Communications in Applied Analysis **14** (2010), no. 1, 1–12

PERIODIC POINTS FOR COMPACT ABSORBING CONTRACTIONS IN EXTENSION TYPE SPACES

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ABSTRACT. Several new periodic point results are presented for self maps in extension type spaces. In particular we discuss compact absorbing contractions.

AMS (MOS) Subject Classification. 47H10

1. INTRODUCTION

Section 2 discusses extension type spaces and maps. In Sections 3 we present new periodic point results in extension type spaces. These results improve those in the literature; see [1–3, 5, 8–11, 14–15] and the references therein. Our results were motivated in part from ideas in [1, 2, 9, 12, 15].

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathfrak{X}$ of maps, $\mathfrak{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathfrak{X}$, and $\mathfrak{X}_c$ the set of finite compositions of maps in $\mathfrak{X}$. We let

$$\mathfrak{F}(\mathfrak{X}) = \{Z : \text{ Fix } F \neq \emptyset \text{ for all } F \in \mathfrak{X}(Z, Z)\}$$

where Fix F denotes the set of fixed points of F.

The class $\mathfrak{B}$ of maps is defined by the following properties:

(i) $\mathfrak{B}$ contains the class $\mathfrak{C}$ of single valued continuous functions;
(ii) each $F \in \mathfrak{B}_c$ is upper semicontinuous and closed valued; and
(iii) $B^n \in \mathfrak{F}(\mathfrak{B}_c)$ for all $n \in \{1, 2, \dots\}$; here $B^n = \{x \in \mathbf{R}^n : \|x\| \leq 1\}$.

The class $\mathfrak{B}$ is essentially due to Ben-El-Mechaiekh and Deguire [6]. $\mathfrak{B}$ includes the class of maps $\mathfrak{U}$ of Park ($\mathfrak{U}$ is the class of maps defined by (i), (iii) and (iv). each $F \in \mathfrak{U}_c$ is upper semicontinuous and compact valued). Thus if each $F \in \mathfrak{B}_c$ is compact valued the class $\mathfrak{B}$ and $\mathfrak{U}$ coincide.

Received March 20, 2009

We also consider the class $\mathfrak{U}_c^\kappa(X,Y)$ (respectively $\mathfrak{B}_c^\kappa(X,Y)$) of maps $F : X \to 2^Y$ such that for each F and each nonempty compact subset K of X there exists a map $G \in \mathfrak{U}_c(K,Y)$ (respectively $G \in \mathfrak{B}_c(K,Y)$) such that $G(x) \subseteq F(x)$ for all $x \in K$.

Theorem 1.1. *T (the Tychonoff cube) is in $\mathfrak{F}(\mathfrak{U}_c^\kappa)$.*

For a subset K of a topological space X, we denote by $Cov_X(K)$ the set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$. Given two maps single valued $f, g : X \to Y$ and $\alpha \in Cov(Y)$, f and g are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_h : X \to Y$ $(0 \leq t \leq 1)$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0,1]$.

The following results can be found in [4, Lemma 1.2 and 4.7].

Theorem 1.2. *Let X be a regular topological space and $F : X \to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

From Theorem 1.2 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [5 pp. 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [13 pp. 199] so such refinements form a cofinal family of open covers). Note also uniform spaces are regular (in fact completely regular) [7 pp. 431] (see also [7 pp. 434]). Note in Theorem 1.2 if F is compact valued then the assumption that X is regular can be removed. For convenience in this paper we will apply Theorem 1.2 only when the space is uniform.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = qp^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ. A multivalued map $\phi : X \to 2^Y$ is said to be determined by a morphism $\{X \stackrel{p}{\Leftarrow} \Gamma \stackrel{q}{\to} Y\}$ provided $\phi(x) = qp^{-1}(x)$ for each $x \in X$; the morphism which determines ϕ is also denoted by ϕ. Note a multivalued map determined by a morphism is upper semicontinuous and compact valued. Finally note every morphism determines a multivalued map but not conversely.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \backslash N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q Tr(f_q).$$

The Euler characteristic $\chi(f)$ is defined to be

$$\chi(f) = \sum_q (-1)^q dim(\tilde{E}_q).$$

Let $Q\{x\}$ denote the integral domain consisting of all formal power series $\sum_{n=0}^{\infty} a_n x^n$ with coefficients $a_n \in Q$ (here Q is a fixed field). The Lefschetz power series $L(f)$ of the Leray endomorphism $f = \{f_q\}$ is an element of $Q\{x\}$ defined by

$$L(f) = \chi(f) + \sum_{n=1}^{\infty} \Lambda(f^n) x^n$$

From [10, pp 325] (see also [12, pp 434]) we know $L(f)$ admits a representation $L(f) = u.v^{-1}$ where u and v are relatively prime polynomials with $deg\, u < deg\, v$ $(u \neq 0)$. We define

$$P(f) = deg\, v.$$

Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star q}\}$ where $f_{\star q} : H_q(X) \to H_q(X)$.

