Communications in Applied Analysis **14** (2010), no. 1, 39–58

FRACTIONAL TRIGONOMETRIC KOROVKIN THEORY

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ABSTRACT. In this article we study quantitatively with rates the trigonometric weak convergence of a sequence of finite positive measures to the unit measure. Equivalently we study quantitatively the trigonometric pointwise convergence of sequence of positive linear operators to the unit operator, all acting on continuous functions on $[-\pi, \pi]$. From there we derive with rates the corresponding trigonometric uniform convergence of the last. Our inequalities for all of the above in their right hand sides contain the moduli of continuity of the right and left Caputo fractional derivatives of the involved function. From our uniform trigonometric Shisha-Mond type inequality we derive the first trigonometric fractional Korovkin type theorem regarding the trigonometric uniform convergence of positive linear operators to the unit. We give applications, especially to Bernstein polynomials over $[-\pi, \pi]$ for which we establish fractional trigonometric quantitative results.

AMS (MOS) Subject Classification. 26A33, 41A17, 41A25, 41A36, 41A80.

1. INTRODUCTION

In this paper among other we are motivated by the following results.

Theorem 1 (P. P. Korovkin [11], (1960)). *Let $L_n : C([-\pi, \pi]) \to C([-\pi, \pi])$, $n \in \mathbb{N}$, be a sequence of positive linear operators. Assume*

$$L_n(1) \overset{u}{\to} 1 \textit{ (uniformly) }, L_n(\cos t) \overset{u}{\to} \cos t, \quad L_n(\sin t) \overset{u}{\to} \sin t, \textit{ as } n \to \infty.$$

Then $L_n f \overset{u}{\to} f$, for every $f \in C([-\pi, \pi])$ that is $2\pi-$ periodic.

Let $f \in C([a, b])$ and $0 \le h \le b - a$. The first modulus of continuity of f at h is given by

$$\omega_1(f, h) = \sup \{|f(x) - f(y)|; x, y \in [a, b], |x - y| \le h\}$$

If $h > b - a$, then we define

$$\omega_1(f, h) = \omega_1(f, b - a).$$

Another motivation is the following.

Theorem 2 (Shisha and Mond [16], (1968)). *Let $L_1, L_2, \ldots,$ be linear positive operators, whose common domain D consists of real functions with domain $(-\infty, \infty)$.*

Received June 15, 2009
1083-2564 $15.00 ©Dynamic Publishers, Inc.

Suppose $1, \cos x, \sin x, f$ *belong to* D*, where* f *is an everywhere continuous,* 2π*-periodic function,with modulus of continuity* ω_1*. Let* $-\infty < a < b < \infty$*, and suppose that for* $n = 1, 2, \ldots, L_n(1)$ *is bounded in* $[a, b]$*.*

Then for $n = 1, 2, \ldots,$

$$\|L_n(f) - f\|_\infty \leq \|f\|_\infty \|L_n(1) - 1\|_\infty + \|L_n(1) + 1\|_\infty \omega_1(f, \mu_n), \tag{1}$$

where

$$\mu_n = \pi \left\| \left(L_n \left(\sin^2 \left(\frac{t-x}{2} \right) \right) \right) (x) \right\|_\infty^{1/2},$$

and $\|\cdot\|_\infty$ *stands for the sup norm over* $[a, b]$*.*

In particular, if $L_n(1) = 1$*, then* (1) *reduces to*

$$\|L_n(f) - f\|_\infty \leq 2\omega_1(f, \mu_n).$$

One can easily see that, for $n = 1, 2, \ldots,$

$$\mu_n^2 \leq \left(\frac{\pi^2}{2} \right) \Big[\|L_n(1) - 1\|_\infty + \|(L_n(\cos t))(x) - \cos x\|_\infty + \|(L_n(\sin t))(x) - \sin x\|_\infty \Big],$$

so the last along with (1) prove Korovkin's Theorem 1 in a quantitative way and with rates of convergence.

One more motivation follows.

Theorem 3 (see [1], p. 217). *Let* $f \in C^n([-\pi, \pi])$*,* $n \geq 1$*, and* μ *a measure on* $[-\pi, \pi]$ *of mass* $m > 0$*. Put*

$$\beta := \left(\int \left(\sin \frac{|t|}{2} \right)^{n+1} \cdot \mu(dt) \right)^{1/(n+1)} \tag{2}$$

and denote by $w := \omega_1(f^{(n)}, \beta)$ *the modulus of continuity of* $f^{(n)}$ *at* β*. Then*

$$\left| \int f \, d\mu - f(0) \right| \leq |f(0)| \cdot |m - 1| + \sum_{k=1}^{n} \frac{|f^{(k)}(0)|}{k!} \cdot \left| \int t^k \mu(dt) \right| + w[m^{1/(n+1)} + \pi/(n+1)] \cdot \frac{\pi^n \beta^n}{n!}.$$

Final motivation is [3]. A great aid for fractional calculus is [15].

In this article we study quantitatively the rate of trigonometric weak convergence of a sequence of finite positive measures to the unit measure given the existence and presence of the left and right Caputo fractional derivatives of the involved function. That is in the right hand sides of the derived inequalities appear the first moduli of continuity of the above mentioned fractional derivatives, see Theorem 23 and Corollary 24.

Then via the Riesz representation theorem we transfer Theorem 23 into the language of quantitative trigonometric pointwise convergence of a sequence of positive linear operators to the unit operator, all operators acting from $C([-\pi,\pi])$ into itself, see Theorem 25, Corollary 26 and Theorem 28.

From there we derive quantitative results with respect to the sup-norm $\|\cdot\|_\infty$, regarding the trigonometric uniform convergence of positive linear operators to the unit. Again in the right hand side of our inequalities we have moduli of continuity with respect to right and left Caputo derivatives of the engaged function. For the last see Theorem 30, a trigonometric Sisha-Mond type result. From there we derive the first trigonometric Korovkin type convergence theorem at the fractional level, see Theorem 31.

We give applications of our fractional trigonometric Sisha-Mond and trigonometric Korovkin theory, see Corollaries 34–36, etc.

This is the first in literature article studying the trigonometric quantitative convergence of positive linear operators to the unit at the fractional level.

