $(\mathcal{P}_\infty)$.

Proof. For $p > N$ let $F_p : \mathcal{C}_0(\Omega) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$F_p(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega \left| \frac{\nabla u}{\lambda_p^{1/p}} \right|^p dx - \langle f_p, u \rangle & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise.} \end{cases}$$

We claim that the family $(F_p)_{p>N}$ Γ-converges in $\mathcal{C}_0(\Omega)$ to F_∞ given by

$$F_\infty(u) := \begin{cases} -\langle f_\infty, u \rangle & \text{if } u \in \mathrm{Lip}(\Omega) \text{ and } |\nabla u| \leq \Lambda_\infty \text{ a.e. in } \Omega \\ +\infty, & \text{otherwise,} \end{cases}$$

with respect to the norm of the uniform convergence. We first show the $\Gamma - \liminf$ inequality, that is:

$$\Gamma - \liminf_{p\to+\infty} F_p \geq F_\infty \tag{3.3}$$

Let $(v_p)_{p>N}$ converging uniformly to v, then we have:

$$\langle f_p, v_p\rangle \to \langle f_\infty, v\rangle. \tag{3.4}$$

We shall prove that $\liminf_{p\to+\infty} F_p(v_p) \geq F_\infty(v)$. We may assume that $\liminf_{p\to+\infty} F_p(v_p) < +\infty$, that is (thanks to (3.4)):

$$M \ := \ \liminf_{p\to+\infty} \left(\frac{1}{p}\int_\Omega \left|\frac{\nabla v_p}{\lambda_p^{1/p}}\right|^p dx\right) < +\infty.$$

It then remains to check that v is Lipschitz continuous and satisfies $|\nabla v| \leq \Lambda_\infty$ a.e. in Ω. Let $N < q < p$, then the $W^{1,q}$-norm of $(\frac{v_p}{\lambda_p^{1/p}})_p$ is bounded. Indeed, as for $t > 0$ the function $s \mapsto \frac{(t^s-1)}{s}$ is monotone increasing on $]0, +\infty[$:

$$\frac{1}{q}\int_\Omega \left|\frac{\nabla v_p}{\lambda_p^{1/p}}\right|^q dx \leq \frac{1}{p}\int_\Omega \left|\frac{\nabla v_p}{\lambda_p^{1/p}}\right|^p dx + (1/q - 1/p)|\Omega|.$$

Then, possibly extracting a subsequence we may assume $\frac{v_p}{\lambda_p^{1/p}} \rightharpoonup \frac{v}{\Lambda_\infty}$ in $W_0^{1,q}(\Omega)$ and then:

$$\left(\int_\Omega \left|\frac{\nabla v}{\Lambda_\infty}\right|^q dx\right)^{1/q} \leq \liminf_{p\to\infty} \left(\int_\Omega \left|\frac{\nabla v_p}{\lambda_p^{1/p}}\right|^q dx\right)^{1/q} \leq (q\, M - |\Omega|)^{1/q}.$$

Letting q go to $+\infty$ we get $|\nabla v| \leq \Lambda_\infty$ almost everywhere on Ω. This concludes the proof of (3.3). The $\Gamma - \limsup$ inequality, i.e. $\Gamma - \limsup_{p\to+\infty} F_p(v) \leq F(v)$, follows by considering the constant sequence $(v_p)_{p\geq 1} := (v)_{p\geq 1}$.

The Proposition now follows as a consequence of Theorem 2.2 and of the uniform convergence of $(u_p)_p$ to u_∞. □

In Proposition 3.6 below, we shall see that the measure σ_∞ plays its role in the classical dual problem $(\mathcal{P}_\infty^*)$ associated to $(\mathcal{P}_\infty)$, and given by

$$(\mathcal{P}_\infty^*) \qquad \min_{\lambda\in\mathcal{P}(\partial\Omega)} \min_{\sigma\in\mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int_{\mathbb{R}^N} |\sigma| :\ -\mathrm{div}(\sigma) = f_\infty - \lambda \text{ in } \mathbb{R}^N\}.$$

To identify $(\mathcal{P}_\infty^*)$ as the dual problem of $(\mathcal{P}_\infty^*)$, we use the classical convex duality:

Proposition 3.3 (Duality for the limit problem)**.** *By convex duality it holds*

$$\min(\mathcal{P}_\infty) = -\min(\mathcal{P}_\infty^*).$$

Moreover the minimum of $(\mathcal{P}_\infty^*)$ *can also be expressed as:*

$$\min(\mathcal{P}_\infty^*) = \min_{\sigma\in\mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int_{\overline{\Omega}} |\sigma| :\ \mathrm{spt}(\sigma) \subset \overline{\Omega},\ -\mathrm{div}(\sigma) \in \mathcal{M}_b(\mathbb{R}^N) \textit{ and } -\mathrm{div}(\sigma) = f_\infty \textit{ in } \Omega\}.$$

The equalities $-\mathrm{div}(\sigma) = f_\infty - \lambda$ in $\mathbb{R}^N$ and $-\mathrm{div}(\sigma) = f_\infty$ in Ω should be understood in the sense of distributions, that is:

$$-\mathrm{div}(\sigma) = f_\infty - \lambda \text{ in } \mathbb{R}^N \quad \text{means: } \int \nabla\varphi \cdot \sigma = \int \varphi \, d(f_\infty - \lambda) \quad \forall \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^N),$$

$$-\mathrm{div}(\sigma) = f_\infty \text{ in } \Omega \quad \text{means: } \int \nabla\varphi \cdot \sigma = \int \varphi \, df_\infty \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega).$$