With Čech homology functor extended to a category of morphisms (see [9, 10]) we have the following well known result (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\rightarrow} Y\} : X \to Y$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$).

Theorem 1.3. *If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then*

$$(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star.$$

Two morphisms $\phi, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [9, pp. 231]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p,q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p,q) \subset \phi$) if the following two conditions hold:

(i). p is a Vietoris map

and

(ii). $q(p^{-1}(x)) \subset \phi(x)$ for any $x \in X$.

Definition 1.4. A upper semicontinuous map $\phi : X \to Y$ is said to be strongly admissible [9] (and we write $\phi \in Ads(X,Y)$) provided there exists a selected pair (p,q) of ϕ with $\phi(x) = q(p^{-1}(x))$ for $x \in X$.

Definition 1.5. A map $\phi \in Ads(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ with $\phi(x) = q(p^{-1}(x))$ for $x \in X$ the linear map $q_\star p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism.

If $\phi : X \to X$ is a Lefschetz map as described above then we define the Lefschetz number (see [9]) $\Lambda(\phi)$ (or $\Lambda_X(\phi)$) by

$$\Lambda(\phi) = \Lambda(q_\star p_\star^{-1}).$$

Also we define

$$\chi(\phi) = \chi(q_\star p_\star^{-1}), L(\phi) = L(q_\star p_\star^{-1}) \text{ and } P(\phi) = P(q_\star p_\star^{-1}).$$

Definition 1.6. A Hausdorff topological space X is said to be a Lefschetz space provided every compact $\phi \in Ads(X,X)$ is a Lefschetz map and $\Lambda(\phi) \neq 0$ implies ϕ has a fixed point.

Theorem 1.7 ([9, 12])**.** *Let $\phi \in Ads(X,X)$ be a Lefschetz map. Then*

(a) $\chi(\phi) = 0$ *implies* $P(\phi) = 0$*;*
(b) $P(\phi) = 0$ *if and only if* $\Lambda(\phi^n) = 0$ *for some natural number* n*;*
(c) *if* $P(\phi) = k \neq 0$ *then for any* $m \in \{0, 1, 2, \dots\}$ *at least one of* $\Lambda(\phi^{m+1}), \dots, \Lambda(\phi^{m+k})$ *is different from zero.*

2. PRELIMINARY FIXED POINT THEORY

We note that some of the fixed point theory presented in this section can be found in [15, 16, 17]. In addition in this section we improve some of the results in [15, 16]. We also establish some new properties (see Remark 2.5 and Remark 2.8) which will be needed in Section 3.

By a space we mean a Hausdorff topological space. Let X and Y be spaces. A space Y is an neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and for any continuous function $f_0 : K \to Y$, there exists a continuous extension $f : U \to Y$ of f_0 over a neighbourhood U of K in X.

In [17] we established the following result.

Theorem 2.1. *Let $X \in NES(compact)$ and $F \in Ads(X,X)$ a compact map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall \alpha \in Cov(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α close and α-homotopic.

Theorem 2.2 ([17])**.** *Let $X \in SANES(compact)$ be a uniform space and $F \in Ads(X,X)$ a compact map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

In fact we obtained a more general result in [17] which contains both Theorem 2.1 and Theorem 2.2.

Let X be a Hausdorff topological space. A map $F \in Ads(X,X)$ is said to be a compact absorbing contraction (written $F \in CACs(X,X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$;
(ii) $F|_Y \in Ads(Y,Y)$ (automatically satisfied) is a compact map with Y a Lefschetz space;

(iii) for every compact $K \subseteq X$ there is an integer $n = n(K)$ such that $F^n(K) \subseteq Y$.

Remark 2.3. Examples of Lefschetz spaces Y are of course NES(compact) and $SANES$(compact) uniform spaces.

