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QUANTITATIVE ESTIMATES IN THE OVERCONVERGENCE OF SOME SINGULAR INTEGRALS

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ABSTRACT. In this paper we obtain quantitative estimates in the overconvergence phenomenon for the classical singular integrals of Gauss-Weierstrass, Poisson-Cauchy and Picard.

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1. INTRODUCTION

Let $\xi > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. If f is 2π-periodic or nonperiodic and bounded on $\mathbb{R}$ or of some exponential or polynomial growth on $\mathbb{R}$, the following singular integrals are well defined :

$$P_\xi(f)(x) = \frac{1}{2\xi}\int_{-\infty}^{+\infty} f(x+t)e^{-|t|/\xi}\,dt = \frac{1}{2\xi}\int_{-\infty}^{+\infty} f(u)e^{-|u-x|/\xi}du,$$

$$Q_\xi(f)(x) = \frac{\xi}{\pi}\int_{-\pi}^{\pi}\frac{f(x+t)}{t^2+\xi^2}\,dt = \frac{\xi}{\pi}\int_{x-\pi}^{x+\pi}\frac{f(u)}{(u-x)^2+\xi^2}du,$$

$$Q^*_\xi(f)(x) = \frac{\xi}{\pi}\int_{-\infty}^{+\infty}\frac{f(x+t)}{t^2+\xi^2}\,dt = \frac{\xi}{\pi}\int_{-\infty}^{+\infty}\frac{f(u)}{(u-x)^2+\xi^2}\,du,$$

$$R_\xi(f)(x) = \frac{2\xi^3}{\pi}\int_{-\infty}^{+\infty}\frac{f(x+t)}{(t^2+\xi^2)^2}\,dt = \frac{2\xi^3}{\pi}\int_{-\infty}^{+\infty}\frac{f(u)}{((u-x)^2+\xi^2)^2}\,du,$$

$$W_\xi(f)(x) = \frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^{\pi} f(x+t)e^{-t^2/\xi}\,dt = \frac{1}{\sqrt{\pi\xi}}\int_{x-\pi}^{x+\pi} f(u)e^{-(u-x)^2/\xi}\,du,$$

$$W^*_\xi(f)(x) = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty} f(x+t)e^{-t^2/\xi}\,dt = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{\infty} f(u)e^{-(u-x)^2/\xi}\,du,$$

Here $P_\xi(f)$ is called of Picard type, $Q_\xi(f)$, $Q^*_\xi(f)$ and $R_\xi(f)$ are called of Poisson-Cauchy type and $W_\xi(f)$, $W^*_\xi(f)$ are called of Gauss-Weierstrass type.

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Concerning the approximation of $f(x)$ by the above real singular integrals (as $\xi \to 0$), many qualitative and quantitative results are known, see e.g. [3], [4], [2], [1], to mention only a few.

A quite natural problem would be the study of the overconvergence phenomenon for these singular integrals, that is the approximation of the analytic function $f(z)$ by the complex singular integrals obtained replacing x by z in the above formulas of definition.

Remark 1.1. For each from the above singular integrals, it is easy to show that by replacing $x \in \mathbb{R}$ with $z \in \mathbb{C}$, the two forms identical in the real case, produce two different complex operators. While the first forms of each operator will appear to have the overconvergence phenomenon, unfortunately the second complex form of each operator is not good for approximation. Indeed, to exemplify this fact we consider the case of the Picard singular integral but the considerations for the other operators are similar. In this case, for the second form we get the complex operator

$$P_\xi(f)(z) = \frac{1}{2\xi}\int_{-\infty}^{+\infty} f(u)e^{-|u-z|/\xi}du$$

and taking $f(z) = 1$, $z = x + iy$, after the successive substitutions $u - x = t$, $\eta = 1/\xi$ and $t = v/\eta$ we obtain