The proof of Proposition 3.3 requires the following Lemma:

Lemma 3.4. *Let $u \in \mathrm{Lip}(\Omega)$ such that $|\nabla u| \le \Lambda_\infty$ a.e. in Ω and $u = 0$ on $\partial\Omega$. Then there exists a sequence $(u_n)_n$ in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ such that for any $n \in \mathbb{N}$:*

$$u_n \to u \textit{ uniformly in } \overline{\Omega}$$

$$u_n \textit{ is } \Lambda_\infty\textit{-Lipschitz and } u_n = 0 \textit{ on a neighborhood of } \partial\Omega.$$

Proof. We denote by $\tilde{u}$ the function u extended by 0 outside Ω. For any $\varepsilon > 0$ we set:

$$\theta_\varepsilon(t) = \begin{cases} 0 & \text{if } |t| \le \Lambda_\infty\varepsilon \\ t - \mathrm{sign}(t)\Lambda_\infty\varepsilon & \text{if } |t| \ge \Lambda_\infty\varepsilon. \end{cases}$$

The function $\theta_\varepsilon \circ \tilde{u}$ remains Λ_∞-Lipschitz and satisfies:

$$\theta_\varepsilon \circ \tilde{u}(x) = 0 \text{ as soon as } d(x, \partial\Omega) \le \varepsilon. \tag{3.5}$$

We now make a standard regularization by convolution setting for any $\varepsilon > 0$ and $n \in \mathbb{N}$:

$$\psi_{n,\varepsilon}(x) = \int_{B(0,1/n)} \rho_n(x)(\theta_\varepsilon \circ \tilde{u})(x - y) \, dy$$

where $\rho_n := \frac{1}{n}\rho(n \times \cdot)$ is a standard mollifier obtained from a function ρ satisfying

$$\rho \in \mathcal{C}^\infty(\mathbb{R}^N, [0, +\infty[), \quad \mathrm{spt}(\rho) \subset B(0,1), \quad \int_{B(0,1)} \rho(x) \, dx = 1.$$

For any $n \ge \frac{2}{\varepsilon}$, the function $\psi_{n,\varepsilon}$ is $\mathcal{C}^1$, Λ_∞-Lipschitz and, by (3.5), equals 0 on $\mathbb{R}^N \setminus \{x \in \Omega, d(x, \partial\Omega) \le \frac{\varepsilon}{2}\}$. Moreover we have the following convergences:

$$\psi_{n,\varepsilon} \to \theta_\varepsilon \circ \tilde{u} \text{ uniformly on } \overline{\Omega} \text{ as } n \to +\infty,$$

$$\theta_\varepsilon \circ \tilde{u} \to \tilde{u} \text{ uniformly on } \overline{\Omega} \text{ as } \varepsilon \to 0.$$

By extracting a diagonal subsequence of $(\psi_{n,\varepsilon})_{n,\varepsilon}$, we get the desired sequence $(u_n)_n$. ☐

Proof of Proposition 3.3. The above lemma allows us to rewrite problem $(\mathcal{P}_\infty)$ in the following way:

$$\min(\mathcal{P}_\infty) = \inf\{- < f_\infty, u >: \; u \in \mathcal{C}^1(\mathbb{R}^N) \cap \mathcal{C}_c(\mathbb{R}^N), \; |\nabla u| \le \Lambda_\infty, \; u = 0 \text{ on } \partial\Omega\}.$$

We introduce the operator $A : \mathcal{C}_c(\mathbb{R}^N) \to \mathcal{C}_c(\mathbb{R}^N)^N$ of domain $\mathcal{C}^1(\mathbb{R}^N) \cap \mathcal{C}_c(\mathbb{R}^N)$ defined as $Au := \nabla u$ for all u in its domain. We also introduce the characteristic functions $\chi_{B_{\Lambda_\infty}}$ and χ_C defined by:

$$\forall \Phi \in \mathcal{C}_c(\mathbb{R}^N)^N, \qquad \chi_{B_{\Lambda_\infty}}(\Phi) = \begin{cases} 0 & \text{if } \ |\Phi(x)| \leq \Lambda_\infty, \quad \forall x \in \mathbb{R}^N \\ +\infty & \text{elsewhere.} \end{cases}$$

$$\forall \varphi \in \mathcal{C}_c(\mathbb{R}^N). \qquad \chi_C(\varphi) = \begin{cases} 0 & \text{if } \ \varphi(x) = 0, \quad \forall x \in \partial\Omega \\ +\infty & \text{elsewhere.} \end{cases}$$

We have:

$$\begin{aligned} \min(\mathcal{P}_\infty) &= -\max\{< f_\infty, u > -(\chi_{B_{\Lambda_\infty}} \circ A + \chi_C)(u) : \ u \in \mathcal{C}_c(\mathbb{R}^N)\} \\ &= -(\chi_{B_{\Lambda_\infty}} \circ A + \chi_C)^*(f_\infty) = -\left((\chi_{B_{\Lambda_\infty}} \circ A)^* \bigtriangledown \chi_C^*\right)^{**}(f_\infty) \end{aligned}$$

where $\bigtriangledown$ is the inf-convolution, that is for all $f \in \mathcal{M}_b^+(\mathbb{R}^N)$:

$$(\chi_{B_{\Lambda_\infty}} \circ A)^* \bigtriangledown \chi_C^*(f) = \inf_{\lambda \in \mathcal{M}_b^+(\mathbb{R}^N)} \{(\chi_{B_{\Lambda_\infty}} \circ A)^*(f - \lambda) + \chi_C^*(\lambda)\}.$$