In approximation theory the involvement of fractional derivatives is very rare, almost nothing exists, with the exception of the recent [3]. The only other fractional articles that exist so far are of V. Dzyadyk [6] of 1959, F. Nasibov [12] of 1962, J. Demjanovic [4] of 1975, and of M. Jaskolski [10] of 1989, all regarding estimates to best approximation of functions by algebraic and trigonometric polynomials.

2. BACKGROUND

We need

Definition 4. Let $\upsilon \geq 0$, $n = \lceil \upsilon \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [5], p. 38, [8], [15] the function

$$D_{*a}^{\upsilon} f(x) = \frac{1}{\Gamma(n-\upsilon)} \int_a^x (x-t)^{n-\upsilon-1} f^{(n)}(t)dt, \tag{3}$$

$\forall x \in [a,b]$, where Γ is the gamma function $\Gamma(\upsilon) = \int_0^\infty e^{-t} t^{\upsilon-1} dt, \upsilon > 0$.

We set $D_{*a}^0 f(x) = f(x), \forall x \in [a,b]$.

Lemma 5 ([3]). *Let $\upsilon > 0$, $\upsilon \notin \mathbb{N}$, $n = \lceil \upsilon \rceil$, $f \in C^{n-1}([a,b])$ and $f^{(n)} \in L_\infty([a,b])$. Then $D_{*a}^{\upsilon} f(a) = 0$.*

Definition 6 (see also [8], [7], [2]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta) d\zeta, \tag{4}$$

$\forall x \in [a,b]$. We set $D_{b-}^0 f(x) = f(x)$.

Lemma 7 ([3]). *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(b) = 0$.*

We also need

Lemma 8 ([3]). *Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$; μ is a positive finite measure on the Borel σ-algebra of $[a,b]$, $x_0 \in [a,b]$. Then*

$$E_{x_0}([a,b]) := \int_{[a,b]} f(x)d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[a,b]} (x-x_0)^k d\mu(x)$$
$$= \frac{1}{\Gamma(\alpha)} \Big\{ \int_{[a,x_0]} \left(\int_x^{x_0} (\zeta - x)^{\alpha-1} (D_{x_0-}^{\alpha} f(\zeta) - D_{x_0-}^{\alpha} f(x_0)) d\zeta \right) d\mu(x)$$
$$+ \int_{(x_0,b]} \left(\int_{x_0}^{x} (x-\zeta)^{\alpha-1} (D_{*x_0}^{\alpha} f(\zeta) - D_{*x_0}^{\alpha} f(x_0)) d\zeta \right) d\mu(x) \Big\}. \qquad (2.5)$$

Convention 9. We assume that

$$D_{*x_0}^{\alpha} f(x) = 0, \quad \text{for } x < x_0, \qquad (6)$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \quad \text{for } x > x_0, \qquad (7)$$

for all $x, x_0 \in (a,b]$.

We mention

Proposition 10 ([3]). *Let $f \in C^n([a,b])$, $n = \lceil \upsilon \rceil$, $\upsilon > 0$. Then $D_{*a}^{\upsilon} f(x)$ is continuous in $x \in [a,b]$.*

Also we have

Proposition 11 ([3]). *Let $f \in C^m([a,b])$, $m = \lceil \upsilon \rceil$, $\upsilon > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

We further mention

Proposition 12 ([3]). *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \qquad (8)$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Proposition 13 ([3]). *Let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)d\zeta, \qquad (9)$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

We need

Proposition 14 ([3]). *Let $g \in C([a,b])$, $0 < c < 1, x, x_0 \in [a,b]$. Define*

$$L(x,x_0) = \int_{x_0}^{x} (x-t)^{c-1} g(t)dt, \quad \text{for } x \geq x_0, \qquad (10)$$

and $L(x,x_0)=0$, *for* $x<x_0$.

Then L *is jointly continuous in* (x,x_0) *on* $[a,b]^2$.

We mention

Proposition 15 ([3]). *Let* $g\in C([a,b])$, $0<c<1$, $x,x_0\in[a,b]$. *Define*

$$K(x,x_0)=\int_x^{x_0}(\zeta-x)^{c-1}g(\zeta)d\zeta,\qquad \text{for } x\le x_0, \tag{11}$$

and $K(x,x_0)=0$, *for* $x>x_0$.

Then $K(x,x_0)$ *is jointly continuous from* $[a,b]^2$ *into* $\mathbb{R}$.

Based on Propositions 14, 15 we derive

Corollary 16 ([3]). *Let* $f\in C^m([a,b])$, $m=\lceil\alpha\rceil$, $\alpha>0$, $x,x_0\in[a,b]$. *Then* $D^\alpha_{*x_0}f(x)$, $D^\alpha_{x_0-}f(x)$ *are jointly continuous functions in* (x,x_0) *from* $[a,b]^2$ *into* $\mathbb{R}$.

We need

Theorem 17 ([3]). *Let* $f:[a,b]^2\to\mathbb{R}$ *be jointly continuous. Consider*

$$G(x)=\omega_1(f(\cdot,x),\delta,[x,b]),$$

$\delta>0, x\in[a,b]$.

Then G *is continuous on* $[a,b]$.

Also it holds

Theorem 18 ([3]). *Let* $f:[a,b]^2\to\mathbb{R}$ *be jointly continuous. Then*

$$H(x)=\omega_1(f(\cdot,x),\delta,[a,x]),$$

$x\in[a,b]$, *is continuous in* $x\in[a,b]$, $\delta>0$.

We make

Remark 19. Let μ be a finite positive measure on Borel σ-algebra of $[-\pi,\pi]$. Let $\alpha>0$, then by Hölder's inequality we obtain ($x_0\in[-\pi,\pi]$),

$$\int_{[-\pi,x_0]}(x_0-x)^\alpha d\mu(x)\le 2^\alpha\left(\int_{[-\pi,x_0]}\left(\frac{(x_0-x)}{2}\right)^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}$$

$$\mu([-\pi,x_0])^{\frac{1}{(\alpha+1)}}\le(2\pi)^\alpha\left(\int_{[-\pi,x_0]}(\sin((x_0-x)/4))^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}\mu([-\pi,x_0])^{\frac{1}{(\alpha+1)}}, \tag{12}$$

by $|t|\le\pi\sin(|t|/2), t\in[-\pi,\pi]$.