$$P_\eta(1)(z) = \int_0^\infty e^{-\sqrt{v^2+\eta^2y^2}}dv < \int_0^\infty e^{-\sqrt{v^2}}dv = 1,$$

for all $\eta > 0$. For any fixed v, y, denoting $F(\eta) = e^{-\sqrt{v^2+\eta^2y^2}}$ we easily get $F'(\eta) < 0$ for all $\eta > 0$, that is $F(\eta)$ is strictly decreasing on $[0, \infty)$. This implies that as function of η (with fixed z), $P_\eta(1)(z)$ is decreasing, therefore from the above inequality it follows $\lim_{\eta\to\infty} P_\eta(f)(z) < 1 = f(z)$ for f the constant function 1.

The aim of this note is to give some quantitative answers to the overconvergence problem for the first complex forms of the above mentioned singular integrals.

2. MAIN RESULTS

The first main result is the following.

Theorem 2.1. *Let $d > 0$ and suppose that $f : S_d \to \mathbb{C}$ is bounded and uniformly continuous in the strip $S_d = \{z = x + iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| \leq d\}$.*

(i) Denoting $P_\xi(f)(z) = \frac{1}{2\xi}\int_{-\infty}^{+\infty} f(z+t)e^{-|t|/\xi}\,dt$, for all $\xi > 0$ and $z \in S_d$ we have

$$|P_\xi(f)(z) - f(z)| \leq \frac{5}{2}\omega_2(f;\xi)_{S_d},$$

where

$$\omega_2(f;\delta)_{S_d} = \sup\{|f(u+t) - 2f(u) + f(u-t)|; u, u-t, u+t \in S_d, |t| \leq \delta\}.$$

(ii) Denoting $R_\xi(f)(z) = \frac{2\xi^3}{\pi}\int_{-\infty}^{+\infty} \frac{f(z+t)}{(t^2+\xi^2)^2}\, dt$, *for all* $\xi > 0$ *and* $z \in S_d$ *we have*

$$|R_\xi(f)(z) - f(z)| \leq C\omega_2(f;\xi)_{S_d},$$

where $C > 0$ *is independent of* z, ξ *and* f.

(iii) Suppose in addition that f *is of Lipschitz class* $\alpha \in (0,1)$ *in* S_d, *that is there exists a constant* $M > 0$ *such that*

$$|f(u) - f(v)| \leq M|u-v|^\alpha, \text{ for all } u, v \in S_d.$$

Denoting $Q^*_\xi(f)(z) = \frac{\xi}{\pi}\int_{-\infty}^{+\infty} \frac{f(z+t)}{t^2+\xi^2}\, dt$, *for all* $\xi > 0$ *and* $z \in S_d$ *we have*

$$|Q^*_\xi(f)(z) - f(z)| \leq C\xi^\alpha,$$

where $C > 0$ *is independent of* z *and* ξ *but depends on* f.

(iv) Denoting $W^*_\xi(f)(z) = \frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty} f(z+t)e^{-t^2/\xi}\, dt$, *for all* $\xi > 0$ *and* $z \in S_d$ *we have*

$$|W^*_\xi(f)(z) - f(z)| \leq C\omega_2(f;\sqrt{\xi})_{S_d},$$

where $C > 0$ *is independent of* z, ξ *and* f.

Proof. (i) If $z \in S_d$ then clearly that for all $t \in \mathbb{R}$ we have $z + t \in S_d$ and since f is bounded in S_d (denote its bound by $M(f)$) it easily follows $|P_\xi(f)(z)| \leq 2M(f)$ for all $z \in S_d$. Therefore $P_\xi(f)(z)$ exists for all $z \in S_d$. Also, the uniform continuity of f on S_d implies that $0 \leq \lim_{\xi\to 0}\omega_2(f;\xi)_{S_d} \leq 2\lim_{\xi\to 0}\omega_1(f;\xi)_{S_d} = 0$.