Remark 2.4. If $Y = U$ is an open subset of X then (iii) could be changed to

(iii)′ for every $x \in X$ there exists an integer $n = n(x)$ such that $F^{n(x)}(x) \subseteq Y = U$.

Remark 2.5. Let $F \in CACs(X, X)$ and let Y be as above. Notice $F^2(Y) \subseteq F(Y) \subseteq Y$, $F^2|_Y \in Ads(Y, Y)$ (see [9, pp. 201]) and $F^2|_Y$ is a compact map. Let K be a compact subset of X and let $n = n(K)$ be as described above. Then if n is even we have $(F^2)^{\frac{n}{2}}(K) \subseteq Y$ whereas in n is odd we have $(F^2)^{\frac{n+1}{2}} = F^{n+1}(K) = F(F^n(K)) \subseteq F(Y) \subseteq Y$. Thus $F^2 \in CACs(X, X)$. Similarly $F^m \in CACs(X, X)$ for every integer m.

Theorem 2.6 ([17])**.** *Let X be a Hausdorff topological space and $F \in CACs(X, X)$. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

In [15, 16] we considered a more general situation. Let X be a compact space. A map $F \in Ads(X, X)$ is said to be a NES(compact) map if for any compact pair (Z, A) and any homeomorphism $g : X \to A$ there exists a neighborhood U of A in Z and a $\Phi \in Ads(U, X)$ with $\Phi|_A = Fg^{-1}$.

Theorem 2.7 ([16])**.** *Let X be a compact space and let $F \in Ads(X, X)$ be a NES (compact) map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

In fact in [15] we generalized this result. A map $F \in Ads(X, X)$ is said to be a compact absorbing contraction (written $F \in MCACs(X, X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$;
(ii) Y is a compact space and $F|_Y \in Ads(Y, Y)$ (automatically satisfied) is a NES (compact) map;
(iii) for every compact $K \subseteq X$ there is a $n = n(K)$ such that $F^n(K) \subseteq Y$.

Remark 2.8. Let $F \in MCACs(X, X)$ and let Y be as above. Consider any compact pair (Z, A) and any homeomorphism $g : Y \to A$. Now there exists a neighborhood U of A in Z and a $\Phi \in Ads(U, Y)$ with $\Phi|_A = Fg^{-1}$. Let $\Psi = F\Phi$. Notice $\Psi \in Ads(U, Y)$ and $\Psi|_A = F\Phi|_A = FFg^{-1} = F^2g^{-1}$. Thus (see also Remark 2.5) $F^2 \in MCACs(X, X)$. Similarly $F^m \in MCACs(X, X)$ for each integer m.

Theorem 2.9 ([15])**.** *Let X be a Hausdorff topological space and $F \in MCACs(X, X)$. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

Our next two results improve those in [15].

A map $F \in \mathfrak{U}_c^\kappa(X,Y)$ is called a $ANES$(compact) map if for any compact pair (Z,A) and any homeomorphism $g : X \to A$ the following holds: for each $\alpha \in Cov(Y)$ there exists a neighborhood U_α of A in Z and a $\Phi_\alpha \in \mathfrak{U}_c^\kappa(U_\alpha,Y)$ such that for each $x \in A$ with $x \in j_{U_\alpha} g \Phi_\alpha(x)$ (here $j_{U_\alpha} : A \hookrightarrow U_\alpha$ is the natural imbedding) there exists $U_x \in \alpha$ such that $g^{-1}(x) \in U_x$ and $Fg^{-1}(x) \cap U_x \neq \emptyset$.

Let X be a compact space and $F \in \mathfrak{U}_c^\kappa(X,X)$ a $ANES$(compact) map. Let $\alpha \in Cov_X(X)$. X is compact so [12] X is homeomorphic to a closed subset of the Tychonoff cube T, so as a result X can be embedded as a closed subset $K^\star$ of T; let $s : X \to K^\star$ be a homeomorphism. Now since $s^{-1} : K^\star \to X$ and since F is a $ANES$(compact) map there exists a neighborhood U_α of $K^\star$ in T and a $\Phi_\alpha \in \mathfrak{U}_c^\kappa(U_\alpha,X)$ such that for each $x \in K^\star$ with $x \in j_{U_\alpha} s, \Phi_\alpha(x)$ (here $j_{U_\alpha} : K^\star \hookrightarrow U_\alpha$ is the natural imbedding) there exists $U_x \in \alpha$ such that $s^{-1}(x) \in U_x$ and $Fs^{-1}(x) \cap U_x \neq \emptyset$. Let $G_\alpha = j_{U_\alpha} s \Phi_\alpha$. Notice $G_\alpha \in \mathfrak{U}_c^\kappa(U_\alpha,U_\alpha)$. We now assume

$$G_\alpha \in \mathfrak{U}_c^\kappa(U_\alpha,U_\alpha) \text{ has a fixed point for each } \alpha \in Cov_X(X). \tag{2.1}$$