Similarly we obtain

$$\int_{(x_0,\pi]}(x-x_0)^\alpha d\mu(x)\le 2^\alpha\left(\int_{(x_0,\pi]}\left(\frac{(x-x_0)}{2}\right)^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}$$

$$\mu((x_0,\pi])^{\frac{1}{(\alpha+1)}}\le(2\pi)^\alpha\left(\int_{(x_0,\pi]}(\sin((x-x_0)/4))^{\alpha+1}d\mu(x)\right)^{\frac{\alpha}{(\alpha+1)}}\mu((x_0,\pi])^{\frac{1}{(\alpha+1)}}. \tag{13}$$

Let now $m = \lceil \alpha \rceil$, $\alpha \in \mathbb{N}$, $\alpha > 0$, $k = 1, \ldots, m-1$. Then again by Hölder's inequality we obtain

$$\int_{[-\pi,\pi]} |x - x_0|^k d\mu(x) \tag{2.14}$$

$$\leq 2^k \left(\int_{[-\pi,\pi]} \left(\frac{|x - x_0|}{2} \right)^{\alpha+1} d\mu(x) \right)^{\frac{k}{(\alpha+1)}} (\mu([-\pi,\pi]))^{\frac{\alpha+1-k}{(\alpha+1)}}$$

$$\leq (2\pi)^k \left(\int_{[-\pi,\pi]} (\sin(|x - x_0|/4))^{\alpha+1} d\mu(x) \right)^{\frac{k}{(\alpha+1)}} \mu([-\pi,\pi])^{\frac{\alpha+1-k}{(\alpha+1)}},$$

Terminology 20. Here $C([-\pi,\pi])$ denotes all the real valued continuous functions on $[-\pi,\pi]$. Let $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. By Riesz representation theorem (see [14], p. 304) we have

$$L_N(f, x_0) = \int_{[-\pi,\pi]} f(t) d\mu_{Nx_0}(t), \tag{15}$$

$\forall\, x_0 \in [-\pi,\pi]$, where μ_{Nx_0} is a unique positive finite measure on a Borel algebra of $[-\pi,\pi]$. Call

$$L_N(1, x_0) = \mu_{Nx_0}([-\pi,\pi]) = M_{Nx_0}. \tag{16}$$

We make

Remark 21 ([3]). Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \upsilon \rceil \upsilon > 0$, $\upsilon \notin \mathbb{N}$.

Then we have

$$|D^{\upsilon}_{*a} f(x)| \leq \frac{\|f^{(n)}\|_\infty}{\Gamma(n - \upsilon + 1)} (x - a)^{n-\upsilon}, \quad \forall\, x \in [a,b]. \tag{17}$$

Thus we observe

$$\omega_1(D^{\upsilon}_{*a} f, \delta) = \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} |D^{\upsilon}_{*a} f(x) - D^{\upsilon}_{*a} f(y)|$$

$$\leq \sup_{\substack{x,y \in [a,b] \\ |x-y| \leq \delta}} \left(\frac{\|f^{(n)}\|_\infty}{\Gamma(n - \upsilon + 1)} (x - a)^{n-\upsilon} + \frac{\|f^{(n)}\|_\infty}{\Gamma(n - \upsilon + 1)} (y - a)^{n-\upsilon} \right) \tag{2.18}$$

$$\leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n - \upsilon + 1)} (b - a)^{n-\upsilon}. \tag{2.19}$$

Consequently

$$\omega_1(D^{\upsilon}_{*a} f, \delta) \leq \frac{2\|f^{(n)}\|_\infty}{\Gamma(n - \upsilon + 1)} (b - a)^{n-\upsilon}. \tag{20}$$

Similarly, let $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, then

$$\omega_1(D^{\alpha}_{b-} f, \delta) \leq \frac{2 \parallel f^{(m)} \parallel_\infty}{\Gamma(m - \alpha + 1)} (b - a)^{m-\alpha}. \tag{21}$$

So for $f \in C^{m-1}([a,b])$, $f^{(m)} \in L_\infty([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$, we find

$$\sup_{x_0 \in [a,b]} \omega_1(D^{\alpha}_{*x_0} f, \delta)_{[x_0,b]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m - \alpha + 1)} (b - a)^{m-\alpha}, \tag{22}$$

and

$$\sup_{x_0\in[a,b]} \omega_1(D^{\alpha}_{x_0-}f,\delta)_{[a,x_0]} \leq \frac{2\|f^{(m)}\|_\infty}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{23}$$

We also make

Remark 22. Let $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, be a sequence of positive linear operators. Using (15) and Hölder's inequality we obtain ($x \in [-\pi,\pi]$, $k = 1,\dots,m-1$, $m = \lceil\alpha\rceil, \alpha \notin \mathbb{N}, \alpha > 0$) for $k = 1,\dots,m-1$ that

$$\left\|L_N\left(|\cdot - x|^k, x\right)\right\|_\infty \leq (2\pi)^k \left(\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{k}{(\alpha+1)}}\right) \|L_N 1\|_\infty^{(\alpha+1-k)/(\alpha+1)}. \tag{24}$$

Notice that for any $x \in [-\pi,\pi]$ we have

$$C([-\pi,\pi]) \ni |\cdot - x|\mathcal{X}_{[-\pi,\pi]}(\cdot) \leq |\cdot - x| \in C([-\pi,\pi]),$$

therefore

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{25}$$

Consequently, by positivity of L_N we obtain

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty. \tag{26}$$

Similarly, for any $x \in [-\pi,\pi]$ we have

$$C([-\pi,\pi]) \ni |\cdot - x|\mathcal{X}_{[x,\pi]} \leq |\cdot - x| \in C([-\pi,\pi]),$$

thus

$$C([-\pi,\pi]) \ni \left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}}{4}\right)\right)^{\alpha+1} \leq \left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1} \in C([-\pi,\pi]). \tag{27}$$

Hence

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty. \tag{28}$$

So if the right hand side of (26), (28) goes to zero, so do their left hand sides.

In fact we notice that

$$\left(\sin\frac{|\cdot - x|}{4}\right)^{\alpha+1} = \left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1} + \left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, \tag{29}$$

for every $x \in [-\pi,\pi]$.