Now, by classical computations, we have that for all $\lambda \in \mathcal{M}_b^+(\mathbb{R}^N)$

$$\begin{aligned} (\chi_{B_{\Lambda_\infty}} \circ A)^*(f - \lambda) &= \inf_{\sigma \in \operatorname{dom} A^*} \{\chi_{B_{\Lambda_\infty}}^*(\sigma) : \ A^*(\sigma) = f - \lambda\} \\ &= \inf_{\sigma \in \mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int |\sigma| : \ -\operatorname{div}(\sigma) = f - \lambda \text{ in } \mathbb{R}^N\} \end{aligned}$$

and:

$$\chi_C^*(\lambda) = \sup_{u \in \mathcal{C}_c(\mathbb{R}^N), \ u = 0 \text{ on } \partial\Omega} < \lambda, u >= \begin{cases} 0 & \text{if } \operatorname{spt}(\lambda) \subset \partial\Omega \\ +\infty & \text{elsewhere.} \end{cases}$$

The inf-convolution thus gives:

$$(\chi_{B_{\Lambda_\infty}} \circ A)^* \bigtriangledown \chi_C^*(f) = \inf_{\lambda \in \mathcal{M}_b^+(\partial\Omega)} \inf_{\sigma \in \mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int d|\sigma| : \ -\operatorname{div}(\sigma) = f - \lambda \text{ in } \mathbb{R}^N\}$$

which happens to be a convex, lower semi-continuous function in f. By consequence:

$$\min(\mathcal{P}_\infty) = -\inf_{\lambda \in \mathcal{M}_b^+(\partial\Omega)} \inf_{\sigma \in \mathcal{M}_b(\mathbb{R}^N)^N} \{\Lambda_\infty \int d|\sigma| : \ -\operatorname{div}(\sigma) = f_\infty - \lambda \text{ in } \mathbb{R}^N\}.$$

We notice that if λ is not a probability then the second infimum is $+\infty$, otherwise it is a minimum. This proves the thesis. $\square$

The previous result of course holds for the approximating problems:

Proposition 3.5 (Duality for the approximating problems)**.** *For every $p > 1$, setting $p' = \frac{p}{p-1}$, by standard duality we have:*

$$\min(\mathcal{P}_p) = -\min(\mathcal{P}_p^*) := -\min_{\sigma \in L^{p'}(\mathbb{R}^N)} \{\frac{1}{p'}\lambda_p^{p'-1}\int_\Omega |\sigma|^{p'}\ dx :\ \operatorname{spt}(\sigma) \subset \overline{\Omega},$$
$$-\operatorname{div}\sigma \in \mathcal{M}_b(\mathbb{R}^N) \textit{ and } -\operatorname{div}\sigma = f_p \textit{ in } \Omega\}. \quad (3.6)$$

Sketch of the proof. As in the proof of Proposition 3.3, it can be proved that:

$$\min(\mathcal{P}_p) = \inf\left\{(G\circ A + \chi_C)(u) - < f_p, u >:\ u \in \mathcal{C}^1(\mathbb{R}^N) \cap \mathcal{C}_c(\mathbb{R}^N)\right\} =$$
$$-(G\circ A + \chi_C)^*(f_p)$$

where $G(\Phi)$ is defined for all $\Phi \in \mathcal{C}_c(\mathbb{R}^N, \mathbb{R}^N)$ by $G(\Phi) = \frac{1}{p\lambda_p}\int |\Phi(x)|^p\ dx$. Its Fenchel transform is for any $\rho \in \mathcal{M}_b(\mathbb{R}^N, \mathbb{R}^N)$:

$$G^*(\sigma) = \begin{cases} \frac{1}{p'}\lambda_p^{p'-1}\int |\underline{\rho}|^{p'}\ dx & \text{if } \rho \ll dx \text{ with } \rho = \underline{\rho}\ dx, \\ +\infty & \text{otherwise.} \end{cases}$$

The rest of the proof follows that of Proposition 3.3. □

It can now be checked that also the dual problems converge that is:

$$\min(\mathcal{P}_p^*) \to \min(\mathcal{P}_\infty^*).$$

More precisely, one has the following:

Proposition 3.6. *The function σ_p defined in* (3.2) *is the unique minimizer of* $(\mathcal{P}_p^*)$*. Moreover, its limit σ_∞ given by Theorem* 3.2 *is a solution of* $(\mathcal{P}_\infty^*)$*. In other words, setting $\lambda_\infty := f_\infty + \operatorname{div}\sigma_\infty$, the couple $(\lambda_\infty, \sigma_\infty) \in \mathcal{P}(\partial\Omega) \times \mathcal{M}_b(\mathbb{R}^N)^N$ minimizes* $(\mathcal{P}_\infty^*)$*.*

Proof. As u_p is an eigenfunction of the p-Laplacian, recalling (3.2), σ_p is admissible for $(\mathcal{P}_p^*)$. Moreover by Lemma 3.1, we have:

$$\min(\mathcal{P}_p) = \frac{1}{p\lambda_p}\int_\Omega |\nabla u_p|^p\ dx - < f_p, u_p > = \frac{1}{p} - \int_\Omega u_p^p(x)\ dx = -\frac{1}{p'},$$

$$\text{and} \qquad \frac{1}{p'}\lambda_p^{p'-1}\int_\Omega |\sigma_p|^{p'}\ dx = \frac{1}{p'\lambda_p}\int_\Omega |\nabla u_p|^p\ dx = \frac{1}{p'}.$$

Then by (3.6), σ_p is a solution of $(\mathcal{P}_p^*)$, the uniqueness follows from the strict convexity of the functional $\sigma \mapsto \int |\sigma|^{p'}\ dx$.