For all $z \in S_d$ we have

$$\begin{aligned} |P_\xi(f)(z) - f(z)| &= \left|\frac{1}{2\xi}\int_0^\infty [f(z+t) - 2f(z) + f(z-t)]e^{-|t|/\xi}dt\right| \\ &\leq \frac{1}{2\xi}\int_0^\infty \omega_2(f;(t/\xi)\xi)_{S_d}e^{-t/\xi}dt \\ &\leq \omega_2(f;\xi)_{S_d}\frac{1}{2\xi}\int_0^\infty [1 + (t/\xi)]^2 e^{-t/\xi}dt = \frac{5}{2}\omega_2(f;\xi)_{S_d}. \end{aligned}$$

For the last equality see [2], pp. 252–253, proof of Theorem 5.2.

(ii) We reason exactly as at the above point (i). We obtain

$$\begin{aligned} |R_\xi(f)(z) - f(z)| &= \left|\frac{2\xi^3}{\pi}\int_0^\infty \frac{[f(z+t) - 2f(z) + f(z-t)]}{(t^2+\xi^2)^2}dt\right| \\ &\leq \frac{2\xi^3}{\pi}\int_0^{+\infty} \frac{\omega_2(f;(t/\xi)\xi)_{S_d}}{(t^2+\xi^2)^2}\, dt \\ &\leq \omega_2(f;\xi)_{S_d}\frac{2\xi^3}{\pi}\int_0^\infty \left[1 + \frac{t}{\xi}\right]^2 \cdot \frac{1}{(t^2+\xi^2)^2}\, dt \leq C\omega_2(f;\xi)_{S_d}, \end{aligned}$$

since by easy calculation we get that

$$\frac{2\xi^3}{\pi}\int_0^\infty \left[1 + \frac{u}{\xi}\right]^2 \cdot \frac{1}{(t^2+\xi^2)^2}\, dt \leq C,$$

where $C>0$ is independent of ξ.

(iii) We get

$$\begin{aligned}|Q_{\xi}^{*}(f)(z)-f(z)| &= \left|\frac{\xi}{\pi}\int_{-\infty}^{\infty}\frac{[f(z+t)-f(z)]}{u^2+\xi^2}\,du\right| \\ &\leq \frac{\xi}{\pi}\int_{-\infty}^{\infty}\frac{|f(z+t)-f(z)|}{t^2+\xi^2}\,dt \\ &\leq 2M\frac{\xi}{\pi}\int_0^{\infty}\frac{t^{\alpha}}{t^2+\xi^2}\,du \\ &= \frac{2M}{\pi}\xi^{\alpha}\int_0^{\infty}\frac{v^{\alpha}}{v^2+1}dv,\end{aligned}$$

where it is easy to prove that $\int_0^{\infty}\frac{v^{\alpha}}{v^2+1}dv<\infty$.

(iv) We get

$$\begin{aligned}|W_{\xi}^{*}(f)(z)-f(z)| &= \left|\frac{1}{\sqrt{\pi\xi}}\int_0^{\infty}[f(z+t)-2f(z)+f(z-t)]e^{-t^2/\xi}dt\right| \\ &\leq \frac{1}{\sqrt{\pi\xi}}\int_0^{+\infty}\omega_2(f;(t/\sqrt{\xi})\sqrt{\xi})_{S_d}e^{-t^2/\xi}dt \\ &\leq \omega_2(f;\sqrt{\xi})_{S_d}\frac{1}{\sqrt{\pi\xi}}\int_0^{\infty}\left[\frac{t}{\sqrt{\xi}}+1\right]^2 e^{-t^2/\xi}dt \leq C\omega_2(f;\sqrt{\xi})_{S_d},\end{aligned}$$

since

$$\frac{1}{\sqrt{\pi\xi}}\int_0^{\infty}\frac{t^2}{\xi}e^{-t^2/\xi}dt=\frac{1}{\sqrt{\pi}}\int_0^{\infty}v^2e^{-v^2}dv<\infty$$

and

$$\frac{2}{\sqrt{\pi\xi}}\int_0^{\infty}\frac{t}{\sqrt{\xi}}e^{-t^2/\xi}dt=\frac{2}{\sqrt{\pi\xi}}\sqrt{\xi}\int_0^{\infty}ve^{-v^2}dv=\frac{2}{\sqrt{\pi}}\int_0^{\infty}ve^{-v^2}dv<\infty.$$