Hence it holds

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \leq \left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty$$

$$+\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty. \tag{30}$$

Consequently, if both

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty,$$

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \to 0,$$

as $N\to+\infty$, then

$$\left\|L_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{\alpha+1},x\right)\right\|_\infty \to 0.$$

3. MAIN RESULTS

We present our first main result

Theorem 23. *Let $f\in AC^m([-\pi,\pi])$, $f^{(m)}\in L_\infty([-\pi,\pi])$, $m=\lceil\alpha\rceil,\alpha\notin\mathbb{N}$, $\alpha>0$; $r_1,r_2>0$, μ is a positive finite measure on the Borel σ-algebra of $[-\pi,\pi]$, $x_0\in[-\pi,\pi]$. Then*

$$\left|\int_{[-\pi,\pi]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{[-\pi,\pi]}(x-x_0)^k d\mu(x)\right|$$

$$\leq\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left\{\left[(\mu([-\pi,x_0]))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r_1}\right]\right.$$

$$\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^\alpha_{x_0-}f,r_1\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}$$

$$+\left[(\mu((x_0,\pi]))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r_2}\right]\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^\alpha_{*x_0}f,r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right\}. \tag{31}$$

Proof. By (5) we obtain

$$E_{x_0}([-\pi,\pi])$$
$$\leq \frac{1}{\Gamma(\alpha)}\left\{\int_{[-\pi,x_0]}\left(\int_x^{x_0}(\zeta-x)^{\alpha-1}|D^\alpha_{x_0-}f(\zeta)-D^\alpha_{x_0-}f(x_0)|d\zeta\right)d\mu(x)\right.$$
$$\left.+\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-\zeta)^{\alpha-1}|D^\alpha_{*x_0}f(\zeta)-D^\alpha_{*x_0}f(x_0)|d\zeta\right)d\mu(x)\right\}=(*). \tag{32}$$

Let $h_1, h_2 > 0$, then

$$(*)\leq\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\int_x^{x_0}(\zeta-x)^{\alpha-1}\left(1+\frac{x_0-\zeta}{h_1}\right)d\zeta\right)\right.\right.$$
$$\left.d\mu(x)\right]\omega_1(D^\alpha_{x_0-}f,h_1)_{[-\pi,x_0]}$$
$$\left.+\left[\int_{(x_0,\pi]}\left(\int_{x_0}^{x}(x-\zeta)^{\alpha-1}\left(1+\frac{\zeta-x_0}{h_2}\right)d\zeta\right)d\mu(x)\right]\omega_1(D^\alpha_{*x_0}f,h_2)_{[x_0,\pi]}\right\}. \tag{33}$$

I.e.

$$E_{x_0}([-\pi,\pi])$$
$$\leq\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^\alpha}{\alpha}+\frac{1}{h_1}\left(\int_x^{x_0}(x_0-\zeta)^{2-1}(\zeta-x)^{\alpha-1}d\zeta\right)\right.\right.\right.$$
$$\left.d\mu(x)\right]\omega_1(D^\alpha_{x_0-}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^\alpha}{\alpha}+\frac{1}{h_2}\int_{x_0}^{x}(x-\zeta)^{\alpha-1}(\zeta-x_0)^{2-1}d\zeta\right)d\mu(x)\right]$$
$$\left.\omega_1(D^\alpha_{*x_0}f,h_2)[x_0,\pi]\right\} \tag{34}$$
$$=\frac{1}{\Gamma(\alpha)}\left\{\left[\int_{[-\pi,x_0]}\left(\frac{(x_0-x)^\alpha}{\alpha}+\frac{1}{h_1}\frac{(x_0-x)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]\right.$$
$$\omega_1(D^\alpha_{x_0-}f,h_1)_{[-\pi,x_0]}$$
$$\left.+\left[\int_{(x_0,\pi]}\left(\frac{(x-x_0)^\alpha}{\alpha}+\frac{1}{h_2}\frac{(x-x_0)^{\alpha+1}}{\alpha(\alpha+1)}\right)d\mu(x)\right]\omega_1(D^\alpha_{*x_0}f,h_2)_{[x_0,\pi]}\right\}. \tag{35}$$

Therefore

$$E_{x_0}([-\pi,\pi])\leq\frac{1}{\Gamma(\alpha)}\left\{\left[\frac{1}{\alpha}\int_{[-\pi,x_0]}(x_0-x)^\alpha d\mu(x)\right.\right.$$
$$\left.+\frac{1}{h_1\alpha(\alpha+1)}\int_{[-\pi,x_0]}(x_0-x)^{\alpha+1}d\mu(x)\right]\omega_1(D^\alpha_{x_0-}f,h_1)_{[-\pi,x_0]}$$
$$+\left[\frac{1}{\alpha}\int_{(x_0,\pi]}(x-x_0)^\alpha d\mu(x)+\frac{1}{h_2\alpha(\alpha+1)}\int_{(x_0,\pi]}(x-x_0)^{\alpha+1}d\mu(x)\right]$$
$$\left.\omega_1(D^\alpha_{*x_0}f,h_2)_{[x_0,\pi]}\right\}. \tag{36}$$

Momentarily we assume positive choices of

$$h_1=r_1\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}>0, \tag{37}$$

$$h_2=r_2\left(\int_{(x_0,-\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}>0. \tag{38}$$

Consequently, by (12),(13) and (36), we obtain

$$E_{x_0}([-\pi,\pi]) \le \frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)} \left\{ \left[(\mu([-\pi,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_1} \right] \left(\frac{h_1}{r_1}\right)^{\alpha} \right.$$

$$\omega_1(D^{\alpha}_{x_0-}f,h_1)_{[-\pi,x_0]} + \left[(\mu((x_0,\pi]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_2} \right] \left(\frac{h_2}{r_2}\right)^{\alpha}$$

$$\left. \omega_1(D^{\alpha}_{*x_0}f,h_2)_{[x_0,\pi]} \right\}, \tag{39}$$

proving (31).