Passing to the limit in the constraint of $(\mathcal{P}_p^*)$, we obtain that the measure σ_∞ satisfies $-\operatorname{div}(\sigma_\infty) = f_\infty$ in Ω. It then remains to prove that

$$\min(\mathcal{P}_\infty^*) \geq \Lambda_\infty \int_\Omega |\sigma_\infty|.$$

Following the proof of Theorem 4.2 in [11], we use the inequality $\frac{s^{p'}}{p'} \geq s - \frac{1}{p}$ for any $s > 0$, and get:

$$\min(\mathcal{P}_p^*) = \frac{1}{p'}\lambda_p^{p'-1}\int |\sigma_p|^{p'}\ dx \geq \lambda_p^{p'-1}\left(\int |\sigma_p|\ dx - \frac{|\Omega|}{p}\right).$$

Then, passing to the limit, by Corollary 3.2, we obtain:

$$\min(\mathcal{P}_\infty^*) \geq \liminf_{p\to+\infty} \lambda_p^{p'-1}\int |\sigma_p|\ dx = \liminf_{p\to+\infty} (\lambda_p^{1/p})^{p'}\int |\sigma_p|\ dx \geq \Lambda_\infty \int |\sigma_\infty|.$$

□

A second Γ-convergence approach. An other way of obtaining the problem $(\mathcal{P}_\infty)$ in a limit process, which we shall use is the following of the paper, is to define for any $p \in]N, +\infty[$ the functional $G_p : \mathcal{M}(\Omega) \times \mathcal{C}_0(\Omega) \to \overline{\mathbb{R}}$ by

$$G_p(g,v) = \begin{cases} -\langle g, v\rangle & \text{if } g \in L^{p'},\ \|g\|_{p'} \leq 1 \text{ and } v \in W_0^{1,p}(\Omega),\ \|\nabla v\|_p \leq \lambda_p^{1/p}, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.7}$$

and

$$G_\infty(g,v) = \begin{cases} -\langle g, v\rangle & \text{if } \int_\Omega d|g| \leq 1 \text{ and } v \in W_0^{1,\infty}(\Omega),\ \|\nabla v\|_\infty \leq \Lambda_\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.8}$$

For $p \in]N, +\infty[$ it happens that the couple (f_p, u_p) is a minimizer of the functional G_p. Indeed by the definitions above and (2.2) it comes

$$-G_p(g,v) = \langle g, v\rangle \leq \ \|g\|_{p'}\|v\|_p \leq \frac{1}{\lambda_p^{1/p}}\|\nabla v\|_p \leq 1 = \langle f_p, u_p\rangle = -G_p(f_p, u_p).$$

We now notice that this property does also hold in the limit $p = +\infty$:

Proposition 3.7. *Let $\alpha > 0$, then the generalized sequence $(G_p)_{N+\alpha<p}$ is equicoercive and Γ-converges to G_∞ with respect to the $(w^* \times \text{uniform})$-convergence. In particular the couple (f_∞, u_∞) is a minimizer of the functional G_∞.*

Proof. We only prove the Γ-convergence, and first show the $\Gamma - \liminf$ inequality, that is:

$$\Gamma - \liminf_{p\to+\infty} G_p \geq G_\infty. \tag{3.9}$$

Let $(g_p, v_p) \in L^{p'}(\Omega) \times W_0^{1,p}(\Omega)$ and $(g, v) \in \mathcal{M}(\Omega) \times \mathcal{C}_0(\Omega)$ such that (g_p, v_p) converges to (g, v) for the $(w^* \times \text{uniform})$-topology. We easily have:

$$-\langle g_p, v_p\rangle = -\int v_p\ dg_p \to -\int v\ dg = -\langle g, v\rangle;\ \int_\Omega d|g| = \lim_{p\to+\infty} \|g\|_{p'} \leq 1.$$

Moreover, for any $\varphi \in \mathcal{C}_c^\infty(\Omega)$, it holds:

$$\left|\int v_p(x)\nabla\varphi(x)\ dx\right| \leq \|\nabla v_p\|_p\|\varphi\|_{p'} \leq \lambda_p^{1/p}\|\varphi\|_{p'}.$$

Passing to the limit as p tends to ∞ this yields:

$$\left|\int v(x)\nabla\varphi(x)\ dx\right| \le \Lambda_\infty\|\varphi\|_1,$$

that is $v \in W_0^{1,\infty}(\Omega)$ and $\|\nabla v\|_\infty \le \Lambda_\infty$. This ends the proof of (3.9).
Let us now prove the $\Gamma - \limsup$ inequality. Take $(g, v) \in \mathcal{M}(\Omega) \times W_0^{1,\infty}(\Omega)$ such that:

$$\int_\Omega d|g| \le 1, \quad \|\nabla v\|_\infty \le \Lambda_\infty.$$

By setting $v_p = \frac{\lambda_p^{1/p}}{\Lambda_\infty} v$, we get a sequence such that:

$$v_p \to v \text{ uniformly}\ , \quad v_p \in W_0^{1,p}(\Omega), \quad \frac{\|\nabla v_p\|_p}{\lambda_p^{1/p}} = \frac{\|\nabla v\|_p}{\Lambda_\infty} \le 1.$$