Thus there exists $x \in U_\alpha$ with $x \in G_\alpha x$. Then there exists $y \in \Phi_\alpha(x)$ with $x = j_{U_\alpha} s(y)$. Note $s(y) \in K^\star$. Now there exists a $U \in \alpha$ with $s^{-1}(x) \in U$ and $Fs^{-1}(x) \cap U \neq \emptyset$. Since $x = j_{U_\alpha} s(y)$ we have $y \in U$ and $F(y) \cap U \neq \emptyset$. As a result F has an α–fixed point. Now apply Theorem 1.2 and we have the following result which improves a result in [15].

Theorem 2.10. *Let X be a uniform compact space and let $F \in \mathfrak{U}_c^\kappa(X,X)$ be a $ANES$(compact) map. In addition assume F is a upper semicontinuous map with compact values. Also assume* (2.1) *holds with K, s, U_α, Φ_α and j_{U_α} as described above. Then F has a fixed point.*

We now discuss Theorem 2.10 for the class $Ads(X,X)$. Let X be a uniform compact space. A map $F \in Ads(X,X)$ is said to be a weakly $ANES$(compact) map if for any compact pair (Z,A) and any homeomorphism $g : X \to A$ the following two conditions hold for each $\alpha \in Cov_X(X)$:

(1) there exists a neighborhood U_α of A in Z and a $\Phi_\alpha \in Ads(U_\alpha,X)$ such that for each $x \in A$ with $x \in j_{U_\alpha} g \Phi_\alpha(x)$ there exists $U_x \in \alpha$ such that $g^{-1}(x) \in U_x$ and $Fg^{-1}(x) \cap U_x \neq \emptyset$,
(2) if (p,q) is any selected pair for F with $qp^{-1}(x) = F(x)$ for $x \in X$ then there exists a selected pair (p''_α, q''_α) of $\Phi_\alpha j_{U_\alpha} g$ with $q''_\alpha (p''_\alpha)^{-1}(x) = \Phi_\alpha j_{U_\alpha} g(x)$ for $x \in X$ and with $(q''_\alpha)_\star (p''_\alpha)_\star^{-1} = q_\star p_\star^{-1}$; here $j_{U_\alpha} : A \hookrightarrow U_\alpha$ is the natural embedding.

Remark 2.11. Let X be a compact space and $F \in Ads(X,X)$ be such that for any compact pair (Z,A) and any homeomorphism $g : X \to A$ we have for each

$\alpha \in Cov_X(X)$ that there exists a neighborhood U_α of A in Z and a continuous function $h_\alpha : U_\alpha \to X$ of g^{-1} such that $h_\alpha|_A$ and g^{-1} are α-homotopic. Then (2) above holds with $\Phi_\alpha = F h_\alpha$. To see this let (p, q) be any selected pair for F with $qp^{-1}(x) = F(x)$ for $x \in X$. Then [9, Theorem 40.6, pp. 201] guarantees that there exists a selected pair $(p'''_\alpha, q'''_\alpha)$ of $F h_\alpha j_{U_\alpha} g$ with $q'''_\alpha (p'''_\alpha)^{-1}(x) = F h_\alpha j_{U_\alpha} g(x)$ for $x \in X$ and with

$$(q'''_\alpha)_\star (p'''_\alpha)_\star^{-1} = q_\star p_\star^{-1} (h_\alpha)_\star (j_{U_\alpha})_\star g_\star .$$

As a result $(q'''_\alpha)_\star (p'''_\alpha)_\star^{-1} = q_\star p_\star^{-1}$ since $h_\alpha j_{U_\alpha} g$ is α-homotopic to i (note $h_\alpha|_A$ and g^{-1} are α-homotopic).