Next we examine the special cases. If

$$\int_{(x_0,\pi]} \left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1} d\mu(x) = 0,$$

then $\sin\left(\frac{x-x_0}{4}\right) = 0$, a.e. on $(x_0,\pi]$, that is $x = x_0$ a.e. on $(x_0,\pi]$, more precisely $\mu\{x \in (x_0,\pi] : x \neq x_0\} = 0$, hence $\mu(x_0,\pi] = 0$. Therefore μ concentrates on $[-\pi,x_0]$. In that case (31) is written and holds as

$$\left| \int_{[-\pi,x_0]} f(x)d\mu(x) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} \int_{[-\pi,x_0]} (x-x_0)^k d\mu(x) \right|$$

$$\le \frac{(2\pi)^{\alpha}}{\Gamma(\alpha+1)} \left\{ \left[(\mu([-\pi,x_0]))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r_1} \right] \right.$$

$$\left(\int_{[-\pi,x_0]} \left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu(x) \right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\omega_1\left(D^{\alpha}_{x_0-}f, r_1 \left(\int_{[-\pi,x_0]} \left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1} d\mu(x) \right)^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_0]} \tag{40}$$

Since $(\pi,\pi] = \emptyset$ and $\mu(\emptyset) = 0$, in the case of $x_0 = \pi$, we get again (40) written for $x_0 = \pi$. So inequality (40) is a valid inequality when

$$\int_{[-\pi,x_0]} \left(\sin\left(\frac{(x_0-x)}{4}\right)\right)^{\alpha+1} d\mu(x) \neq 0.$$

If additionally we assume that

$$\int_{[-\pi,x_0]} \left(\sin\left(\frac{(x_0-x)}{4}\right)\right)^{\alpha+1} d\mu(x) = 0,$$

then $\sin(\frac{x_0-x}{4}) = 0$, a.e. on $[-\pi,x_0]$, that is $x = x_0$ a.e. on $[-\pi,x_0]$, which means $\mu\{x \in [-\pi,x_0] : x \neq\} = 0$. Hence $\mu = \delta_{x_0}M$, where δ_{x_0} is the unit Dirac measure and $M = \mu([-\pi,\pi]) > 0$.

In the last case we obtain L.H.S (40) = R.H.S (40) = 0, that is (40) is valid trivially.

At last we go the other way around. Let us assume that

$$\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)=0,$$

then reasoning similarly as before, we get that μ over $[-\pi,x_0]$ concentrates at x_0. That is $\mu=\delta_{x_0}\mu([-\pi,x_0])$, on $[-\pi,x_0]$.

In the last case (31) is written and holds as

$$\left|\int_{(x_0,\pi]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{(x_0,\pi]}(x-x_0)^k d\mu(x)\right|$$

$$\leq\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left\{\left[(\mu((x_0,\pi]))^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r_2}\right]\right.$$

$$\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^\alpha_{*x_0}f,r_2\left(\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right\}. \tag{41}$$

If $x_0=-\pi$, then (41) can be redone and rewritten, just replace $(x_0,\pi]$ by $[-\pi,\pi]$ all over. So inequality (41) is valid when

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\neq 0.$$

If additionally we assume that

$$\int_{(x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)=0,$$

then as before $\mu(x_0,\pi]=0$. Hence (41) is trivially true, in fact L.H.S (41) = R.H.S (41) = 0. The prof of (31) now is completed in all possible cases. □

We continue in a special case.

In the assumptions of Theorem 23, when $r=r_1=r_2>0$, and by calling $M=\mu([-\pi,\pi])\geq\mu([-\pi,x_0]),\mu((x_0,\pi])$, we get

Corollary 24. *It holds*

$$\left|\int_{[-\pi,\pi]}f(x)d\mu(x)-\sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}\int_{[-\pi,\pi]}(x-x_0)^k d\mu(x)\right|$$

$$\leq\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[M^{\frac{1}{(\alpha+1)}}+\frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\omega_1\left(D^{\alpha}_{x_0-}f, r\left(\int_{[-\pi,x_0]}\left(\sin\left(\frac{x_0-x}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}$$

$$+\left(\int_{[x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f, r\left(\int_{[x_0,\pi]}\left(\sin\left(\frac{x-x_0}{4}\right)\right)^{\alpha+1}d\mu(x)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right]. \tag{42}$$

Based on Theorem 23, Corollary 24 and (15), we obtain

Theorem 25. *Let $f \in AC^m([-\pi,\pi]), f^{(m)} \in L_\infty([-\pi,\pi])$, $m = \lceil\alpha\rceil, \alpha \notin \mathbb{N}$, $\alpha > 0$; $r > 0$, and $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $n \in \mathbb{N}$, a sequence of positive linear operators, $x_0 \in [-\pi,\pi]$. Then*

$$\left|L_N(f,x_0) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((x-x_0)^k, x_0)\right|$$

$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\omega_1\left(D^{\alpha}_{x_0-}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}$$

$$+\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\omega_1\left(D^{\alpha}_{*x_0}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\right]. \tag{43}$$

Corollary 26 (to Theorem 25). *It holds*

$$\left|L_N(f,x_0) - \sum_{k=0}^{m-1}\frac{f^{(k)}(x_0)}{k!}L_N((x-x_0)^k, x_0)\right|$$

$$\leq \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\omega_1\left(D^{\alpha}_{x_0-}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}\right.$$

$$\omega_1\left(D^{\alpha}_{*x_0}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\Bigg]$$

$$\left(L_N\left(\left(\sin\left(\frac{|x-x_0|}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{\alpha}{(\alpha+1)}}. \tag{44}$$

We make

Remark 27. Let $f \in AC([-\pi,\pi])$, $f' \in L_\infty([-\pi,\pi])$, $0 < \alpha < 1$, $x_0 \in [-\pi,\pi]$; $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, sequence of positive linear operators. Then by Theorem 25 and

$$|L_N(f,x_0) - f(x_0)| \le |L_N(f,x_0) - f(x_0)L_N(1,x_0)| + |f(x_0)||L_n(1,x_0) - 1|, \tag{45}$$

we obtain

Theorem 28. *Let* $f \in AC([-\pi,\pi])$, $f' \in L_\infty([-\pi,\pi])$,$0 < \alpha < 1$, $r > 0$, $x_0 \in [-\pi,\pi]$; $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, *sequence of positive linear operators. Then*

$$|L_N(f,x_0) - f(x_0)| \le |f(x_0)|\,|L_N(1,x_0) - 1|$$
$$+\frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[(L_N(1,x_0))^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$
$$\Bigg[\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\omega_1\left(D^{\alpha}_{x_0-}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[-\pi,x_0]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[-\pi,x_0]}$$
$$+\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\left(\frac{\alpha}{\alpha+1}\right)}$$
$$\omega_1\left(D^{\alpha}_{*x_0}f, r\left(L_N\left(\left(\sin\left(\frac{|x-x_0|\mathcal{X}_{[x_0,\pi]}(x)}{4}\right)\right)^{\alpha+1}, x_0\right)\right)^{\frac{1}{(\alpha+1)}}\right)_{[x_0,\pi]}\Bigg]. \tag{46}$$