To build a sequence of measures $g_p \in L^{p'}(\Omega)$ satisfying $\|g\|_{p'} \le 1$, we make a regularization by convolution:

$$\forall x \in \mathbb{R}^N, \qquad g_p(x) := \int \rho_p(x-y)\ dg(y)$$

where $\rho_p := \frac{1}{p}\rho(p \times \cdot)$ is a standard mollifier obtained as in the proof of Lemma 3.4. We thus get a family $(g_p)_{p>N}$ in $\mathcal{C}_c^\infty(\mathbb{R}^N)$ such that:

$$g_p \stackrel{*}{\rightharpoonup} g \text{ in } \mathcal{M}(\Omega) \text{ and } \|g_p\|_{p'} \le \int d|g| \le 1.$$

Finally, from the properties of $(v_p)_p$ and $(g_p)_p$, we have:

$$\lim_{p\to+\infty} G_p(g_p, v_p) = G(g, v).$$

□

4. THE LINK WITH AN OPTIMAL TRANSPORT PROBLEM.

A reader familiar with the Monge-Kantorovich or optimal transportation problem already recognized in problems $(\mathcal{P}_\infty)$ and $(\mathcal{P}_\infty^*)$ two of its dual formulations. Let us introduce this connection shortly. One of the advantages in exploiting this connection is that sometime it is possible to compute explicitly or numerically the value of the Wasserstein distance introduced below. For example, we will use this explicit computability in section §5 to prove that $\Lambda_\infty = 1/R_1$.

Given two probability measures α and ν on $\overline{\Omega}$ the Monge problem (with the Euclidean norm as cost) is the following minimization problem:

$$\inf\left\{\int_{\overline{\Omega}} |x - T(x)| d\alpha\ :\ T_\sharp\alpha = \nu\right\} \tag{4.1}$$

where the symbol $T_\sharp\mu$ denotes the push forward of α through T (i.e. $T_\sharp\alpha(B) := \alpha(T^{-1}(B))$ for every Borel set B). A Borel map T such that $T_\sharp\alpha = \nu$ is called a transport of α to ν and it is called an optimal transport if it minimizes (4.1). It may happens that the set of transports of α to ν is empty (e.g. $\alpha = \delta_0$ and $\nu = \frac{1}{2}(\delta_1+\delta_{-1})$ or that the minimum is not achieved (e.g. $\alpha = \mathcal{H}^1_{\{0\}\times[0,1]}$, $\nu = \frac{1}{2}(\mathcal{H}^1_{\{-1\}\times[0,1]} + \mathcal{H}^1_{\{1\}\times[0,1]})$. To deal with these situations in the '40 Kantorovich proposed the following relaxation of the problem above

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|d\gamma : \pi^1_\sharp\gamma = \alpha,\ \pi^2_\sharp\gamma = \nu\right\}. \tag{4.2}$$

A measure γ such that $\pi^1_\sharp\gamma = \alpha$, $\pi^2_\sharp\gamma = \nu$ is called a transport plan of α to ν. Notice that by the direct method of the Calculus of Variations the minimum in (4.2) is achieved. The minimal value is usually called Wasserstein distance of α and ν and it is denoted by $\mathcal{W}_1(\nu, \alpha)$.

Let $f_\infty \in \mathscr{P}(\Omega)$ be the measure defined in Lemma 3.2, and consider its Wasserstein distance from $\mathscr{P}(\partial\Omega)$, i.e. the following variational problem defined on $\mathscr{P}(\partial\Omega)$

$$\inf_{\nu\in\mathscr{P}(\partial\Omega)} \mathcal{W}_1(f_\infty, \nu). \tag{4.3}$$

With the usual abuse of notations, we shall denote by $\mathcal{W}_1(f_\infty, \mathscr{P}(\partial\Omega))$ the infimum in (4.3). We can also rewrite it as

$$\mathcal{W}_1(f_\infty, \mathscr{P}(\partial\Omega)) \;=\; \inf\left\{\int_{\overline{\Omega}\times\overline{\Omega}} |x-y|d\gamma : \pi^1_\sharp\gamma = f_\infty,\ \pi^2_\sharp\gamma \in \mathscr{P}(\partial\Omega)\right\} \tag{4.4}$$

The following proposition is a variant of the classical Kantorovich duality (see for example theorem 1.3 of [30]) and it will help us to connect problems (4.4) with problems $(\mathcal{P}_\infty)$ and $(\mathcal{P}^*_\infty)$.

Proposition 4.1. *The following equalities hold*

$$\mathcal{W}_1(f_\infty, \mathscr{P}(\partial\Omega)) \;=\; -\frac{1}{\Lambda_\infty}\min(\mathcal{P}_\infty) \;=\; \frac{1}{\Lambda_\infty}\min(\mathcal{P}^*_\infty). \tag{4.5}$$

An other way of expressing the link between the limit quantities obtained in Lemma 3.2 and the optimal transportation theory is via the following Theorem 4.2, which is the main result of this section and expresses in a useful way the primal-dual optimality conditions coming from Proposition 3.3.