Remark 2.12. Let X be a compact space and $F \in Ads(X, X)$ be such that for any compact pair (Z, A) and any homeomorphism $g : X \to A$ we have for each $\alpha \in Cov_X(X)$ that there exists a neighborhood U_α of A in Z and a continuous function $h_\alpha : U_\alpha \to X$ of g^{-1} such that $h_\alpha|_A$ and g^{-1} are α-close. In addition assume for each $x \in A$ with $x \in j_{U_\alpha} g \Phi_\alpha(x)$ and $h_\alpha(x) \in U_x$, $F(h_\alpha(x)) \cap U_x \neq \emptyset$ for some $U_x \in \alpha$ there exists a $U \in \alpha$ with $g^{-1}(x) \in U$ and $F(g^{-1}(x)) \cap U \neq \emptyset$. Then (1) above holds with $\Phi_\alpha = F h_\alpha$. To see this suppose $x \in A$ with $x \in j_{U_\alpha} g \Phi_\alpha(x)$. Let $y = h_\alpha(x)$ so $y \in h_\alpha j_{U_\alpha} g F(y)$ i.e. $y = h_\alpha j_{U_\alpha} g(q)$ for some $q \in F(y)$. Now since $h_\alpha j_{U_\alpha} g$ and i are α–close there exists $U \in \alpha$ with $h_\alpha j_{U_\alpha} g(q) \in U$ and $i(q) \in U$ i.e. $q \in U$ and $y = h_\alpha j_{U_\alpha} g(q) \in U$. Thus $y \in U$ and $F(y) \cap U \neq \emptyset$ since $q \in F(y)$. As a result

$$h_\alpha(x) \in U \text{ and } F(h_\alpha(x)) \cap U \neq \emptyset .$$

By assumption there exists $U_x \in \alpha$ with $g^{-1}(x) \in U_x$ and $F(g^{-1}(x)) \cap U_x \neq \emptyset$.

Exactly the same proof as in [15, Theorem 2.2] (except here we use Theorem 2.10 above) gives the following result.

Theorem 2.13. *Let X be a uniform compact space and let $F \in Ads(X, X)$ be a weakly $ANES$(compact) map. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

A map $F \in Ads(X, X)$ is said to be a approximate compact absorbing contraction (written $F \in ACACs(X, X)$) if there exists $Y \subseteq X$ such that

(i) $F(Y) \subseteq Y$;
(ii) Y is a compact uniform space and $F|_Y \in Ads(Y, Y)$ (automatically satisfied) is a weakly ANES(compact) map;
(iii) for every compact $K \subseteq X$ there is a $n = n(K)$ such that $F^n(K) \subseteq Y$.

Theorem 2.14 ([15])**.** *Let X be a Hausdorff topological space and assume $F \in ACACs(X, X)$. Then $\Lambda(F)$ is well defined and if $\Lambda(F) \neq 0$ then F has a fixed point.*

Remark 2.15. As above we can generalize the definition of stronger $ANES$(compact) map in [15] and obtain a stronger Theorem 2.3 in [15] for the class Ad. Let X be a compact space and we say $F \in Ad(X, X)$ is a strongly $ANES$(compact) map if for any compact pair (Z, A) and any homeomorphism $g : X \to A$ the following two conditions hold for each $\alpha \in Cov_X(X)$:

(3) there exists a neighborhood U_α of A in Z and a $\Phi_\alpha \in Ad(U_\alpha, X)$ such that for each $x \in A$ with $x \in j_{U_\alpha} g\Phi_\alpha(x)$ there exists $U_x \in \alpha$ such that $s^{-1}(x) \in U_x$ and $Fg^{-1}(x) \cap U_x \neq \emptyset$,

(4) if (p, q) is any selected pair for F then there exists a selected pair (p'_α, q'_α) of Φ_α with $(q'_\alpha)_\star (p'_\alpha)_\star^{-1} (j_{U_\alpha})_\star g_\star = q_\star p_\star^{-1}$; here $j_{U_\alpha} : A \hookrightarrow U_\alpha$ is the natural embedding.