We make

Remark 29. We observe that

$$R.H.S(43) \le \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$
$$\Bigg[\left\|L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\sup_{x\in[-\pi,\pi]} \omega_1 \left(D^{\alpha}_{x-}f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{[-\pi,x]}$$

$$+ \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left. \sup_{x\in[-\pi,\pi]} \omega_1 \left(D^{\alpha}_{*x}f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{[x,\pi]} \right] =: \theta. \quad (47)$$

So that

$$Z := \left\| L_N(f, x_0) - \sum_{k=0}^{m-1} \frac{f^{(k)}(x_0)}{k!} L_N((\cdot - x)^k, x) \right\|_{\infty} \leq \theta. \quad (48)$$

We further observe that

$$|L_N(f,x) - f(x)| \leq Z + |f(x)| \; |L_N(1,x) - 1| + \sum_{k=0}^{m-1} \frac{|f^{(k)}(x)|}{k!} \left| L_N((\cdot - x)^k, x) \right|$$

$$\leq |f(x)| \, |L_N(1,x) - 1| + \sum_{k=1}^{m-1} \frac{|f^{(k)}(x)|}{k!} \left| L_N((\cdot - x)^k, x) \right| + \theta. \quad (49)$$

We have proved the main result, a Shisha-Mond type trigonometric inequality at the fractional level.

Theorem 30. *Let $f \in AC^m([-\pi,\pi])$, $f^{(m)} \in L_\infty([-\pi,\pi])$, $m = \lceil \alpha \rceil$, $\alpha \notin \mathbb{N}$, $\alpha > 0$,$r > 0$ and $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, a sequence of positive linear operators, $x \in [-\pi,\pi]$. Then*

$$\|L_N f - f\|_\infty \leq \|f\|_\infty \|L_N 1 - 1\|_\infty + \sum_{k=1}^{m-1} \frac{\|f^{(k)}\|_\infty}{k!} \|L_N((\cdot - x)^k, x)\|_\infty$$

$$+ \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[\|L_N(1)\|_\infty^{1/(\alpha+1)} + \frac{2\pi}{(\alpha+1)r} \right]$$

$$\left[\left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\left(\frac{\alpha}{\alpha+1}\right)} \right.$$

$$\sup_{x\in[-\pi,\pi]} \omega_1 \left(D^{\alpha}_{x-}f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{[-\pi,x]}$$

$$+ \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left. \sup_{x\in[-\pi,\pi]} \omega_1 \left(D^{\alpha}_{*x}f, r \left\| L_N \left(\left(\sin\left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty}^{\frac{1}{\alpha+1}} \right)_{[x,\pi]} \right]. \quad (50)$$

Next we derive the following trigonometric Korovkin type convergence result at fractional level.

Theorem 31. *Let $\alpha \notin \mathbb{N}$, $\alpha > 0$, $m = \lceil \alpha \rceil$, and $L_N : C([-\pi,\pi]) \to C([-\pi,\pi])$, $N \in \mathbb{N}$, a sequence of positive linear operators. Assume $L_N 1 \overset{n}{\to} 1$ (uniformly), and*

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty} \to 0,$$

as $N \to \infty$. Then $L_N f \overset{u}{\to} f$, $\forall\ f \in AC^m([-\pi,\pi])$, $f^{(m)} \in L_\infty([-\pi,\pi])$. (The second condition means $\left(L_N \left(\left(\sin \left(\frac{|\cdot - x|}{4} \right) \right)^{\alpha+1} \right) \right)(x) \overset{u}{\to} 0$, $x \in [-\pi,\pi]$.)

Proof. Since $\|L_N 1 - 1\|_\infty \to 0$ we get $\|L_N 1 - 1\|_\infty \le K$, for some $K > 0$. We write $L_N 1 = L_N 1 - 1 + 1$, hence

$$\|L_N 1\|_\infty \le \|L_N 1 - 1\|_\infty + \|1\|_\infty \le K + 1, \quad \forall\ N \in \mathbb{N}.$$

That is $\|L_N 1\|_\infty$ is bounded. Se we are using inequality (50). By assumption $\|L_N((\sin(\frac{|\cdot - x|}{4}))^{\alpha+1}, x)\|_\infty \to 0$, as $N \to \infty$ and (24) we get $\|L_N(|\cdot - x|^k, x)\|_\infty \to 0$ for $k = 1, \ldots, m-1$. Also by (26) and (28) we obtain that

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty},$$

and

$$\left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[x,\pi]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_{\infty} \to 0,$$

as $N \to \infty$.

Additionally by (22) and (23) we get that

$$\sup_{x \in [-\pi,\pi]} \omega_1 \left(D^\alpha_{x-} f, \cdot \right)_{[-\pi,x]}, \quad \sup_{x \in [-\pi,\pi]} \omega_1 (D^\alpha_{*x} f; \cdot)_{[x,\pi]} \le \frac{2\|f^{(m)}\|_\infty}{\Gamma(m - \alpha + 1)} (2\pi)^{m-\alpha},$$

so they are bounded.

Thus based on the above, from (50), we derive that $\|L_N f - f\|_\infty \to 0$, proving the claim. ∎

We make

Remark 32. Based on Corollary 16 and Theorem 17, 18, given that $f \in C^m([-\pi,\pi])$, we obtain that,

$$(i) \quad \sup_{x \in [-\pi,\pi]} \omega_1 \left(D^\alpha_{x-} f, r \left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|_\infty^{\frac{1}{\alpha+1}} \right)_{[-\pi,x]}$$

$$= \omega_1 \left(D^\alpha_{x_1-} f, r \left\| L_N \left(\left(\sin \left(\frac{|\cdot - x| \mathcal{X}_{[-\pi,x]}(\cdot)}{4} \right) \right)^{\alpha+1}, x \right) \right\|^{\frac{1}{(\alpha+1)}} \right)_{[-\pi,x_1]} \to 0, \quad (51)$$

as

$$\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0,$$

when $N \to \infty$, for some $x_1 \in [-\pi, \pi]$.