Theorem 4.2. *The limits $(u_\infty, f_\infty, \sigma_\infty, \xi_\infty, \mu_\infty)$ obtained in Lemma 3.1 satisfy:*

$$\begin{cases} \sigma_\infty &=& \xi_\infty\,\mu_\infty \\ \xi_\infty &=& \Lambda_\infty^{-1}\nabla_{\mu_\infty}u_\infty, \quad \mu_\infty - a.e.\ in\ \overline{\Omega}, \\ -\mathrm{div}(\nabla_{\mu_\infty}u_\infty\,.\,\mu_\infty) &=& \Lambda_\infty\, f_\infty, \quad in\ the\ sense\ of\ distributions\ in\ \Omega, \\ |\nabla_{\mu_\infty}u_\infty| &=& \Lambda_\infty, \quad \mu_\infty - a.e.\ in\ \overline{\Omega}. \end{cases} \tag{4.6}$$

In the above result $\nabla_{\mu_\infty}u_\infty$ denotes the tangential gradient of u_∞ to the measure μ_∞ (see Definition 4.6 for details).

The proof of Theorem 4.2 requires to perform an integration by parts with respect to a measure. In order to do that we introduce, shortly, the notion of tangent space to a measure and of tangential gradient to a measure. This notion has first been introduced by Bouchitté, Buttazzo and Seppecher in [12], the case of interest here is developed in [23]: we now recall the main points tools in our setting.

Let us define the set

$$\mathcal{N} := \left\{ \begin{array}{l} \xi \in \mathrm{L}^\infty_{\mu_\infty}(\mathbb{R}^N, \mathbb{R}^N) : \ \exists (u_n)_n, \ u_n \in \mathcal{C}^1(\mathbb{R}^N), \\ \qquad u_n \to 0 \text{ uniformly on } \mathbb{R}^N, \ \nabla u_n \stackrel{*}{\rightharpoonup} \xi \text{ in } \sigma\left(\mathrm{L}^\infty_{\mu_\infty}, \mathrm{L}^1_{\mu_\infty}\right) \end{array} \right\} \tag{4.7}$$

where $\sigma\left(\mathrm{L}^\infty_{\mu_\infty}, \mathrm{L}^1_{\mu_\infty}\right)$ denotes the weak star topology of $\mathrm{L}^\infty_{\mu_\infty}(\mathbb{R}^N, \mathbb{R}^N)$. We notice that when μ_∞ is not absolutely continuous with respect to the Lebesgue measure, this set is not necessarily reduced to zero.

The following results and notions may be found in [23]:

Proposition 4.3. *There exists a multi-function T_{μ_∞} from $\mathbb{R}^N$ to $\mathbb{R}^N$ such that:*

$$\eta \in \mathcal{N}^\perp \Leftrightarrow \eta(x) \in T_{\mu_\infty}(x) \ \mu_\infty - a.e.x.$$

Definition 4.4. For μ_∞ – a.e. x, we call $T_{\mu_\infty}(x)$ the tangent space to μ_∞ at x and denote by $P_{\mu_\infty}(x, \cdot)$ the orthogonal projection on $T_{\mu_\infty}(x)$.

Proposition 4.5. *Let $u \in \mathrm{Lip}(\mathbb{R}^N)$, there exists a unique function ξ in $\mathrm{L}^\infty_{\mu_\infty}$ such that*

$$\left. \begin{array}{ll} (u_n) \in \mathrm{Lip}(\mathbb{R}^N), & \textit{equiLipschitz} \\ u_n \to u, & \textit{uniformly on } \mathbb{R}^N \end{array} \right\} \Rightarrow P_{\mu_\infty}(\cdot, \nabla u_n(\cdot)) \stackrel{*}{\rightharpoonup} \xi.$$

Definition 4.6. The function ξ appearing in the last proposition is called tangential gradient of u to μ_∞ and is denoted by $\nabla_{\mu_\infty} u$.

Proposition 4.7 (Integration by parts formula)**.** *Let $\Psi \in \mathrm{Lip}(\mathbb{R}^N)$ and $\theta \in \mathrm{L}^1_{\mu_\infty}(\mathbb{R}^N, \mathbb{R}^N)$ such that $-\mathrm{div}(\theta \mu_\infty)$ belongs to $\mathcal{M}_b(\mathbb{R}^N)$. Then*

$$\theta(x) \in T_{\mu_\infty}(x) \ \mu_\infty - a.e., \quad \textit{and} \quad - < \mathrm{div}(\theta\mu_\infty), \Psi >= \int \theta \cdot \nabla_{\mu_\infty} \Psi \, d\mu_\infty.$$

In the previous results, we have defined the tangential gradient of functions in $\mathrm{Lip}(\mathbb{R}^N)$. As we are dealing with functions on $\mathrm{Lip}(\Omega)$, we will also need the following

$$u \in \mathrm{Lip}(\mathbb{R}^N), \quad u = 0 \ \mu_\infty\text{-a.e. in } \Omega \Rightarrow \nabla_{\mu_\infty} u = 0 \ \mu_\infty\text{-a.e. in } \Omega$$

so that the tangential gradient of any function u in $\mathrm{Lip}(\Omega)$ is well defined via the restriction of the tangential gradient of any of its Lipschitz extension to $\mathbb{R}^N$.