□

Remark 2.2. Simple examples of functions f satisfying Theorem 2.1 are given by any finite sum of the form $T_m(z)=\sum_{k=0}^{m}a_k\cos(kz)+b_k\sin(kz)$, where $a_k,b_k\in\mathbb{C}$, which for each fixed m is bounded in S_d together with its derivatives of any order. These follow from the relationships $[\sin(z)]'=\cos(z)$, $[\cos(z)]'=-\sin(z)$, for all $z\in\mathbb{C}$ and from the inequalities valid for all for all $z=x+iy\in S_d$ and $k=0,1,2,...,$

$$|\sin(kz)|\leq e^{k|y|}\leq e^{kd},\ |\cos(kz)|\leq e^{k|y|}\leq e^{kd}.$$

These inequalities follow from $\sinh(y)=(e^y-e^{-y})/2$ for $y\in\mathbb{R}$, which implies

$$|\sin(z)|=\sqrt{\sin^2(x)+\sinh^2(y)}=\sqrt{(e^{2y}+e^{-2y})/2-cos^2(x)}\leq e^{|y|}.$$

Replacing now z by kz we obtain the required inequality in the statement for sin. For the upper estimate of $|\cos(z)|$ we take into account that

$$\begin{aligned}|\cos(z)| &= |\sin(\pi/2-z)|=|\sin(\pi/2-x-iy)| \\ &= \sqrt{\cos^2(x)+\sinh^2(y)}\end{aligned}$$

$$= \sqrt{(e^{2y} + e^{-2y})/2 - sin^2(x)} \leq e^{|y|},$$

which proves the inequality for $|\cos(kz)|$ too.

Remark 2.3. If in Theorem 2.1 f is analytic in S_d and we have $|f''(z)| \leq M$ for all $z \in S_d$, then by the mean value theorem in complex analysis, for all $z \in S_d$ and $\xi > 0$ we obtain the more concrete estimates

$$|P_\xi(f)(z) - f(z)| \leq C\xi, , \ |R_\xi(f)(z) - f(z)| \leq C\xi^2, \ |W_\xi^*(f)(z) - f(z)| \leq C\xi.$$

In the cases of the other two singular integrals (on the interval $[-\pi, \pi]$) we can state the following local/pointwise estimates.

Theorem 2.4. *Let $d > 0$ and suppose that $f : S_d \to \mathbb{C}$ is continuous in the strip $S_d = \{z = x + iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| \leq d\}$.*

(i) Denoting $Q_\xi(f)(z) = \frac{\xi}{\pi}\int_{-\pi}^{\pi} \frac{f(z+t)}{t^2+\xi^2}\,dt$, for all $\xi > 0$ and $z \in S_d$ we have

$$|Q_\xi(f)(z) - f(z)| \leq \left[1 + \frac{1}{\pi}\ln(\pi^2+1)\right] \frac{\omega_2(f;\xi)_{[z-\pi,z+\pi]}}{\xi} + \xi\frac{2}{\pi^2}\|f\|_{S_d},$$

where $[z-\pi, z+\pi] = \{(z-\pi)(1-\lambda) + \lambda(z+\pi); \lambda \in [0,1]\}$ and for $0 \leq \delta \leq \pi$

$$\omega_2(f;\delta)_{[z-\pi,z+\pi]}$$
$$= \sup\{|f(u+t) - 2f(u) + f(u-t)|; u, u-t, u+t \in [z-\pi, z+\pi], t \in \mathbb{R}, |t| \leq \delta\}.$$