Similarly

$$(ii) \qquad \sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x} f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}$$

$$= \omega_1\left(D^\alpha_{*x_2} f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{(\alpha+1)}}\right)_{[x_2,\pi]} \to 0, \qquad (52)$$

as

$$\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty \to 0,$$

when $N \to \infty$, for some $x_2 \in [-\pi, \pi]$.

Corollary 33. *Here* $L_N : C([-\pi, \pi]) \to C([-\pi, \pi])$, $N \in \mathbb{N}$, *positive linear operators. Let* $0 < \alpha < 1$, $r > 0$, $f \in AC([-\pi, \pi])$, $f' \in L_\infty([-\pi, \pi])$. *Then*

$$\|L_N f - f\|_\infty \le \|f\|_\infty \|L_N 1 - 1\|_\infty + \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)}\left[\|L_N(1)\|_\infty^{\frac{1}{(\alpha+1)}} + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-} f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[-\pi,x]}\right\}$$

$$+ \left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x} f, r\left\| L_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}\right\}\right]. \qquad (53)$$

4. APPLICATION

Consider the Bernstein polynomials on $[-\pi,\pi]$ for $f \in C([-\pi,\pi])$:

$$(B_N f)(x) = \sum_{k=0}^{N} \binom{N}{k} f\left(-\pi + \frac{2\pi k}{N}\right) \left(\frac{x+\pi}{2\pi}\right)^k \left(\frac{\pi - x}{2\pi}\right)^{N-k},$$

$N \in \mathbb{N}$, any $x \in [-\pi,\pi]$. There are positive linear operators from $C([-\pi,\pi])$ into itself. Here let $0<\alpha<1$, $r>0$ and take $f \in AC([-\pi,\pi])$, $f' \in L_\infty([-\pi,\pi])$. Setting $g(t) = f(2\pi t - \pi)$, $t \in [0,1]$, we have $g(0) = f(-\pi)$, $g(1) = f(\pi)$, and

$$(B_N g)(t) = \sum_{k=0}^{N} \binom{N}{k} g\left(\frac{k}{N}\right) t^k (1-t)^{N-k} = (B_N f)(x), x \in [-\pi,\pi].$$

Here $x = \varphi(t) = 2\pi t - \pi$ is an $1-1$ and onto map from [0,1] onto $[-\pi,\pi]$. Clearly here $g \in AC([0,1])$ and $g' \in L_\infty([0,1])$.

Notice also that

$$(B_N((\cdot - x)^2))(x) = [(B_N((\cdot - t)^2))(t)](2\pi)^2 = \frac{(2\pi)^2}{N} t(1-t)$$

$$= \frac{(2\pi)^2}{N}\left(\frac{x+\pi}{2\pi}\right)\left(\frac{\pi - x}{2\pi}\right) = \frac{1}{N}(x+\pi)(\pi - x) \le \frac{\pi^2}{N}, \forall x \in [-\pi,\pi].$$

I.e.

$$(B_N((\cdot - x)^2))(x) = \le \frac{\pi^2}{N}, \quad \forall\, x \in [-\pi,\pi].$$

In particular $(B_N 1)(x) = 1$, $\forall\, x \in [-\pi,\pi]$.

Applying Corollary 33 we get

Corollary 34. *It holds*

$$\|B_N f - f\|_\infty \le \frac{(2\pi)^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{2\pi}{(\alpha+1)r}\right]$$

$$\left[\left\| B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{x-} f, r \left\| B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[-\pi,x]}\right\}$$

$$+ \left\| B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\left(\frac{\alpha}{\alpha+1}\right)}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D^\alpha_{*x} f, r \left\| B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\alpha+1}, x\right)\right\|_\infty^{\frac{1}{\alpha+1}}\right)_{[x,\pi]}\right\}\right], \quad (54)$$

$\forall N \in \mathbb{N}$.

Next let $\alpha = \frac{1}{2}$, and $r = \frac{1}{\alpha+1}$, that is $r = \frac{2}{3}$. Notice $\Gamma(\frac{3}{2}) = \frac{\sqrt{\pi}}{2}$.

Corollary 35. *Let* $f \in AC([-\pi,\pi])$, $f' \in L_\infty([-\pi,\pi])$, $n \in \mathbb{N}$. *Then*

$$\|B_N f - f\|_\infty \le 2\sqrt{2}(2\pi+1)\left[\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{1}{3}}\right.$$

$$\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{x-}^{\frac{1}{2}} f, \frac{2}{3}\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[-\pi,x]}\right\}$$

$$+\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{1}{3}}$$

$$\left.\left\{\sup_{x\in[-\pi,\pi]} \omega_1\left(D_{*x}^{\frac{1}{2}} f, \frac{2}{3}\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty^{\frac{2}{3}}\right)_{[x,\pi]}\right\}\right], \quad (55)$$

$\forall\ N \in \mathbb{N}$.

By $|\sin x| < |x|$, $\forall\ x \in \mathbb{R} - \{0\}$, in particular $\sin x \le x$, for $x \ge 0$, we get

$$\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{3/2} \le \left(\frac{|\cdot - x|}{4}\right)^{3/2} = \frac{1}{8}|\cdot - x|^{3/2}.$$

Hence

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{\frac{3}{2}}, x\right)\right\|_\infty \le \frac{1}{8}\left\|B_N\left(|\cdot - x|^{\frac{3}{2}}, x\right)\right\|_\infty. \quad (56)$$

We observe that

$$B_N(|\cdot - x|^{3/2}, x) = \sum_{k=0}^N \left|x + \pi - \frac{2\pi k}{N}\right|^{3/2}\binom{N}{K}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi - x}{2\pi}\right)^{N-k}$$

(by discrete Hölder's inequality)

$$\le \left[\sum_{k=0}^N \left(x + \pi - \frac{2\pi k}{N}\right)^2\binom{N}{k}\left(\frac{x+\pi}{2\pi}\right)^k\left(\frac{\pi - x}{2\pi}\right)^{N-k}\right]^{3/4}$$

$$= \left(B_N\left((\cdot - x)^2, x\right)\right)^{3/4} \le \frac{\pi^{3/2}}{N^{3/4}}, \ \forall \quad x \in [-\pi,\pi]. \quad (57)$$

Consequently it holds

$$\|B_N(|\cdot - x|^{3/2}, x)\|_\infty \le \frac{\pi^{3/2}}{N^{3/4}}, \quad (58)$$

and

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|}{4}\right)\right)^{3/2}, x\right)\right\|_\infty \le \frac{\pi^{3/2}}{8N^{3/4}}, \ \forall \quad N \in \mathbb{N}. \quad (59)$$