Proof of Theorem 4.2. Using the duality relation between $(\mathcal{P}_\infty)$ and $(\mathcal{P}_\infty^*)$ and the optimality of $\sigma_\infty = \xi_\infty \mu_\infty$ and u_∞ (see Theorem 3.2 and Proposition 3.6), we get:

$$\int_\Omega u_\infty(x)\; df_\infty(x) = \Lambda_\infty \int_{\overline{\Omega}} |\xi_\infty(x)|\; d\mu_\infty(x). \tag{4.8}$$

By Proposition 4.7, as $-\mathrm{div}(\sigma_\infty) \in \mathcal{M}_b(\mathbb{R}^N)$ and u_∞ is zero outside Ω, we can make an integration by parts an get:

$$\int_\Omega u_\infty(x)\; df_\infty(x) = \langle -\mathrm{div}(\xi_\infty \mu_\infty), u_\infty \rangle_{\mathcal{M}_b(\mathbb{R}^N), \mathcal{C}_c(\mathbb{R}^N)} = \int_{\overline{\Omega}} \nabla_{\mu_\infty} u_\infty \cdot \xi_\infty \; d\mu_\infty.$$

Using (4.8), we get:

$$\int_{\overline{\Omega}} (\nabla_{\mu_\infty} u_\infty \cdot \xi_\infty) - \Lambda_\infty |\xi_\infty| \; d\mu_\infty = 0. \tag{4.9}$$

The constraint $|\nabla u_\infty| \leq \Lambda_\infty$ a.e. in Ω is reformulated using the definitions of T_{μ_∞} and ∇_{μ_∞} as a constraint on $\nabla_{\mu_\infty} u_\infty$ by saying (see [23], Lemma 4.13 and proof of Theorem 5.1):

$$\exists \zeta \in \mathrm{L}^\infty_{\mu_\infty}(\mathbb{R}^N, \mathbb{R}^N) \text{ such that } \begin{cases} \zeta(x) \in T_{\mu_\infty}(x)^\perp, & \mu_\infty\text{-a.e.} x \in \overline{\Omega} \\ |\nabla_{\mu_\infty} u_\infty(x) + \zeta(x)| \leq \Lambda_\infty, & \mu_\infty\text{-a.e.} x \in \overline{\Omega}. \end{cases}$$

As $\xi_\infty(x) \in T_{\mu_\infty}(x)$ μ_∞-a.e, we have:

$$\nabla_{\mu_\infty} u_\infty(x) \cdot \xi_\infty(x) = (\nabla_{\mu_\infty} u_\infty(x) + \zeta(x)) \cdot \xi_\infty(x) \leq \Lambda_\infty |\xi_\infty(x)| \; \mu_\infty\text{-a.e.} x \in \overline{\Omega}.$$

Combining this with (4.9), we obtain $\nabla_{\mu_\infty} u_\infty(x) \cdot \xi_\infty(x) = \Lambda_\infty |\xi_\infty(x)|$ $\mu_\infty-$almost everywhere and consequently:

$$|\nabla_{\mu_\infty} u_\infty| = \Lambda_\infty, \quad \xi_\infty = \frac{\nabla_{\mu_\infty} u_\infty}{\Lambda_\infty} \quad \mu_\infty - \text{ a.e. in } \overline{\Omega}.$$

The second equality in (4.6) then follows from $\sigma_\infty = \Lambda_\infty^{-1} \nabla_{\mu_\infty} u_\infty \,.\, \mu_\infty$. □

5. SOME PROPERTIES OF THE LIMITS

In this section we will use the optimal transport problem to investigate more properties of u_∞ and f_∞ and to give an alternative way of identifying Λ_∞ which we hope will be useful in the future.

We shall denote by $d_\Omega(x)$ the distance of a point x of Ω from $\partial\Omega$ and we recall the notation

$$R_1 = \sup\{r |\; \exists\; x_0 \text{ s.t. } B(x_0, r) \subset \Omega\}.$$

The main theorem is the following:

Theorem 5.1. *The limits u_∞, f_∞ and Λ_∞ satisfies the following:*

1. *f_∞ maximizes $\mathcal{W}_1(\cdot, \mathcal{P}(\partial\Omega))$ in $\mathscr{P}(\Omega)$,*
2. *$\Lambda_\infty = \frac{1}{R_1}$,*
3. *$spt(f_\infty) \subset argmax\; u_\infty \subset argmax\; d_\Omega$.*

Proof of Theorem 5.1. By Theorem 2.2 and Proposition 3.7 the couple (f_∞, u_∞) minimizes G_∞ or, which is equivalent, maximizes

$$\max\{\langle g, v\rangle \mid \int_\Omega d|g| \leq 1,\ v \in \mathrm{W}_0^{1,\infty}(\Omega),\ \|\nabla v\|_\infty \leq \Lambda_\infty\}$$
$$= \max_{g\in\mathscr{P}(\Omega)} \max\{\langle g, v\rangle \mid v \in \mathrm{W}_0^{1,\infty}(\Omega),\ \|\nabla v\|_\infty \leq \Lambda_\infty\}$$
$$= \max_{g\in\mathscr{P}(\Omega)} \Lambda_\infty \mathcal{W}_1(g, \mathscr{P}(\partial\Omega)).$$

We now remark that $\max_{g\in\mathscr{P}(\Omega)} \mathcal{W}_1(g, \mathscr{P}(\partial\Omega)) = R_1$ and that the maximal value is achieved exactly by the probability measures concentrated on the set $\{x \in \Omega \mid d_\Omega(x) = R_1\} = argmax\ d_\Omega$. Then $\mathcal{W}_1(f_\infty, \mathscr{P}(\partial\Omega)) = R_1$ and f_∞ is concentrated on the set $argmax\ d_\Omega$. Then from $1 = \Lambda_\infty \mathcal{W}_1(f_\infty, \mathscr{P}(\partial\Omega)) = \Lambda_\infty R_1$ it follows $\Lambda_\infty = \frac{1}{R_1}$.