(ii) Denoting $W_\xi(f)(z) = \frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^{+\pi} f(z+t)e^{-t^2/\xi}\,dt$, for all $\xi > 0$ and $z \in S_d$ we have

$$|W_\xi(f)(z) - f(z)| \leq \frac{C\omega_2(f;\xi)_{[z-\pi,z+\pi]}}{\xi} + 2\|f\|_{S_d}\sqrt{\xi} \cdot \frac{1}{\pi\sqrt{\pi}},$$

where $C > 0$ is independent of z, ξ and f.

Proof. (i) Fix $z \in S_d$. Evidently that for all $t \in \mathbb{R}$ we have $z + t \in S_d$ and the continuity of f in S_d evidently implies that $Q_\xi(f)(z)$ exists for all $z \in S_d$. Also, the uniform continuity of f on $[z-\pi, z+\pi]$ implies

$$0 \leq \lim_{\xi \to 0} \omega_2(f;\xi)_{[z-\pi,z+\pi]} \leq 2 \lim_{\xi \to 0} \omega_1(f;\xi)_{[z-\pi,z+\pi]} = 0.$$

For all $z \in S_d$ we have

$$Q_\xi(f)(z) - f(z) = \frac{\xi}{\pi}\int_0^{\pi} \frac{f(z+t) - 2f(z) + f(z-t)}{t^2+\xi^2}\,dt - f(z)E(\xi),$$

where

$$|E(\xi)| = E(\xi) = 1 - \frac{2\xi}{\pi}\int_0^{\pi} \frac{dt}{t^2+\xi^2} = 1 - \frac{2}{\pi}\operatorname{arctg}\frac{\pi}{\xi} \leq \frac{2}{\pi^2}\xi$$

(for the last estimate $|E(\xi)| \leq \frac{2}{\pi^2}\xi$ see e.g. [2], p. 257).

It follows

$$|Q_\xi(f)(z)-f(z)|\le \frac{\xi}{\pi}\int_0^\pi \frac{|f(z+t)-2f(z)+f(z-t)|}{t^2+\xi^2}\,dt+\|f\|_{S_d}|E(\xi)|$$
$$\le \frac{\xi}{\pi}\int_0^\pi \frac{\omega_2(f;(t/\xi)\xi)_{[z-\pi,z+\pi]}}{t^2+\xi^2}\,dt+\|f\|_{S_d}\cdot|E(\xi)|$$
$$\le \frac{\xi}{\pi}\cdot\omega_2(f;\xi)_{[z-\pi,z+\pi]}\cdot\int_0^\pi\left[1+\frac{t}{\xi}\right]^2\frac{1}{t^2+\xi^2}\,dt+\|f\|_{S_d}\cdot|E(\xi)|.$$

Reasoning as in the proof of Theorem 3.1, pp. 257–258 in [2], we arrive at the desired estimate in statement.

(ii) We reason exactly as at the above point (i). We can write

$$W_\xi(f)(z)-f(z)=\frac{1}{\sqrt{\pi\xi}}\int_0^\pi[f(z+t)-2f(z)+f(z-t)]e^{-t^2/\xi}\,dt$$
$$-f(z)\left[1-\frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^\pi e^{-t^2/\xi}\,dt\right].$$

Here

$$\left|f(z)\left[1-\frac{1}{\sqrt{\pi\xi}}\cdot\int_{-\pi}^\pi e^{-t^2/\xi}\,dt\right]\right|=\left|f(z)\left[1-\frac{2}{\sqrt{\pi\xi}}\int_0^\pi e^{-t^2/\xi}\,dt\right]\right|$$
$$=\left|f(z)\left[\frac{2}{\sqrt{\pi\xi}}\int_0^\infty e^{-t^2/\xi}\,dt-\frac{2}{\sqrt{\pi\xi}}\int_0^\pi e^{-t^2/\xi}\,dt\right]\right|$$
$$=|f(z)|\cdot\left|\frac{2}{\sqrt{\pi\xi}}\int_\pi^\infty e^{-t^2/\xi}\,dt\right|\le\|f\|_{S_d}\cdot\frac{2}{\sqrt{\pi\xi}}\int_\pi^\infty\frac{\xi}{t^2}\,dt=2\|f\|_{S_d}\sqrt{\xi}\cdot\frac{1}{\pi\sqrt{\pi}}.$$