Therefore we get

$$\left\|B_N\left(\left(\sin\left(\frac{|\cdot - x|\mathcal{X}_{[-\pi,x]}(\cdot)}{4}\right)\right)^{3/2}, x\right)\right\|_\infty,$$

$$\left\| B_N\left(\left(\sin\left(\frac{|\cdot-x|\mathcal{X}_{[x,\pi]}(\cdot)}{4}\right)\right)^{3/2},x\right)\right\|_\infty$$

$$\leq \left\| B_N\left(\left(\sin\left(\frac{|\cdot-x|}{4}\right)\right)^{3/2},x\right)\right\|_\infty \leq \frac{\pi^{3/2}}{8N^{3/4}},\ \forall \quad N\in\mathbb{N}. \tag{60}$$

We have proved

Corollary 36. *Let $f\in AC([-\pi,\pi])$, $f'\in L_\infty([-\pi,\pi])$, $N\in\mathbb{N}$. Then*

$$\|B_Nf-f\|_\infty \leq \frac{(2\pi+1)\sqrt{2\pi}}{\sqrt[4]{N}}\left[\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{x-}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[-\pi,x]}\right.$$

$$\left.+\sup_{x\in[-\pi,\pi]}\omega_1\left(D_{*x}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[x,\pi]}\right]. \tag{61}$$

So as $N\to\infty$ we derive that $B_Nf\overset{u}{\to}f$ with rates.

Discussion 37. From (61), Corollary 16, and Theorems 17, 18 we obtain that

$$\|B_Nf-f\|_\infty \leq \frac{(2\pi+1)\sqrt{2\pi}}{\sqrt[4]{N}}\left[\omega_1\left(D_{x_1-}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[-\pi,x_1]}\right.$$

$$\left.+\omega_1\left(D_{*x_2}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[x_2,\pi]}\right], \tag{62}$$

for some $x_1,x_2\in([-\pi,\pi])$, $f\in C^1([-\pi,\pi])$.

Therefore

$$\|B_Nf-f\|_\infty \leq \left(\frac{2\pi+1}{\sqrt[4]{N}}\right)(\sqrt{2\pi})\left[\omega_1\left(D_{x_1-}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[-\pi,x]}\right.$$

$$\left.+\omega_1\left(D_{*x_2}^{1/2}f,\frac{\pi}{6\sqrt{N}}\right)_{[-\pi,\pi]}\right]. \tag{63}$$

Further we assume that $D_{x_1-}^{1/2}f$ and $D_{*x_2}^{1/2}$ are Lipschitz functions of order 1, that is

$$\left|D_{x_1-}^{1/2}f(x)-D_{x_1-}^{1/2}f(y)\right|\leq K_1|x-y|, \tag{64}$$

and

$$\left|D_{*x_2-}^{1/2}f(x)-D_{*x_2-}^{1/2}f(y)\right|\leq K_2|x-y|, \tag{65}$$

$\forall x,y\in[-\pi,\pi]$, and $K_1,K_2>0$. Then from (63) we obtain

$$\|B_Nf-f\|_\infty \leq \frac{\pi\sqrt{2\pi}\,(2\pi+1)}{6N^{3/4}}(K_1+K_2). \tag{66}$$

REFERENCES

[1] G. A. Anastassiou, *Moments in Probability and Approximation Theory*, Pitman Research Notes in Math., Vol. 287, Longman Sci. & Tech., Harlow, U.K., 1993.

[2] G. A. Anastassiou, On right fractional calculus, accepted 2008, in *Chaos, Solitons and Fractals.*

[3] G. A. Anastassiou, Fractional Korovkin theory, accepted 2009, in *Chaos, Solitons and Fractals.*

[4] Jurii Dem'janovič, Approximation by local functions in a space with fractional derivatives. (Lithuanian English summaries), *Differencial'nye Uravnenja i Primenen, Trudy Sem. Processy*, **103**(1975), 35–49.

[5] Kai Diethelm, *Fractional Differential Equations,* online:http://www.tu-bs.de/~diethelm/lehre/f-dg102/fde-skript.ps.gz

[6] V. K. Dzyadyk, On the best trigonometric approximations in the L metric of some functions (Russian), *Dokl. Akad. Nauk SSSR*, **129** (1959), 19–22.

[7] A. M. A. El-Sayed and M. Gaber, On the finite Caputo and finite Riesz derivatives,*Electronic Journal of Theoretical Physics*, Vol. **3**, No. 12 (2006), 81–95.

[8] G. S. Frederico and D. F. M. Torres, Fractional Optimal Control in the sense of Caputo and the functional Noether's theorem, *International Mathematical Forum*, Vol. **3**, No. 10 (2008), 479–493.

[9] R. Gorenflo and F. Mainardi, *Essentials of Fractional Calculus,* 2000, Maphysto Center, http://www.maphysto.dk/oldpages/events/LevyCAC2000/MainardiNotes/fm2k0a.ps

[10] Miroslaw Jaskólski, Contributions to fractional calculus and approximation theory on the square, *Funct. Approx. Comment. Math.*, **18**(1989), 77–89.

[11] P. P. Korovkin, *Linear Operators and Approximation Theory,* Hindustan Publ. Corp. Delhi, India, 1960.

[12] F. G. Nasibov, On the degree of best approximation of functions having a fractional derivative in the Riemann-Liouville sense. (Russian-Azerbaijani summary), *Izv. Akad. Nauk Azerbaĭdžan. SSR. Ser. Fiz.-Mat. Tehn. Nauk*, (1962), No. 3, 51–57.

[13] T. Popoviciu, Sur l'approximation des functions convexes d'ordre superieur,*Mathematica (Cluj)*, **10**(1935), 49–54.

[14] H. L. Royden, *Real Analysis,* 2nd Edition, Macmillan, New York, 1968.

[15] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives, Theory and Applications,* (Gordon and Breach, Amsterdam, 1993) [English translation from the Russian, Integrals and Derivatives of Fractional Order and Some of Their Applications (Nauka i Tekhnika, Minsk, 1987)].

[16] O. Shisha and B. Mond, The degree of approximation to periodic functions by linear positive operators, *Journal of Approximation Theory*, **1**(1968), 335–339.