Let us now prove $argmax\ u_\infty \subset argmax\ d_\Omega$.
For $x \in \Omega$, let $y \in \partial\Omega$ be a projection of x on $\partial\Omega$, we have:

$$u_\infty(x) = u_\infty(x) - u_\infty(y) \leq ||\nabla u_\infty||_\infty |x - y| = \frac{1}{R_1} d_\Omega(x).$$

Now, if x is in $argmax\ u_\infty$, $u_\infty(x) = 1$ and using the inequality above we get $1 \leq \frac{1}{R_1} d_\Omega(x)$ which implies $d_\Omega(x) = R_1$.

Finally, let us show that $spt\ f_\infty \subset argmax\ u_\infty$.
Assume x is a point out of $argmax\ u_\infty$. Then it exists a ball $B(x, r)$ centered at x of radius r on which $u_\infty < 1 - \alpha$ with $\alpha > 0$. As $u_p \to u_\infty$ uniformly, for p big enough we have $u_p < 1 - \frac{\alpha}{2}$ on $B(x, r)$. This statement implies:

$$\begin{aligned} \int_{B(x,r)} df_\infty(y) &\leq \liminf_{p\to+\infty} \int_{B(x,r)} f_p(y)\ dy \\ = \liminf_{p\to+\infty} \int_{B(x,r)} u_p(y)^{p-1} dy &\leq \liminf_{p\to+\infty} (1 - \alpha/2)^{p-1} \omega_N r^N = 0. \end{aligned}$$

Consequently $x \notin spt\ f_\infty$.

□

Remark 5.2. Examples are given in [25] to illustrate that u_∞ may differ from d_Ω, but it is still an open question whether one has $argmax u_\infty = argmax\ d_\Omega$. In this respect, a close understanding on the transport problem $(\mathcal{P}_\infty)$ may yield that $spt(f_\infty) = argmax\ d_\Omega$ and thus answer this question.

Next step would be to investigate some PDE properties of u_∞ with the aim of understanding in which region is satisfied each part of the equation (1.1). We can give some partial results on that.

Definition 5.3. For each $x \in \Omega$ we define its projection on $\partial\Omega$ as

$$p_{\partial\Omega}(x) = \{z \in \partial\Omega \mid |x - z| = d_\Omega(x)\}.$$

The transport set T is given by

$$T = \{[x, y] \mid x \in spt(f_\infty) \text{ and } y \in p_\Omega(x)\}. \tag{5.1}$$

The transport set plays a crucial role in the theory of optimal transportation because it is the set on which the transport takes actually place. It should also play a role in dividing the open set Ω in regions in which u_∞ satisfies different equations. The next proposition below goes in this direction.

Proposition 5.4. *The function u_∞ is differentiable in $T \setminus (spt(f_\infty) \cup \partial\Omega)$ moreover it satisfies $-\Delta_\infty u_\infty \le 0$ in the viscosity sense on $T \setminus (spt(f_\infty) \cup \partial\Omega)$.*

Proof. Let $x_0 \in T \backslash spt(f_\infty)$. There exists $(y_1, y_2) \in spt(f_\infty) \times \partial\Omega \subset argmax\ (u_\infty) \times \partial\Omega$ such that $x_0 \in]y_1, y_2[$. The closure of the segment $]y_1, y_2[$ is called a transport ray and for each $z \in]y_1, y_2[$, u_∞ satisfies

$$u_\infty(z) = \Lambda_\infty |z - y_2| = u_\infty(y_1) - \Lambda_\infty |z - y_1|.$$

It follows by a classical argument (see for example Proposition 4.2 of [1]) that u_∞ is differentiable on this segment and that $|\nabla u_\infty(z)| = \Lambda_\infty$ for all $z \in]y_1, y_2[$. As $x_0 \notin argmax\ u_\infty$ one get

$$\Lambda_\infty u(x_0) < |\nabla u(x_0)| = \Lambda_\infty. \tag{5.2}$$

By [25], u_∞ is a viscosity sub-solution of

$$\min\{\frac{|\nabla u(x)|}{|u(x)|} - \Lambda_\infty, -\Delta_\infty u\} = 0,$$

i.e. $\forall x \in \Omega$ and for all smooth φ such that $\varphi \ge u_\infty$ in Ω and $\varphi(x) = u_\infty(x)$ one has

$$\min\{\frac{|\nabla \varphi(x)|}{|\varphi(x)|} - \Lambda_\infty, -\Delta_\infty \varphi(x)\} \le 0.$$

The differentiability of u_∞ at x_0 together with (5.2) implies that for every φ as above

$$\min\{\frac{|\nabla \varphi(x_0)|}{|\varphi(x_0)|} - \Lambda_\infty, -\Delta_\infty \varphi(x_0)\} = \min\{\frac{|\nabla u_\infty(x_0)|}{|u_\infty(x_0)|} - \Lambda_\infty, -\Delta_\infty \varphi(x_0)\} \le 0,$$

and then $-\Delta_\infty \varphi(x_0) \le 0$ which is, by definition, $-\Delta_\infty u_\infty(x_0) \le 0$ in the viscosity sense. □

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