Reasoning as in [2], p. 258), this implies

$$|W_\xi(f)(z)-f(z)|\le\frac{1}{\sqrt{\pi\xi}}\int_0^\pi\omega_2(f;t)_{[z-\pi,z+\pi]}e^{-t^2/\xi}\,dt+2\|f\|_{S_d}\sqrt{\xi}\frac{1}{\pi\sqrt{\pi}}$$
$$\le\frac{C\omega_2(f;\xi)_{[z-\pi,z+\pi]}}{\xi}+2\|f\|_{S_d}\sqrt{\xi}\cdot\frac{1}{\pi\sqrt{\pi}},$$

which proves the theorem. □

In what follows, for $P_\xi(f)(z)$ and $W^*_\xi(f)(z)$ we will consider the weighted approximation on S_d, which seems to be more natural because S_d is unbounded in $\mathbb{C}$.

For this purpose, first we need some general notations. Let $w:S_d\to\mathbb{R}_+$ be a continuous weighted function in S_d, with the properties that $w(z)>0$ for all $z\in S_d$ and $\lim_{|z|\to\infty}w(z)=0$. Define the space

$$C_w(S_d)=\{f:S_d\to\mathbb{C}; f \text{ is continuous in } S_d \text{ and } \|f\|_w<\infty\},$$

where $\|f\|_w:=\sup\{w(z)|f(z)|; z\in S_d\}$.

Also, for $f\in C_w(S_d)$ define the weighted modulus of smoothness

$$\omega_{2,w}(f;t)_{S_d}=\sup\{w(z)|f(z+h)-2f(z)+f(z-h)|; z\in S_d, h\in\mathbb{R}, |h|\le t\}.$$

Remark 2.5. This modulus of smoothness has the properties : a) it is increasing as function of t ; b) $\omega_{2,w}(f;0)_{S_d}=0$; c) $\omega_{2,w}(f;\lambda t)_{S_d}\le(\lambda+1)^2\omega_{2,w}(f;t)_{S_d}$, for all $\lambda,t\ge 0$. (The proofs are similar to those for the functions of real variable in [4], p. 234).

Theorem 2.6. *Let $d>0$ and suppose that $f:S_d\to\mathbb{C}$ is continuous in the strip $S_d=\{z=x+iy\in\mathbb{C};\, x\in\mathbb{R},\ |y|\le d\}$.*

Let the Freud-type weight $w(z)=e^{-q|z|}$ with $q>0$ fixed and $f\in C_w(S_d)$. Denoting $P_\xi(f)(z)=\frac{1}{2\xi}\int_{-\infty}^{+\infty}f(z+t)e^{-|t|/\xi}\,dt$ and $W_\xi^*(f)(z)=\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty}f(z+t)e^{-t^2/\xi}\,dt$, we have

$$\|P_\xi(f)-f\|_w\le\frac{5}{2}\omega_{2,w}(f;\xi)_{S_d},\ \textit{for all } 0<\xi<1/q$$

and

$$\|W_\xi^*(f)-f\|_w\le C\omega_{2,w}(f;\sqrt{\xi})_{S_d},\ \textit{for all } 0<\xi<1,$$

where $C>0$ is independent of z, ξ and f.

Proof. The continuity of f in S_d implies the continuity of $P_\xi(f)$ and $W_\xi^*(f)$ in S_d.