It is worth mentioning here also that we can also improve Theorem 2.2 in [16]. A map $F \in \mathfrak{U}_c^\kappa(X, Y)$ is called a AES(compact) map if for any compact pair (Z, A) and any homeomorphism $g : X \to A$ for each $\alpha \in Cov(Y)$ there exists $\Phi_\alpha \in \mathfrak{U}_c^\kappa(Z, Y)$ such that for each $x \in A$ with $x \in jg\Phi_\alpha(x)$ (here $j : A \hookrightarrow Z$ is the natural imbedding) there exists $U_x \in \alpha$ such that $s^{-1}(x) \in U_x$ and $Fg^{-1}(x) \cap U_x \neq \emptyset$.

Theorem 2.16. *Let X be a uniform compact space and suppose $F \in \mathfrak{U}_c^\kappa(X, X)$ is a AES(compact) map. In addition assume F is upper semicontinuous map with compact values. Then F has a fixed point.*

Proof. Let $\alpha \in Cov_X(X)$. From Theorem 1.2 it suffices to show F has an α–fixed point. We know [12] that X can be embedded as a closed subset $K^\star$ of T; let $s : X \to K^\star$ be a homeomorphism. Let $j : K^\star \hookrightarrow T$ be an inclusion. Now since $s^{-1} : K^\star \to X$ and since F is a AES(compact) map there exists $\Phi_\alpha \in \mathfrak{U}_c^\kappa(T, X)$ such that for each $x \in K^\star$ with $x \in js\Phi_\alpha(x)$ there exists $U_x \in \alpha$ such that $s^{-1}(x) \in U_x$ and $Fs^{-1}(x) \cap U_x \neq \emptyset$. Let $G_\alpha = js\Phi_\alpha$ and note $G_\alpha \in \mathfrak{U}_c^\kappa(T, T)$ so Theorem 1.1 guarantees that there exists $x \in T$ with $x \in G_\alpha x$. Then there exists $y \in \Phi_\alpha(x)$ with $x = js(y)$. Note $s(y) \in K^\star$. Now there exists a $U \in \alpha$ with $s^{-1}(x) \in U$ and $Fs^{-1}(x) \cap U \neq \emptyset$. Since $x = js(y)$ we have $y \in U$ and $F(y) \cap U \neq \emptyset$. As a result F has an α–fixed point. □

3. PERIODIC POINTS

Let X be a Hausdorff topological space. A point $x \in X$ is said to be a periodic point for a map $F : X \to 2^X$ with period n if $x \in F^n(x)$.

Theorem 3.1. *Let X be a Hausdorff topological space and $F \in CACs(X, X)$. Suppose $\chi(F) \neq 0$ or $P(F) \neq 0$. Fix $m \in \{0, 1, \dots\}$. Then F has a periodic point with period n where $m + 1 \leq n \leq m + P(F)$.*

Proof. We know for Theorem 2.6 that F is a Lefschetz map. Now $P(F) \neq 0$ (see Theorem 1.7 (a)). We now know for Theorem 1.7 (c) that there exists a n, $m + 1 \leq$

$n \le m + P(F)$ with $\Lambda(F^n) \ne 0$. From Remark 2.5 we have $F^n \in CACs(X, X)$. As a result Theorem 2.6 guarantees that F^n has a fixed point. □

Theorem 3.2. *Let X be a Hausdorff topological space and $F \in MCACs(X, X)$. Suppose $\chi(F) \ne 0$ or $P(F) \ne 0$. Fix $m \in \{0, 1, \dots\}$. Then F has a periodic point with period n where $m + 1 \le n \le m + P(F)$.*

Proof. We know for Theorem 2.9 that F is a Lefschetz map and also we know that $\Lambda(F^n) \ne 0$ for some n where $m + 1 \le n \le m + P(F)$. From Remark 2.8 we have $F^n \in MCACs(X, X)$. As a result Theorem 2.9 guarantees that F^n has a fixed point. □

Theorem 3.3. *Let X be a Hausdorff topological space and $F \in ACACs(X, X)$. Suppose $\chi(F) \ne 0$ or $P(F) \ne 0$. Fix $m \in \{0, 1, \dots\}$ and suppose $F^n \in ACACs(X, X)$ for any n with $m + 1 \le n \le m + P(F)$. Then F has a periodic point with period n where $m + 1 \le n \le m + P(F)$.*

Proof. We know for Theorem 2.14 that F is a Lefschetz map and also we know that $\Lambda(F^n) \ne 0$ for some n where $m + 1 \le n \le m + P(F)$. By assumption we have $F^n \in ACACs(X, X)$. As a result Theorem 2.14 guarantees that F^n has a fixed point. □

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