We have

$$\begin{aligned}|w(z)P_\xi(f)(z)|&=\left|\frac{1}{2\xi}\int_{-\infty}^{+\infty}w(z+t)f(z+t)\frac{w(z)}{w(z+t)}e^{-|t|/\xi}\,dt\right|\\&\le\|f\|_w\frac{1}{2\xi}\int_{-\infty}^{\infty}e^{|t|(q-1/\xi)}dt\le C_{\xi,q}\|f\|_w.\end{aligned}$$

Passing to supremum after $z\in S_d$ it follows that $P_\xi(f)\in C_w(S_d)$, for all $0<\xi<1/q$.

Also, for all $z\in S_d$ we obtain

$$\begin{aligned}w(z)|P_\xi(f)(z)-f(z)|&=\left|\frac{1}{2\xi}\int_0^\infty w(z)[f(z+t)-2f(z)+f(z-t)]e^{-|t|/\xi}dt\right|\\&\le\frac{1}{2\xi}\int_0^\infty\omega_{2,w}(f;(t/\xi)\xi)_{S_d}e^{-t/\xi}dt\\&\le\omega_{2,w}(f;\xi)_{S_d}\frac{1}{2\xi}\int_0^\infty[1+(t/\xi)]^2e^{-t/\xi}dt=\frac{5}{2}\omega_{2,w}(f;\xi)_{S_d}.\end{aligned}$$

Passing to supremum with $z\in S_d$ we get the desired estimate for $\|P_\xi(f)-f\|_w$.

In the case of $W^*(f)(z)$ first we get

$$\begin{aligned}|w(z)W_\xi^*(f)(z)|&=\left|\frac{1}{\sqrt{\pi\xi}}\int_{-\infty}^{+\infty}w(z+t)f(z+t)\frac{w(z)}{w(z+t)}e^{-t^2/\xi}\,dt\right|\\&\le\|f\|_w\frac{2}{\sqrt{\pi\xi}}\int_0^\infty e^{t(q-t/\xi)}dt.\end{aligned}$$

But

$$\int_0^\infty e^{t(q-t/\xi)}dt=\int_0^{q+1}e^{t(q-t/\xi)}dt+\int_{q+1}^\infty e^{t(q-t/\xi)}dt,$$

and for $0 < \xi \leq 1$ and $t \geq q+1$ we get $t(q-t/\xi) \leq -t$ and $e^{t(q-t/\xi)} \leq e^{-t}$, which implies

$$\int_{q+1}^{\infty} e^{t(q-t/\xi)} dt \leq \int_{q+1}^{\infty} e^{-t} dt = e^{-(q+1)}.$$

In conclusion, from the above considerations we get $|w(z)W_{\xi}^{*}(f)(z)| \leq C_{\xi,q}\|f\|_w$ and passing to supremum with $z \in S_d$ it immediately follows $W_{\xi}^{*}(f) \in C_w(S_d)$, for all $0 < \xi \leq 1$.

For the estimate, for all $z \in S_d$ we obtain

$$\begin{aligned} w(z)|W_{\xi}^{*}(f)(z) - f(z)| &= \left| \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} w(z)[f(z+t) - 2f(z) + f(z-t)] e^{-t^2/\xi} dt \right| \\ &\leq \frac{1}{\sqrt{\pi\xi}} \int_0^{+\infty} \omega_{2,w}(f; (t/\sqrt{\xi})\sqrt{\xi})_{S_d} e^{-t^2/\xi} dt \\ &\leq \omega_{2,w}(f; \sqrt{\xi})_{S_d} \frac{1}{\sqrt{\pi\xi}} \int_0^{\infty} \left[\frac{t}{\sqrt{\xi}} + 1\right]^2 e^{-t^2/\xi} dt \\ &\leq C\omega_{2,w}(f; \sqrt{\xi})_{S_d}, \end{aligned}$$

(for the last inequality see the proof of Theorem 2.1 (iv)). □

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